Stochastic Stability in Assignment Problems

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Overview and Literature

Two-sided matching markets are important!

▷ buyers and sellers
▷ workers and firms
▷ marriage and college admissions markets

But (decentralized) outcomes are not always stable
- by blocking agents sometimes can get better outcomes.

We consider a decentralized process where blocking pairs can randomly form.

Does such a dynamic process end (in a stable outcome) at some point? If yes, which ones? Even if agents make mistakes?
Overview and Literature

Shapley and Shubik (1972): seminal paper on assignment problems showing that the set of stable outcomes (equivalent to the core) exists and has nice structural properties.

Furthermore, there exist (blocking) paths to such stable outcomes with each step of the path involving players rematching with new partners to obtain payoffs higher than those they currently obtain with their current partners. Such paths to stability results were recently obtained by

- Biró, Bomhoff, Golovach, Kern, and Paulusma (2013),
- Chen, Fujishige, and Yang (2012),
- Klaus and Payot (2013), and
- Nax, Pradelski, and Young (2013).
But which core outcomes are most likely?

We can start at some initial outcome and see which core outcome is most likely to be reached under a dynamic process.

Predictions then depend on the initial outcome...

...and on the exact probabilities with which agents are chosen to rematch.
Overview

We analyze a perturbed dynamic...

...so that agents can make mistakes and rematch with a partner inferior to their current one.

(i) uniform errors: all errors are equally likely &

(ii) stepped errors: we differentiate between error costs $\delta \in (0, 1)$ (for indifferences) and 1 (for payoff losses).

More generally, the existence of 1-errors (at cost 1) and 0-errors (at cost $\delta$) suffices for our results.

Stochastic stability then serves as a prediction for which outcomes are likely to be realized in the long run.
Assignment Problems

- **two-sided market**: $W$ and $F$ are sets of workers and firms.

- $(w,f)$: a *pair* formed by worker $w \in W$ and firm $f \in F$.

- **one-to-one problem (unit demand)**: a worker can work for at most one firm and a firm can employ at most one worker.

- $v(w,f) \in \mathbb{N}_0$: *value* that $w$ and $f$ create by forming a pair.

- $v(i,i) = 0$: *reservation value* of agent $i \in W \cup F$.

- A (two-sided one-to-one) assignment problem is a triple $(W,F,v)$. 

Assignment Problems

- **A matching** $\mu$ is a function $\mu : W \cup F \rightarrow W \cup F$ of order two (that is, $\mu(\mu(i)) = i$), such that

  for $w \in W$, if $\mu(w) \neq w$, then $\mu(w) \in F$ and

  for $f \in F$, if $\mu(f) \neq f$, then $\mu(f) \in W$.

  We write $\mu(i) = j$ or $(i,j) \in \mu$ (possibly $i = j$) if agents $i$ and $j$ are matched.

- **An outcome** (for $(W, F, v)$) is a pair $(\mu, u)$ where $\mu$ is a matching and $u$ is a payoff vector, such that

  (i) if $(w,f) \in \mu$, then $u_w + u_f = v(w,f)$, and

  (ii) if $(i,i) \in \mu$, then $u_i = v(i,i) = 0$.

  $\Rightarrow \sum_{i \in W \cup F} u_i = \sum_{(i,j) \in \mu} v(i,j)$
Assignment Problems

- An outcome \((\mu, u)\) [a payoff vector \(u\)] is **core stable** if
  
  (a) \((\mu, u)\) is **individually rational**, i.e.,
  
  \[
  \text{for all } i \in W \cup F, \ u_i \geq 0 \text{ and}
  \]
  
  (b) there are no **blocking pairs** for \((\mu, u)\), i.e.,
  
  \[
  \text{for all } (w,f) \in W \times F, \ u_w + u_f \geq v(w,f).
  \]

- A matching \(\mu\) is **optimal** if, for all matchings \(\mu'\),

  \[
  \sum_{(i,j) \in \mu} v(i,j) \geq \sum_{(i,j) \in \mu'} v(i,j).
  \]

  A worker \(w\) and a firm \(f\) are **optimal partners** if there exists an optimal matching \(\mu\) such that \((w,f) \in \mu\).

- Loosely speaking:

  \((\mu, u)\) stable \(\iff \mu\) is optimal.
Blocking Paths

A blocking path is a sequence of outcomes \((\mu^1, u^1), \ldots, (\mu^k, u^k)\) such that

\[(\mu^l, u^l) \rightarrow (\mu^{l+1}, u^{l+1}) :\]

- a blocking pair \((w_l, f_l)\) is matched and splits value \(v(w_l, f_l)\) such that

\[u_{w_l}^{l+1} \geq u_{w_l}^l \text{ and } u_{f_l}^{l+1} \geq u_{f_l}^l\]

with at least one strict inequality,

- their former partners are single and receive zero payoffs,

- all the other agents are matched to the same partners and obtain the same payoffs than at outcome \((\mu^l, u^l)\).

A blocking path \((\mu^1, u^1), \ldots, (\mu^k, u^k)\) leads to stability if the last outcome \((\mu^k, u^k)\) is stable.

A blocking path to stability always exists!
Unperturbed Dynamic

We define the following Markov process on the set of outcomes. Under the unperturbed dynamic, every period

- a worker and a firm, or a single agent, is chosen randomly [with full support],

- randomly [with full support] some share of payoffs (were the pair or single agent to match) is proposed, and

- if these payoffs are
  
  (i) at least as high as their existing payoffs for both agents and

  (ii) strictly higher for at least one of the agents,

  then let them match at these payoffs;

- otherwise, let the outcome remain as before.

Finiteness implies that the unperturbed dynamic always leads to stability.
Perturbed Dynamic

We define the following Markov process on the set of outcomes. Under the perturbed dynamic, every period

- a worker and a firm, or a single agent, is chosen randomly [with full support],
- randomly [with full support] some share of payoffs (were the pair or single agent to match) is proposed, and
- if these payoffs are
  (i) at least as high as their existing payoffs for both agents and
  (ii) strictly higher for at least one of the agents
  then let them match at these payoffs;
- otherwise, let them match at these payoffs with probabilities $\varepsilon(costs)$ depending either on uniform or stepped errors.
Perturbed Dynamic

An error process is **stepped** if for any two outcomes $o \neq o'$ such that $o'$ is obtained from $o$,

$$c_{(i,j)}(o, o') = \begin{cases} 
0 & o \rightarrow o' \text{ by weak blocking}, \\
1 & o \rightarrow o' \text{ with some payoff loss}, \\
\delta, & 0 < \delta < 1 \quad o \rightarrow o' \text{ with indifference}.
\end{cases}$$

The two most common error specifications are **Uniform Errors** (Young, 1993) and **Logit Errors** (Blume, 1993).

Our results for **stepped errors** will imply corresponding results for logit error and weakly payoff monotone error dynamics.

In fact, it will suffice to only consider so-called **1-errors** (at cost 1) and **0-errors** (at cost $\delta$).
Stochastic Stability

For a perturbed blocking dynamic we obtain

- the invariant distribution $p(\varepsilon) = \text{the (unique!) long-run probability distribution over outcomes}$,

- the limit invariant distribution $p^* = \lim_{\varepsilon \to 0} p(\varepsilon)$, and

- stochastically stable outcomes:

  an outcome is stochastically stable if it is in the support of $p^*$. The set $SS(W, F, v)$ denotes the set of stochastically stable outcome.

We show that for assignment problems, only the least cost of leaving a stable outcome matters (because by construction, other outcomes can be reached at zero costs via the unperturbed dynamic).
Preliminary Results

An agent who loses a payoff of 1 (the smallest payoff loss possible) makes a 1-error.

We summarize two important preliminary results as follows.

**Theorem (Moving closer)**

Take two stable outcomes $o, o' \in \text{Core}(W, F, v)$. Then, a 1-error suffices to move to a stable outcome that is either payoff closer or match closer to $o'$ than $o$ is.
Main Results

Theorem (No selection for uniform errors)

*If the error process is uniform, then $SS(W, F, v, c) = Core(W, F, v)$.*

Does a similar result apply for stepped errors?

No, because with multiple optimal partners there can be errors with zero payoff loss to those making the error [They are errors because they are not part of the unperturbed dynamic].

To simplify terminology, we refer to such zero payoff errors (at cost $\delta$) as 0-errors.
Main Results

Example (Selection for logit errors)

If the error process is logit, then $SS(W, F, v, c) \not\subseteq Core(W, F, v)$ is possible.

- stable $u_{w_3} < 4 \rightarrow$ stable $\tilde{u}_{w_3} \geq 4$ costs $\delta$,
- stable $u_{w_3} \geq 4 \rightarrow$ any other stable $\tilde{u}_{w_3}$ costs at least $\delta + 1$.
- stochastic stability $\Rightarrow u_{w_3} \geq 4$ (in fact $\Leftrightarrow$).
Main Results

For agents with more than one optimal partner, a 0-error can be used to replicate a 1-error.

This then implies that all stable payoffs for agents with more than one optimal partner can be reached at minimal costs $\delta$.

Then, based on our preliminary theorems, stochastic stability does not select over matchings or payoffs for agents with more than one optimal partner.

**Theorem (No selection for agents with $>1$ optimal partner)**

For all $(\tilde{\mu}, \tilde{u}) \in \text{Core}(W, F, v)$, there exists $(\tilde{\mu}, u^*) \in SS(W, F, v, c)$ such that for all agents $i$ with more than one optimal partner, $u^*_i = \tilde{u}_i$. 
Main Results

If there is a unique optimal matching, then there are almost no payoffs allowing for 0-errors,...

...except for those at which the value function constraint is binding for two currently unmatched agents (i.e., \( \mu(i) \neq j \) and \( u_i + u_j = v(i,j) \)).

Hence, boundary stable outcomes can be left via a 0-error.

However, interior stable outcomes require an error of cost at least \( \delta \) to move elsewhere.

Theorem (Almost no selection if unique optimal matching)

*If there is a unique optimal matching, then the interior of the core equals the stochastically stable set.*