Strong belief and agreeing to disagree

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Abstract

In this paper, we extend Aumann’s agreement theorem to a framework where beliefs are modeled by a conditional probability system à la Battigalli and Siniscalchi (1999). Indeed, if posterior beliefs are derived from a common prior that puts positive probability to some state at which the posteriors are common strongly believed, then these posteriors are equal conditional on each conditioning event that contains this state.

Keywords: Conditional belief hierarchies, strong belief, agreement theorem.

1. Introduction

According to the famous agreement theorem, if two agents have a common prior, and their posteriors for an event are commonly believed at some state that receives positive probability by the prior, then they necessarily agree on the same posterior beliefs (Aumann, 1976).\(^1\) The main contribution of this result is twofold: on the one hand, it has been been crucial for epistemically characterizing solution concepts such as for instance Nash equilibrium (Aumann and Brandenburger, 1995), while at the same time, it has helped us to understand the role of asymmetric information on betting, trading and speculation (Milgrom and Stokey, 1982; Sebenius and Geanakoplos, 1983).\(^2\)

Recent advancements in the theory of games with incomplete information, as well as in the epistemic approach to game theory, have recognized that it is often important to take into account the players’ beliefs conditional on very unlikely events. For instance, the standard characterizations

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\(^1\)Notice that Aumann’s original result is slightly weaker, as it is in fact stated in terms of knowledge which is stronger than belief.

\(^2\)For a survey on the agreement theorem, see Bonanno and Nehring (1997).
of iterated admissibility in normal form games (Brandenburger et al., 2008), or rationalizability in extensive form games (Battigalli and Siniscalchi, 2002) rely on this type of beliefs. In fact, there are two ways of modeling beliefs given unlikely events in game theory – lexicographic probability systems and conditional probability systems – both of them generalizing the standard probabilistic beliefs. In a recent paper, Bach and Perea (2013) prove Aumann’s agreement theorem with lexicographic beliefs. In this paper, we complete the analysis by extending the agreement theorem to the framework of conditional probability systems. In fact we show that if two agents have a common prior, and their posteriors for an event are common strongly believed at some state that receives positive probability by the prior, then they necessarily agree on the same posterior beliefs conditional on every event that contains this state.

2. Conditional probability systems and strong belief

Let \((X, \mathcal{A})\) be a measurable space, with \(\mathcal{B} \subseteq \mathcal{A} \setminus \{\emptyset\}\) being a collection of non-empty conditioning events. A mapping \(\mu(\cdot|\cdot) : \mathcal{A} \times \mathcal{B} \to [0, 1]\) is called a conditional probability system (CPS) on \((X, \mathcal{B})\) whenever it satisfies the following conditions:

\((C_1)\) \(\mu(B|B) = 1\), for all \(B \in \mathcal{B}\);

\((C_2)\) \(\mu(\cdot|B)\) is a probability measure over \((X, \mathcal{A})\), for all \(B \in \mathcal{B}\);

\((C_3)\) \(\mu(A|C) = \mu(A|B) \cdot \mu(B|C)\), for all \(A \subseteq B \subseteq C\) with \(A \in \mathcal{A}\) and \(B, C \in \mathcal{B}\).

According to the usual interpretation, a CPS describes the conditional beliefs (given each conditioning hypothesis in \(\mathcal{B}\)) of an agent who satisfies Bayes rule whenever possible. As usual, \(\Delta^\mathcal{B}(X)\) denotes the space of all CPS’s on \((X, \mathcal{B})\). Standard beliefs are a special case that arises when the collection of conditioning events is the singleton \(\mathcal{B} = \{X\}\), i.e, standard beliefs are essentially unconditional beliefs. Conditional probability systems were first introduced by Rényi (1955).

**Definition 1.** We say that an event \(A \in \mathcal{A}\) is strongly believed whenever it is the case that \(\mu(A|B) = 1\) for each \(B \in \mathcal{B}\) with \(A \cap B \neq \emptyset\).

Strong belief was introduced by Battigalli and Siniscalchi (2002), and generalizes standard notion of probability-1 belief in the case of conditional probability systems. That is, in the degenerate case of standard beliefs, where the collection of conditioning events is the singleton \(\mathcal{B} = \{X\}\), strong belief coincides with probability-1 belief.

\(^3\)For a discussion on the relationship between the two, we refer to Tsakas (2012).
Following Battigalli and Siniscalchi (1999) we extend conditional probability systems to an interactive setting. Consider a finite set of agents $I$ with typical elements $i$ and $j$. Throughout the paper, for simplicity and without loss of generality, we assume that $I = \{a, b\}$ contains two agents. Let $\Theta$ be a countable space of uncertainty, with $\mathcal{B}$ being the common collection of non-empty conditioning events. Each agent forms a hierarchy of conditional beliefs, consisting of a CPS over $\Theta$ (first order conditional beliefs), a CPS over the opponent’s first order conditional beliefs (second order conditional beliefs), and so on.

Hierarchies of conditional beliefs are very complex objects, which makes it very hard to handle and sometimes even to describe. Following Harsanyi’s type-based construction, Battigalli and Siniscalchi (1999) proposed an indirect way of representing hierarchies of conditional beliefs. First, for an arbitrary measurable space $Y$, let $\Delta^B(\Theta \times Y)$ denote the space of CPS’s over $(\Theta \times Y, \mathcal{B} \times Y)$, where $\mathcal{B} \times Y := \{B \times Y | B \in \mathcal{B}\}$ is the collection of conditioning events in $\Theta \times Y$ induced by $\mathcal{B}$. Formally, $\mathcal{B} \times Y$ contains the cylinders generated by $\mathcal{B}$.

We define a type space model as a tuple $(\Theta, \mathcal{B}, (T_i)_{i \in I}, (g_i)_{i \in I})$, where $T_i$ is a finite space of types and $g_i : T_i \to \Delta^B(\Theta \times T_j)$ is a function mapping each type $t_i \in T_i$ to a CPS over $\Theta \times T_j$. For each $t_i \in T_i$ and for each $B \in \mathcal{B}$, let $g_i^B(t_i) \in \Delta(\Theta \times T_j)$ denote $t_i$’s beliefs over $\Theta \times T_j$ given the conditioning event $B \times T_j$. In the degenerate case with a single conditioning event $\mathcal{B} = \{\Theta\}$, the function $g_i$ maps each $t_i \in T_i$ to a single Borel probability measure in $\Delta(\Theta \times T_j)$ similarly to Harsanyi (1967-68).

Throughout the paper, let $\Omega := \Theta \times T_a \times T_b$ be the corresponding state space. Each state $\omega \in \Omega$ is a complete description of all relevant features of the environment, as it determines an element $\theta(\omega) := \text{proj}_\omega \omega$ of the underlying space of uncertainty, as well as a type $t_i(\omega) := \text{proj}_{T_i} \omega$ for each $i \in I$. Then, for each $B \subseteq \Theta$ let $[B] := \{\omega \in \Omega : \theta(\omega) \in B\}$, and for each $t_i \in T_i$ let $[t_i] := \{\omega \in \Omega : t_i(\omega) = t_i\}$. Finally, for each $E \subseteq \Theta \times T_j$ let $[E] := \{\omega \in \Omega : (\theta(\omega), t_j(\omega)) \in E\}$.

**Definition 2.** We say that $i$’s conditional beliefs are derived from a prior $\pi \in \Delta(\Omega)$ if for all $t_i \in T_i$ and $B \in \mathcal{B}$ with $\pi([B] \cap [t_i]) > 0$ it is the case that

$$g^B_i(t_i)(E) = \frac{\pi([E] \cap [B] \cap [t_i])}{\pi([B] \cap [t_i])}$$

for every $E \subseteq \Theta \times T_j$. We say that $\pi \in \Delta(\Omega)$ is a common prior if every agent’s conditional beliefs are derived from $\pi$.

For each $F \subseteq \Omega$, let $F_{\Theta \times T_j} := \text{proj}_{\Theta \times T_j} F$. We say that $i$ strongly believes in $F \subseteq \Omega$ at $\omega \in \Omega$ whenever $t_i(\omega)$ strongly believes in $F_{\Theta \times T_j}$, i.e., whenever $g^B_i(t_i(\omega))(F_{\Theta \times T_j}) = 1$ for each $B \in \mathcal{B}$ with $F_{\Theta \times T_j} \cap (B \times T_j) \neq \emptyset$. 

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3. Generalized agreement theorem

According to Aumann’s famous agreement theorem, in the degenerate framework of standard beliefs, if the beliefs of two agents are derived from a common prior and their posteriors at some state that receives positive probability by the common prior are common belief, then these posteriors must be equal. In this paper, we generalize this result to the framework of conditional beliefs.

Indeed, assume that the conditional beliefs of two agents are derived from a common prior which assigns positive probability to some state at which (i) the posterior beliefs are common strong belief, and (ii) the state belongs to the conditioning event $B$. Then, each player’s posterior beliefs given the conditioning event $B$ must be equal.

We first define the types of player $i$ that assign probability $q^B_i \in [0,1]$ to $A \in \mathcal{A}$ given $B \in \mathcal{B}$ by

$$Q^B_i := \{ t_i \in T_i : g^B_i(t_i)(A \times T_j) = q^B_i \},$$

while the types that assign probabilities $(q^B_i)_{B \in \mathcal{B}} \in [0,1]^\mathcal{B}$ to $A$ given each conditioning event in $\mathcal{B}$ are the ones contained in

$$Q_i := \bigcap_{B \in \mathcal{B}} Q^B_i.$$

We define the event $Q := \Theta \times Q_a \times Q_b$. Then, for each $i \in I$, we iteratively define the set of types that satisfy up to $k$-fold strong belief in $Q$:

$$SB^1_i(Q) := \bigcap_{B \in \mathcal{B}} \{ t_i \in Q_i : (\Theta \times Q_j) \cap (B \times T_j) \neq \emptyset \Rightarrow g^B_i(t_i)(\Theta \times Q_j) = 1 \}$$

$$SB^{k+1}_i(Q) := \bigcap_{B \in \mathcal{B}} \{ t_i \in SB^k_i(Q) : (\Theta \times SB^k_j(Q)) \cap (B \times T_j) \neq \emptyset \Rightarrow g^B_i(t_i)(\Theta \times SB^k_j(Q)) = 1 \}$$

while $CSB_i(Q) := \bigcap_{k>0} SB^k_i(Q)$ contains $i$’s types that satisfy common strong belief in the event that for every $j \in I$ it is the case that $j$’s posterior beliefs given each $B \in \mathcal{B}$ put probability $q^B_j$ to $A$. Then, define

$$CSB(Q) := \Theta \times CSB_a(Q) \times CSB_b(Q).$$

**Main Theorem.** Suppose that there is a common prior $\pi \in \Delta(\Omega)$ such that $\pi(\{B\} \cap CSB(Q)) > 0$. Then, $q^B_a = q^B_b$.

**Proof.** Since $CSB_i(Q) \subseteq Q_i$, it follows that $g^B_i(t_i)(A \times T_j) = q^B_i$ for all $t_i \in CSB_i(Q)$ and all $B \in \mathcal{B}$. Now, let $B \in \mathcal{B}$ be such that $\pi(\{B\} \cap CSB(Q)) > 0$, and observe that $\{B\} \cap CSB(Q) = B \times CSB_a(Q) \times CSB_b(Q)$. Hence, for each $i \in I$ there exists some $t_i \in CSB_i(Q)$ such that
Recall that by hypothesis $\pi([B] \cap CSB(Q)) > 0$, thus implying that

$$q_i^B = \frac{\pi([A] \cap [B])}{\pi([B] \cap CSB(Q))},$$

and notice that $q_i^B$ does not depend on $i$. Hence, $q_i^B = q^B$, which completes the proof.

Notice that our assumption about not only $CSB(Q)$, but also $[B]$ receiving positive probability by $\pi$ is a crucial one, as illustrated by the following example.

**Example 1.** Let $\Theta = \{\theta_1, \theta_2, \theta_3\}$ be the underlying space of uncertainty, with $B = \{\{\theta_1\}, \{\theta_2, \theta_3\}\}$ be the common collection of conditioning events of the two agents in $I = \{a, b\}$. Moreover, consider the type spaces model $(\Theta, B, T_a, T_b, g_a, g_b)$ such that $T_a = \{t_{a}^{0}\}$, $T_b = \{t_{b}^{1}, t_{b}^{2}\}$, with $g_a^{(\theta_1)}(t_{a}^{0})$ being uniformly distributed over $\{(\theta_1, t_{b}^{1}), (\theta_1, t_{b}^{2})\}$ and $g_b^{(\theta_2, \theta_3)}(t_{b}^{k})$ being uniformly distributed over $\{(\theta_2, t_{b}^{1}), (\theta_2, t_{b}^{2})\}$, while at the same time $g_a^{(\theta_1)}(t_{a}^{0}) = 1$ and $g_b^{(\theta_2, \theta_3)}(t_{b}^{k})(\{\theta_3, t_{b}^{0}\}) = 1$ for each $k \in \{1, 2\}$. Notice that the conditional beliefs are derived from the common prior $\pi \in \Delta(\Theta \times T_a \times T_b)$ which is uniformly distributed over $\{(\theta_1, t_{a}^{0}, t_{b}^{1}), (\theta_1, t_{a}^{0}, t_{b}^{2})\}$. Now, consider the event $A = \{\theta_1, \theta_2\}$, and observe that it is common strongly believed that $a$ puts probability 1 to $A$ both given $\{\theta_1\}$ and given $\{\theta_2, \theta_3\}$, whereas $b$ puts probability 1 to $A$ given $\{\theta_1\}$ and probability 0 to $A$ given $\{\theta_2, \theta_3\}$. Hence, $a$ and $b$ completely disagree on their probabilistic assessment over $A$ given the conditioning event $\{\theta_2, \theta_3\}$. 

Notice that in the previous example disagreement occurs only given the conditioning event that receives 0 probability by the common prior, whereas the two agents agree given the conditional event that receives positive probability by the common prior, in accordance to our Main Theorem.

**References**


