Abstract

This paper analyzes a model of asset pricing under asymmetric information with multiple informed traders. Informed traders may have private information of two types: information on the fundamental asset value and information concerning an exogenous persistent demand shock. They need not be symmetrically informed; some may be better informed about the asset value, while others better informed about the demand shock. Results demonstrate that rational informed traders have incentives to initially exacerbate the demand shock, undertaking a form of front running. Surprisingly, these incentives exist even when an informed trader has no information about the value of the asset. These trading strategies give rise to periods of inflated prices. Additionally, trade flows of the informed traders in the model mimic the trade flows of well informed investors during the technology bubble and the South Sea bubble.

1 Introduction

Asset prices perform a crucial role in the efficient allocation of capital and resources. Prices are determined by a number of factors including the presence of private information. Existing literature has primarily focused on information about fundamentals, such as asset dividends, and cases of a single informed trader, or multiple symmetrically informed traders. This paper shows that non-fundamental information, information with no direct relation to an asset’s value, and asymmetries among informed traders are important for asset pricing and can explain such phenomena as extended periods of inflated and deflated prices. In doing so, the paper builds a rigorous, dynamic micro-structure model in which long lived traders compete to exploit both fundamental and non-fundamental information.
The model in this paper builds on the model of Kyle (1985). A single asset is traded dynamically over a fixed window of time, and pays a dividend at the close of trading. At each point in time strategic informed traders and liquidity traders submit an order flow to the market. A group of competitive market makers observe the aggregate order flow and clear the market at a price equal to the expected value of the asset, conditional on the current and past history of the aggregate order flow. This paper diverges from the standard framework in two key ways. First, it allows for multiple informed traders with a wide range of asymmetric signal structures. Second, informed traders may have two sources of private information: fundamental information which is modeled by private information about the dividend, and non-fundamental information which takes the form of a persistent demand shock. The first dimension of information is standard, while the second dimension is used to capture forms of “soft” information, such as exogenous shocks to firms’ liquidity requirements. This demand shock is modeled as a mean demand from liquidity traders at any point in time.

This paper characterizes the linear equilibrium for a broad set of signal structures. More precisely, under the assumption of a (small) quadratic adjustment cost, it characterizes the linear equilibrium by a system of first order differential equations with boundary conditions. Including a small cost permits the application of standard dynamic programming techniques, and ensures that an informed trader’s strategy is well defined. The results suggest that, as the quadratic cost becomes small, the information about the fundamental value is revealed to the market, for all information structures. Further, if a signal about the asset value is shared among two or more traders, then it will be revealed almost instantaneously. This is the case even if they have additional information. The model shows the presence of market manipulation, where an informed trader trades against their private estimate of the value of the asset, making a short term loss in order to fool the market and increase future profits. Furthermore, the paper shows how an informed trader may choose to acquire a less informative signal about the asset fundamentals in order to reduce competition among traders.

Contrarian behavior, or market manipulation, is rare in the literature on insider trading. In both Kyle’s (1985) paper, and the multiple informed trader extensions (Foster and Viswanathan (1996) in discrete time, and Back et al. (2000) in continuous time), informed traders always trade to correct market mispricings. They sell the asset when it is overpriced by the market, and buy an asset when it is underpriced by the market. Thus, the average of the informed traders’ actions acts to correct asset prices, pushing the price towards the fundamental value of the asset.

In this model, contrarian behavior can occur when an informed trader manipulates the market by amplifying the demand shock. This form of market manipulation was shown in the case of a single informed trader with perfect private information about both the asset fundamentals and the non-fundamental demand shock (Sadzik and Woolnough 2014). Sadzik and Woolnough demonstrate that as a monopolist of both forms of information, the informed trader can take fuller advantage of their fundamental information by pushing the demand shock to the wrong side of the market. This paper goes a step further and builds a stronger result, demonstrating that this form of market manipulation is even of benefit to an informed trader who has no information about fundamentals. By trading strongly with the liquidity trader near the beginning of trading, an informed trader pushes (on average) the market’s mis-estimation of the demand shock and mis-estimation of asset value to opposite sides of

---

1A similar adjustment cost is also used by and Subrahmanyam (1998) and Gao and Ou-Yang (2014).
each other. For example, if the demand shock is positive the informed could buy. This pushes up the price and, on average, the asset will be over priced. The price will be corrected by other informed traders with information about value, but as long as this happens gradually the informed trader can offset the demand shock for the rest of the trading window and make a profit (see Section 3.3).

Market manipulation can also be caused by differing levels of information dispersion. When some forms of information are known to a large group of traders, competition to exploit this information becomes fierce, and this information is heavily traded on. Interestingly, some traders may be better informed, and realize that this information is incorrect, but are still willing to trade on this information anyway. This stops their superior information being revealed to traders with the common information.

To illustrate this, consider a situation with two informed traders and two types of private fundamental information, common information and exclusive information. Both traders receive a signal about the common information, but the exclusive information is held by just one, better informed, trader. For simplicity, assume that the exclusive information is a perfect signal of the asset value. In equilibrium, both traders initially trade very heavily on the common signal. For the better informed trader, trading heavily on the common signal plays two roles. When the signal is on the correct side of the market, this helps the informed trader capture the opportunity for profit before the other informed trader. When the signal is on the wrong side of the market, trading heavily on the common information makes a loss. However, this is still optimal for the better informed trader as it stops the other informed trader inferring the true value of the asset. Surprisingly, this situation can also lead to a more direct form of market manipulation. The heavy trading on the common information makes the price very responsive to the order flow, relatively. This makes market manipulation relatively cheap, as the better informed trader can easily distort the price. (See Section 3.2)

In Section 4, an example of a symmetric equilibrium is provided to illustrate how informed traders’ desires to inflate the demand shock can create price paths similar to bubbles. Prices are initially pushed up for strategic reasons. Informed traders then start to sell and the price collapses. In equilibrium, if informed traders initially observe a positive demand shock, they accentuate this shock by initially going long in the stock. This not only has the effect of increasing that market’s estimate of the value, it also increases the estimates of the other informed traders. As the price rises, each informed trader will estimate that the asset is over priced (knowing there is a bubble), but will not be aware of the full size. Eventually informed traders will cash out and start to dump the asset. As this happens, the price is stabilized temporarily by the exogenous demand shock, but eventually the selling pressure causes the price to crash. This forms trading patterns where the informed traders first buy with the bubble and then start selling out before the peak. Whether this constitutes a true price bubble or not depends on the definition. The price always reflects the market makers best estimate of the asset value, thus from the point of view of the market there is no bubble. However, this does provide a rational justification of trading patterns observed in the data where well informed investors ride a price bubble, buying as the price is rising before starting to sell out near the peak. Similar trading patterns where documented for hedge funds during the technology bubble of the late nineties (see Brunnermeier and Nagel 2004) and by Hoare’s bank during the south sea bubble of 1720 (Temin and Voth 2004).

---

2This was first shown by Foster and Viswanathan 1994
3In this example, informed traders are ex-ante symmetric but have imperfectly correlated signals of the value of the demand shock and the asset value.
Asymmetries of private information often arise exogenously, as sources of information are varied and diverse. This paper also highlights how asymmetries may arise endogenously as informed traders have incentives to differentiate their information so as to reduce competition, even if this differentiation is simply a garbling of information. To highlight this, consider a game where two informed traders have access to a wide range of signal precisions, including a perfect signal of the asset value. Traders first choose a level of signal precision. Signal precision is then observed by the market and trade occurs as in the model. Results show that this game has a unique equilibrium in which both traders choose imperfect signals. This has important implications for market efficiency as it means informed traders acquire inefficiently low amounts of information and this in turn makes prices less informative (see Section 3.1).

This paper is closely related to versions of Kyle (1985) with multiple informed traders, such as Foster and Viswanathan (1996) and Holden and Subrahmanyan (1992) in discrete time, and Back et al. (2000) in continuous time. These papers assume that informed traders are ex-ante symmetric, although they may receive imperfectly correlated signals. This paper relaxes this assumption by allowing any form of correlation structure, as well as allowing traders to have multiple signals. The Information structure used in this paper is most similar to that used by Lambert et al. (2014). However, the papers differ in the trading environment. This paper considers a dynamic trading environment, whereas Lambert et al. (2014) considers a single trading period. The single period in Lambert et al. (2014) facilitates the proof of the existence and uniqueness of the analytically solved equilibrium. Bernhardt and Miao (2004) characterize a set of necessary and sufficient conditions for an equilibrium in a model with a finite trading periods. Some key differences between this paper and Bernhardt and Miao (2004) is inclusion of non-fundamental information in the form persistent liquidity demand, and introduction of continuous time trading with a quadratic cost. The latter allows for numerical solutions to be found using standard techniques used for solving ordinary differential equations.

An important aspect of markets is their ability to aggregate information. Kyle (1985) showed that for a single informed trader, as the time between trades goes to zero the trader's information is fully revealed by the close of trading. This result was extend to multiple informed traders, for a specific set of signal structures, by Foster and Viswanathan (1996) and Back et al. (2000). Ostrovsky (2012) allows for a broad class of information structures and shows that information is revealed as long as a separability condition on the information structure is satisfied. Critically, Ostrovsky shows that this result holds for any form of equilibrium and does not depend on the linear equilibrium structure, usually assumed. A stronger result has been shown when more than one informed trader has perfect information about the fundamental value of the asset. In this case information about the value of the asset is revealed almost instantaneously as the time between trades goes to zero (See Holden and Subrahmanyan 1992). The results of this paper suggest that these results are robust to the introduction of non-fundamental information in the form of a market demand shock. This paper also suggest that when two informed traders have the same partial information about the asset value, this information is fully revealed almost instantaneously as the cost of trade becomes small, and this is true even when both informed traders have better information that contradicts this partial information.

This paper also sits within the literature on market manipulation, since it considers how traders use information to manipulate asset prices. One form of market manipulation is shown in Medrano
and Vives (2001), here a large informed trader seeks to slow down the revelation of information by offsetting the trades of smaller informed traders. Although the large informed trader trades against their information of the asset value, the net affect of informed traders is still to correct prices. Another form of market manipulation involves the informed trader using a mixed strategy which puts positive probability on the trader trading against their private information, see Chakraborty and Yilmaz (2004) and Back and Baruch (2004). In these cases the informed trader acts on average to correct the price, but using a mixed strategy helps to slow down information revelation. In contrast to these three papers, this paper shows cases where informed traders have a strictly optimal strategy to trade against their private information, and the net affect of informed traders is to drive the price away from the fundamental value of the asset. Similar forms of market manipulation are found by Sadzik and Woolnough (2014) and Gao and Ou-Yang (2014). However, unlike these papers, the results of this paper do not rely on predictable patterns in behavioral noise trade.

Foster and Viswanathan (1994) consider a model with ex-ante asymmetry of traders, that is where one of the informed traders is willing to trade against their private information in order to slow the learning of the other trader. Foster and Viswanathan consider the case of two traders; the first receives a signal of the asset value, whilst the second learns the true value of the asset and the first agent’s signal. Foster and Viswanathan find that both informed traders initially trade heavily on the signal and this can lead to market manipulation by the second trader, as the value and signal are not always on the same side of the price. This special case is considered in Section 3.2. Under the exact nesting of information used by Foster and Viswanathan, competition becomes so intense that the signal is almost instantaneously revealed when the quadratic cost is small. However, by relaxing common knowledge of the signal, a similar result can be obtained. In fact a stronger result is observed in this paper: under some parameters the second informed trader actually trades with negative intensity on the mispriced asset. This creates a more direct form of market manipulation as the informed trader increases their selling in reaction to the asset being under priced, and increases their buying of an asset which is overpriced.

The paper also adds to the literature on asset pricing with multidimensional uncertainty. Many forms of multidimensional private information have been considered in the literature, examples of these are uncertainty over proportion of informed investors, information quality, or whether an event has occurred (Romer 1993, Avery and Zemsky 1998, Li 2013, Back et al. 2014, Banerjee and Green 2014). These models show how a single dimension of information flow can struggle to aggregate additional dimensions of information. This can lead to large price corrections as additional dimensions of information are aggregated over time. Two recent papers which employ similar forms of multidimensional private information are Gao and Ou-Yang (2014) and Sadzik and Woolnough (2014). Both papers consider cases where liquidity trade follows a predictable pattern and find that a single informed trader will use this information strategically to destabilize the price. By relaxing the assumption of a single informed trader, this paper shows that the “pump-and-dump” found by Sadzik and Woolnough are weakened by the effects of competition and learning. One implication of this is that, unlike in the case of a single informed trader, full revelation of the demand shock does not occur by the close of trading.

The remainder of the paper is structured as follows. Section two of the paper sets out the model and characterizes the equilibrium. Section three provides numerical examples and discusses the trading
strategies of the informed traders. Section four considers the existence of price bubbles and the trading patterns of informed traders during these bubbles. The proofs of results in the paper are left until the Appendix.

2 Model

2.1 Model Setup

An asset is traded in continuous time over a period running from 0 to 1. The fundamental value of the asset is $v$. At time 1 the fundamental value of the asset is revealed and trading ceases. There are three types of traders in the market, informed traders, liquidity traders and market makers. Informed traders are agents who are better informed than the market. They utilize this information to make profit. Liquidity traders are behavioral traders whose trade flow is given exogenously. One explanation for their trade is that these traders represent a large number of risk averse investors who trade for the purpose of liquidity. Market makers are rational traders with no additional market information. They use public information, such as the aggregate order flow, to inform their beliefs. Market makers are modeled implicitly; they are assumed to be willing to buy and sell the asset at its expected value conditional on public information.

The structure of the market is as follows, informed and liquidity traders submit order flows to the market. It is assumed that only the aggregate order flow and the price are observable to the market. Individual order flows are not observed. Having observed the aggregate order flow, market makers clear the market at a price, $p(t)$, equal to the expected value of the asset conditional on the information available to the market.

The order flow of Liquidity traders is an exogenous process with quadratic variation. Liquidity traders serve two roles in the model. First, the quadratic variation helps obfuscate the trade of informed traders. This facilitates trade in the model and ensures the market does not break down due to adverse selection. Liquidity trade is also assumed to carry a persistent trend which is unknown to the market. For example, a persistent trend could be some exogenous market demand shock that affects many firms at once. Or alternatively, it could represent a form of investor sentiment which is unrelated to the fundamental value of the asset ($v$). The order flow of liquidity traders follows the process $dq_t$, where

$$dq_t = mdt + dB_t$$

$m$ represents investor sentiment and $B_t$ is standard Brownian motion. The fundamental asset value, $v$, and the exogenous demand shock, $m$, are assumed to be joint normally distributed.

There are $n$ informed traders. At time zero, each informed trader receives a vector of signals. Let $X_i$ represent signals of informed trader $i \in \{1, ..., n\}$, where $X_i$ is a vector of length $\tau_i$. Additionally, let
X_0 represent a vector of \( \tau_0 \) signals which are not observed by any agent in the model. Note that \( \tau_0 \) may be zero, in which case \( X_0 \) can be ignored. The complete set of signals is represented by \( X \), where \( X \) is a vector of length \( \tau = \sum_{i=0}^{n} \tau_i \) given by

\[
X = \begin{pmatrix}
X_0 \\
X_1 \\
\vdots \\
X_n
\end{pmatrix}
\]

It is assume that \( X \) is joint normally distributed with mean \( \bar{X}(0) \) and variance \( \Sigma(0) \). Further, assume that there are two vectors \( \alpha \) and \( \mu \) such that

\[
v = \alpha'X
\]

\[
m = \mu'X
\]

This form of writing the signals is rather general, and most signal structures can be written in this form. The set of signals, \( X_0 \), allows for information hidden from the market. For example, if the market is imperfectly informed about \( m \) then residual uncertainty about \( m \) could be contained in the signal \( X_0 \). \( X_0 \) could represent the true values of \( v \) and \( m \), or it could represent the residual uncertainty. A drawback of this method is that there is no unique way to represent a set of signals. The main example which will be consider is one where each informed trader receives imperfect signals about \( v \) and \( m \).

**Example 1**: The value of \( v \) and \( m \) are independent and normally distributed with means \( p_0 \) and 0 and variances \( \sigma_v^2 \) and \( \sigma_m^2 \) respectively. There are \( n \) informed trader, and each trader \( i \in 1, ..., n \) receives a pair of signals form

\[
v_i = v + \vartheta_i
\]

\[
m_i = m + v_i
\]

where \( \vartheta_i \) and \( v_i \) are mean zero. Setting \( X_0 = \begin{pmatrix} v \\ m \end{pmatrix} \) and \( X_i = \begin{pmatrix} v_i \\ m_i \end{pmatrix} \), then \( \alpha = \begin{pmatrix} 1 & 0 & \cdots \end{pmatrix}' \),

\[
\mu = \begin{pmatrix} 0 & 1 & \cdots \end{pmatrix}' \text{ and } \bar{X}_i(0) = \begin{pmatrix} p_0 \\ 0 \end{pmatrix}.
\]

The setup of the model also allows for many other signal structures. The following are just some

---

4The further assumption that trader \( i \)'s signals are not collinear and have an invertible covariance matrix.

5To illustrate this, consider the following example. Let \( v \) and \( m \) be independent standard normal random variables. A single informed trader gets a perfect signal of \( v \) and a partial signal of \( m \) which takes the form \( m_1 = m + \epsilon \), where \( \epsilon \) is standard normal and jointly independent of \( v \) and \( m \). Let \( X_1 = \begin{pmatrix} v & m_1 \end{pmatrix}' \). There are two possible ways to write \( X_0 \). One way is to let \( X_0 = (m) \), then \( \mu = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}' \). An alternative is to let \( X_0 = (m_2) \), where \( m_2 = m - \epsilon \). In this case \( \mu = \begin{pmatrix} 0 & 0.5 & 0.5 \end{pmatrix}' \).
**Example 2** (Kyle 1985): The value of the asset \( v \) is normally distributed with mean \( p_0 \) and variance \( \sigma_v^2 \) and \( m \) is constant and equal to zero. There is a single informed trader with a perfect signal of \( v \). In this case \( X = X_1, \bar{X}(0) = p_0, \Sigma(0) = \sigma_v^2, \alpha = 1 \) and \( \mu = 0 \).

**Example 3:** The value of \( v \) and \( m \) are independent and normally distributed with means \( p_0 \) and 0 and variances \( \sigma_v^2 \) and \( \sigma_m^2 \) respectively. A single informed trader gets a a perfect signal of \( v \) only. In this case \( X_1 = v \) and \( X_0 = m, \bar{X}(0) = \begin{pmatrix} 0 \\ p_0 \end{pmatrix}, \Sigma(0) = \begin{pmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_m^2 \end{pmatrix}, \alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) and \( \mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

**Example 4** (Sadzik and Woolnough 2014): The value of \( v \) and \( m \) are normally distributed with means \( p_0 \) and 0 respectively, the covariance of \( v \) and \( m \) is given by \( \Sigma_0 \). A single informed trader gets a a perfect signal of \( v \) and \( m \). In this case \( X = X_1 = \begin{pmatrix} v \\ m \end{pmatrix}, \bar{X}(0) = \begin{pmatrix} p_0 \\ 0 \end{pmatrix}, \Sigma(0) = \Sigma_0, \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \mu = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

**Example 5** (Asymmetric information): The value of \( v \) and \( m \) are independent and normally distributed with means \( p_0 \) and 0 and variances \( \sigma_v^2 \) and \( \sigma_m^2 \) respectively. There are two informed traders, denoted 1 and 2 respectively. Trader 1 has a perfect signal about \( v \) and trader 2 has a perfect signal of \( m \). In this case \( X_1 = v \) and \( X_2 = m, \bar{X}(0) = \begin{pmatrix} p_0 \\ 0 \end{pmatrix}, \Sigma(0) = \begin{pmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_m^2 \end{pmatrix}, \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \mu = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

Informed traders are risk neutral and trade to maximize expected profit. The price of the asset at time \( t \) is given by \( p(t) \), this is set by the market makers. It is assumed that informed traders can trade at the market price, but pay an adjustment cost which is increasing in the aggregate size of the order. Profit for an informed trader \( i \in \{1, \ldots, n\} \) is given by

\[
\Pi_i = \int_0^1 \left( (v - p(t))u_i(t) - \frac{\epsilon}{2} u_i(t)^2 \right) dt \tag{1}
\]

where \( u_i(t) \) is their order flow and \( \epsilon \) represents some quadratic cost of trade. Profit is the value of their asset holdings at time \( t \), minus the cost of their trades.

Including an adjustment cost prevents traders from trading with infinite intensities. This guarantees the existence of an optimal strategy, even off the equilibrium path, and allows the use of standard dynamic programming techniques. It also guarantees the existence of an equilibrium, where none would exist if the adjustment cost was zero. A well known example of this is the case of two informed traders with perfect information about the fundamental \( v \). Kyle (1985) presents the continuous time version of the insider trading model as the limit of a discrete time trading model, when the time
between trades goes to zero. A drawback of the discrete model is its level of tractability, this is especially true when moving to models of asymmetric signals. The continuous time model with a quadratic adjustment share many similarities with the discrete time trading model, but benefits from greater tractability tractable.

It is assumed that individual orders of informed trader’s are not observable to the market. Instead, it is assumed that the market cannot only observe the total market order flow at time \( t \) and the public history of prices. The total market order flow by \( dQ_t \), where

\[
    dQ_t = \left( \sum_{i=1}^{n} u_i(t) \right) dt + dq_t
\]

Market makers observe and the aggregate order flow and use this to update their beliefs. At each point in time they set a price, \( p(t) \), such that the price is equal to the value of the asset, conditional on the public history. Let the filtration of the market makers be represented by \( \mathcal{F}_t = (Q_s, p(s))_{s \in [0,t]} \). The informed traders also observe prices and the public order flow. Additionally, each informed trader \( i \in \{1, ..., n\} \) observes individual private signal \( X_i \), and their own order flow \( u_i(t) \). The filtration of informed trader \( i \) is represented by \( \mathcal{G}_i(t) = (X_i, u_i(s), Q_s, p(s))_{s \in [0,t]} \).

**Definition 1:** A strategy for the market makers is an \( \mathcal{F}_t \)-adapted process \( P = \{p(t)\}_{t \in [0,1]} \), and a strategy for each informed agent \( i \in \{1, ..., n\} \) is a \( \mathcal{G}_i(t) \)-adapted process \( U_i = \{u_i(t)\}_{t \in [0,1]} \).

The profile \((P, U_1, ..., U_n)\) is an equilibrium if

- (i) given \((U_1, ..., U_n)\), for all \( t \in [0,1] \), \( p(t) = E [p|U_1, ..., U_n, \mathcal{F}_t] \)
- (ii) for all agents \( i \in \{1, ..., n\} \), given \( P \) and all \( U_j \) for \( j \neq i \), \( U_i \) maximizes the expected profit \( \Pi_i \).

This paper will consider equilibrium where, on the equilibrium path, each informed trader \( i \in \{1, ..., n\} \) plays a strategy which is linear in their vector of signals \( X_i \) and the expected values of \( X_i \), are conditional on the market makers filtration, \( \mathcal{F}_t \). This is uses a guess and verify method for finding the equilibrium. Having each informed traders strategy depend only on their own signals avoids the problem where informed traders need to forecast the forecasts of other informed trader.\(^6\) This can lead to and infinite regress, as each informed trader may have to also forecast other informed traders forecasts, and so on.

To help define a linear equilibrium, let \( \bar{X}_i(t) = E [X_i|U_1, ..., U_n, \mathcal{F}_t] \). Additionally, let \( \bar{X}(t) = E [X|U_1, ..., U_n, \mathcal{F}_t] \) and \( \Sigma(t) = \text{Var} [X|U_1, ..., U_n, \mathcal{F}_t] \).

**Definition 2:** A linear equilibrium is an equilibrium in which, on the equilibrium path, each informed trader \( i \in \{1, ..., n\} \) plays a strategy of the form

\[
    u_i(t) = \bar{G}_i(t)' \left( X_i - \bar{X}_i(t) \right)
\]

\(^6\)This idea is from Foster, F., and S. Viswanathan (1996)
where $G_i(t)$ is a continuous function mapping $[0,1]$ to $\mathbb{R}_{\tau_i}$.

In more general forms of linear strategies an informed trader may wish to also condition trade on their expectations of other traders’ signals. However, in equilibrium this information will be collinear with their own private signals. This allows an informed trader’s strategy to be reduced to one which only depends on their own signals.

### 2.2 General conditions for equilibrium

Solving for an equilibrium can be broken down into three part, solving the market makers filtering problem, solving each informed traders filtering problem and solving each informed traders optimal control. The first step in solving for equilibrium is to consider the strategy, and updating process, of the market makers. When each informed trader plays a linear strategy of the form given in Definition 2, the market maker’s problem becomes a standard linear filtering problem. The markets makers update their beliefs about $X$ by observing

$$dQ_t = G(t)'(X - \bar{X}(t))dt + \mu'\bar{X}(t)dt + dB_t$$

where

$$G(t) = \begin{pmatrix} G_0(t) \\ \vdots \\ G_n(t) \end{pmatrix} + \mu$$

and $G_0(t)$ is a $\tau_0 \times 1$ vector of zeros.

As each informed trader uses a strategy that is linear solely in their own private signals, the market makers problem reduces to a straight forward linear filtering problem.

**Proposition 1. Updating by the market makers**

*Fix the linear strategies \{U_1,...,U_n\} of informed traders \{1,...,n\}. Conditional on these strategies, the market makers set a price

$$p(t) = \alpha'\bar{X}(t)$$

The market makers updates the of the conditional expectation and conditional variance of $X$, are given by

$$d\bar{X}(t) = L(t)(dQ_t - \mu'\bar{X}(t)dt)$$

$$d\Sigma(t) = -L(t)L(t)'dt$$

where

$$L(t) = \Sigma(t)G(t)$$
with the initial conditions $X(0)$ and $\Sigma(0)$.

**Proof.** See Appendix \( \square \)

Given the linear form of each informed trader’s strategy, Proposition 1 becomes an application of the Kalman-Bucy filter. There are two key elements in proposition 1. Firstly equation 3 helps in writing each informed trader’s problem as a linear optimal control. Second, equations 4 and 5 for part of the system of differential equations which allows us to characterize the equilibrium. Equations 4 gives in the change in state variable $\Sigma(t)$, while equation 5 gives a set of linear equations which link the informed traders strategy with that of the market makers.

Next consider the filtering problem of informed trader $i \in \{1, ..., n\}$. Informed trader $i$ receives initial signal $X_i$. They are able to observe the aggregate order flow $Q_i$, and their own order flow $u_i(t)$. Similar to the market makers, each informed trader uses this information to update their beliefs about $X$. Assume that all other informed traders $j \neq i$ play the equilibrium strategy given in Definition 2 and define $\tilde{X}(t) = E \left[ X \mid \mathcal{G}_i(t), P, \{U_j\}_{j \neq i} \right]$ and $\tilde{\Sigma}(t) = Var \left[ X \mid \mathcal{G}_i(t), P, \{U_j\}_{j \neq i} \right]$.

The informed trader $i$ observes $u_i(t)$ and $dQ_i$, where

$$dQ_i = \tilde{G}_i(t)'(X - \tilde{X}(t))\,dt + \mu'\tilde{X}(t)\,dt + u_i(t)\,dt + dB_t$$

where $\tilde{G}_i(t)$ is the vector of trader’s strategies with trader $i$’s own strategy set to zero.\(^7\)

The following notation will be useful when consider the updating problem of informed trader $i \in \{1, ..., n\}$. Let $H_i$ be a $\tau \times \tau_i$ matrix such that $H_i'X = X_i$.\(^8\) Informed trader $i$’s updates their estimates of $X$ using similar method to the market makers.

**Proposition 2. Updating by informed traders**

Consider informed trader $i \in \{1, ..., n\}$, conditional on their initial signal $X_i$, $X$ is normal distributed with mean and variance, $\tilde{X}^i(0)$ and variance $\tilde{\Sigma}^i(0)$

$$\tilde{X}^i(0) = \tilde{X}(0) + \Sigma(0)H_i(H_i'\Sigma(0)H_i)^{-1}(X_i - \tilde{X}_i(0))$$

$$\tilde{\Sigma}^i(0) = \Sigma(0) - \Sigma(0)H_i(H_i'\Sigma(0)H_i)^{-1}H_i'\Sigma(0)$$

Fix the linear strategies of informed traders $j \neq i$, informed trader $i$ updates their estimates of the mean and variance of $X$ according to the following

$$d\tilde{X}^i(t) = K_i(t)d\tilde{B}_i(t)$$

\(^7\) $\tilde{G}_i(t)' = \left( \begin{array}{cccc} G_0(t) & \cdots & G_{i-1}(t) & 0_{i-1} \\ G_{i+1}(t) & \cdots & G_{n}(t) & \end{array} \right)$

\(^8\) Another way to write $\tilde{G}_i(t)$ is, $\tilde{G}_i(t) = G(t) - H_iG_i(t)$. 

11
\begin{equation}
d\tilde{\Sigma}_i(t)/dt = -K_i(t)K_i(t)'
\end{equation}

where \( K_i(t) = \tilde{\Sigma}_i(t)\tilde{G}_i(t) \) and \( \tilde{B}_i(t) = \int_0^t \tilde{G}_i'(X - \tilde{X}_i(s)) \, ds + B_t \) is standard Brownian motion.

\textit{Proof.} See Appendix \( \square \)

Proposition 2 follows similar logic to proposition 1. For informed trader \( i \), \( u_i(t) \), \( \tilde{X}(t) \) and \( Q_t \) are observable, from these \( d\tilde{B}_i \) is equal to

\[
d\tilde{B}_i = dQ_t - \tilde{G}_i(t)' \left( \tilde{X}^i(t) - \tilde{X}(t) \right) dt - u_i(t) dt
\]

Now turn to the maximization problem of informed trader \( i \). Their goal is to maximize their expected profit. Taking the strategy of the market makers, \( P \), and the strategy of the other informed traders, \( \{U_j \}_{j \neq i} \), as given, expected profit is given by

\[
J(U_i) = E \left[ \int_0^1 \left( u_i(t)\alpha^T(X - \tilde{X}(t)) - \frac{\epsilon}{2} u_i(t)^2 \right) dt \bigg| P, \{U_j \}_{j \neq i} \right]
\]

subject to \( u_i(t) \) being \( G_i(t) \) measurable.

When \( \epsilon > 0 \), the problem is a linear regulator problem with an imperfectly observed state variable \( X \). Applying the separation principle, the informed trader’s maximization problem is equivalent to maximizing

\[
\tilde{J}(u) = E \left[ \int_0^1 u_i(t)\alpha^T(X - \tilde{X}_i(t)) - \frac{\epsilon}{2} u_i(t)^2 \right] dt
\]

This will be shown more formally in proposition 4. Combining proposition 1 and 2, the updating for \( (\tilde{X}^i(t) - \tilde{X}_i) \) is given by

\[
d \left( \tilde{X}^i(t) - \tilde{X}_i \right) = -L_i\tilde{G}_i(t)'(\tilde{X}^i(t) - \tilde{X}(t))dt - L_tu_i(t)dt + (K_i(t) - L_t) d\tilde{B}_i
\]

Thus, the optimal control problem for informed trader \( i \) becomes a stochastic linear regulator problem with observed state variable \( (\tilde{X}^i(t) - \tilde{X}_i) \). The optimal control for informed trader \( i \) is of the form,

\[
u_i(t) = B_i(t) \left( \tilde{X}^i(t) - \tilde{X}(t) \right)
\]

where \( B_i(t) \) is a \( 1 \times \tau \) vector.

The goal is to simultaneously solve the optimal control of each informed trader, and the filtering of the market maker. Proposition 3 helps reduce the strategy of each informed trader to a strategy which depends only on their own signals.
**Proposition 3. Collinearity of informed trader’s estimates**

Fix the strategy of informed trader \( i \in \{0, \ldots, n\} \), and fix the strategies of traders \( j \neq i \). On the equilibrium path, informed trader \( i \)'s conditional expectation of \( X \) at time \( t \) can be written as the linear combination of their signals, \( X_i \), and the market makers conditional expectation of \( X \). Namely,

\[
\tilde{X}^i(t) - \bar{X}(t) = \eta_i(t) (X_i - \bar{X}_i(t))
\]

where \( \eta_i(t) = \Sigma(t)H_i (H_i'\Sigma(t)H_i)^{-1} \).

The logic behind Proposition 3 as the follows. If at time \( t \) the market makers was to learn an informed trader \( i \)'s initial signals and the history of their order flow, then they would have the same filtration as the informed trader. Thus, they would have the same conditional expectation of \( X \) as informed trader \( i \). This means that in equilibrium, \( \tilde{X}^i(t) \) can be found by using the informed traders signal, \( X_i \), to update the market makers estimates at time \( t \).

If informed trader \( i \) has played and equilibrium strategy up until time \( t \), then Proposition 3 means that the optimal strategy at time \( t \) can be written as

\[
u_i(t) = G_i(t)' (X_i - \bar{X}_i(t))
\]

where

\[
G_i(t)' = B_i(t) \eta_i(t)
\]

The next Proposition sets out a set of necessary and sufficient conditions for the linear strategy, given in Definition 2, to be optimal for the informed trader informed trader \( i \in 1, \ldots, n \).

**Proposition 4. Informed trader’s optimal control**

For informed trader \( i \in \{1, \ldots, n\} \) the following are necessary and sufficient conditions for

\[
u_i(t) = G_i(t)' (X_i - \bar{X}_i(t))
\]

(6)

to be an optimal strategy,

\[
B_i(t) = \frac{1}{\epsilon} (\alpha' - 2L' A_i(t))
\]

(7)

\[
G_i(t)' = B_i(t) \eta_i(t)
\]

(8)

where \( A_i(t) \) is a \( \tau \times \tau \) matrix solved by the ordinary differential equation

\[
\dot{A}_i(t) = -\frac{\epsilon}{2} B_i(t)'B_i(t) + \tilde{G}_i(t)L(t)'A_i(t) + A_i(t)L(t)\tilde{G}_i(t)'
\]

(9)
with the boundary condition that $A_i(1)$ is a $\tau \times \tau$ matrix of zeros.

Theorem 1 sets out the key variables of an equilibrium as the solution to a set of ordinary differential equation with boundary conditions. The key components of this are $\Sigma_t$ and $A_1(t), \ldots, A_n(t)$. Combining these with Equation (5) and $L_t = \Sigma_t G_t$, gives a set of $(2\tau)$ equations, which are linear in the $(2\tau)$ unknowns which make up $G_t$ and $L_t$.

**Theorem 1. Characterization of equilibrium**

The equilibrium of the model is characterized by a first order differential equation in $\Sigma(t)$ and $A_1(t), \ldots, A_n(t)$. The rates of are given by

\[
\begin{align*}
\dot{S}(t) &= -L(t)L'(t) \\
\dot{A}_i(t) &= -\frac{\epsilon}{2}B_i(t)'B_i(t) + \tilde{G}_i(t)L(t)'A_i(t) + A_i(t)L(t)\tilde{G}_i(t)' \\
\end{align*}
\]

with boundary conditions $S(0) = \Sigma_0$ and $A_i(1) = 0$ for all $i$. At each time $t$, $L(t)$, $G(t)$ and $B_1(t), \ldots, B_n(t)$ can be found as the unique solution to the following system of equations

\[
\begin{align*}
G(t)' &= (\alpha'\eta(t) + \epsilon\mu')(\epsilon I_\tau + 2S(t)A(t)) \\
L(t) &= S(t)G(t) \\
B_i(t) &= (\alpha' - 2L(t)'A_i(t))
\end{align*}
\]

where

\[
\eta_i(t) = S(t)H_i(H_i'S(t)H_i)^{-1}
\]

and

\[
A(t) = \begin{pmatrix} A_0(t) & A_1(t)\eta_1(t) & \cdots & A_n(t)\eta_n(t) \end{pmatrix}
\]

\[
\eta(t) = \begin{pmatrix} \eta_0(t) & \eta_1(t) & \cdots & \eta_n(t) \end{pmatrix}
\]

where $A_0(t)$ and $\eta_0(t)$ are $\tau \times \tau_i$ zero matrices.

Theorem 1 follows from Proposition 1 through 4 which show that these conditions are sufficient and necessary for an equilibrium.

When there is a single informed trader and only private information about the value of the asset, this model becomes the continuous time model by Kyle (1985), but with the addition of a quadratic cost. In this special case it is possible to find the equilibrium of the model in closed form.
Corollary 1. When $n = 1$, $v \sim N\left(0, \sigma_v^2\right)$, $m$ is deterministic and $X = X_1 = v$, there exists a unique equilibrium where,

$$
\Sigma(t) = \sigma_v^2 \frac{1 - t + g}{1 + g}
$$

$$
A_1(t) = \frac{\sqrt{1 + g}}{2\sigma_v} - \frac{\epsilon (1 + g)}{2\sigma_v^2 (1 - t + g)}
$$

where $g = \frac{\epsilon}{2\sigma_v^2} \left( \sqrt{\epsilon^2 + 4\sigma_v^2} \right)$.

Proof. See Appendix

Corollary 1 provides the equilibrium for a single informed trader and private information about the value. From this it is easy to see that, in this case, the model with a quadratic adjustment cost converges to the continuous time solution in Kyle (1985) when cost goes to zero.

3 Analysis of equilibrium with two informed traders

A lot can be learned just by considering the case of two informed trader. This simplifies the analysis greatly while still capturing many aspects of competition among informed traders and asymmetric signal structures. This section is broken up into four parts. The first two part consider models with no exogenous demand shock, instead they focus on models with asymmetric information. The first part looks at how decreases in signal precision can help informed traders decrease competition. When informed trader have perfectly correlated signals about the asset value, competition between trader’s drives away all profit. In the case of two informed traders, if one trader was to get an imperfect signals instead, then this would decrease competition and allow both traders to make positive profit.

The second part of this section looks at asymmetric signal structures. The more correlated informed traders’ signals are, the more competitively they trade on the information. If informed traders have multiple sources of information, then the sources where information is highly correlated will be more heavily competed on initially. This can lead to informed traders trading against their private estimates of the asset value, as the most correlated signal is not necessarily the most informative.

The last two parts of this section consider non-fundamental information. Part three considers the strategic trading of an informed trader with only non-fundamental information. It shows that the informed trader, in general, acts to offset the demand shock. However, early in the trading window, the informed may be willing to trade with the demand shock accentuating the demand shock, later profiting from the created price distortion by selling directly to the demand shock.
3.1 Signal Garbling

Holden and Subrahmanyan (1992) show that when two traders have perfect information about the asset value, as the period between trades goes to zero, the asset value becomes fully revealed at time zero and informed traders make zero profit. Thus, when two, or more, informed traders have perfect information about the asset value, $v$, competition between them drives away all profit. However, as Backe et al (2000) show, when informed traders have imperfectly correlation signals, the effect of competition is reduced and informed traders are able to make positive profits. The informed traders benefit from the mutual garbling of their own signals, as the reduction in signal accuracy reduces competition.

This brings up an interesting question, would an informed traders choose to receive a less informative signal? Consider a technology which could garble an informed trades signal by adding some white noise. If using this technology was observable, then it may be optimal for an informed trader to partially garble their signal. If two informed traders each have a perfect signal of the asset value $v$, one of these trader could increase profits by reducing the accuracy of their own signal by adding white noise. This breaks common knowledge and reduces the competitiveness, but it would also reduce how heavily the other informed trader would compete, even though the other informed trader still receives perfect information of the asset value.

Consider the following model, there are two informed trader labeled 1 and 2. The first informed trader gets a perfect signal of the asset value $v$. The second informed trader gets a signal equal to $v$ plus some white noise.

\[ X_1 = v \]
\[ X_2 = v + \vartheta_2 \]

where $\vartheta \sim N(0, \sigma^2_2)$. Note that informed trader 1 knows the value of $\sigma^2_2$, but does not observe $\vartheta_2$. When $\sigma^2_2 = 0$, both traders have perfect information about $v$. When $\sigma^2_2 > 0$, informed trader 2’s signal contains some garbling. For this example, assume that there is only private information about the fundamental value of the asset, the demand shock is known to be zero.
Figure 1: Variance of the asset value: Garbled versus perfect signal

Figure 1 shows the variance of the asset value, \( v \), conditional on the market makers filtration, \( \mathcal{F}_t \), for two different value of \( \sigma^2 \): the perfect signal is when \( \sigma^2 = 0 \) and the imperfect signal, \( \sigma^2 = 1 \). The initial variance of \( v \) is 1, and each graph is made for several values of the quadratic trading cost. As can be seen, when \( \sigma^2 = 0 \) (the graph on the left) as the quadratic cost becomes small, the value of \( v \) is almost completely revealed at the very beginning of trading. However, when \( \sigma^2 \) is non zero, competition between informed traders is reduced and information is released more gradually over time.

When \( \sigma^2 = 0 \) is zero, competition between the informed traders quickly reveals the asset value and drives informed traders’ profits down to zero. However, when trader 2’s signal is garbled, \( \sigma^2 \) is none zero, competition is reduce and traders make positive profit even when the quadratic cost becomes small. Garbling of trader 2’s signal provides a positive benefit to each trader. The benefit to trader 1 exceeds the benefit to trader 2, but trader 2 would still be willing to garble their own signal as it increase their personal profit as well.

The key to this result is the observability of \( \sigma^2 \). By observing that \( \sigma^2 \) is positive, informed trader 1 knows that informed trader 2 is unable to steal all the profits. This means that trader 1 is willing to be more patient when trading on their own private information. This means that even when information acquisition is costless for trader 2, they may choose not to get perfect information. In making this choice, the informed trader 2 takes into account the affect the accuracy of his information has on the strategy of the informed trader 1, and the affect this has on the strategy of the market makers.
Figure 2 shows the ex-ante expected profit for informed traders 1 and 2 under different levels of signal garbling, variance for $\vartheta_2$. The profit of informed trader 1 increases as trader 2’s signal becomes less accurate. The ex-ante profit of informed trader 2 is hump shaped. Initially informed trader 2 benefits from having a less informative signal, as this reduces competition between the traders. However, after a point, as this inaccuracy increases ex-ante profit declines. Thus, when trader 1 has a perfect signal of $v$ trader 2 has an optimal level of garbling for their own private signal.

Consider a two stage game between two informed traders. $v$ is normally distributed with mean $p_0$ and variance 1. Each trader gets a signal of the form

$$X_1 = v + \vartheta_1$$
$$X_2 = v + \vartheta_2$$

where $\vartheta_1$ and $\vartheta_2$ are independent and normally distributed with mean 0 and variance $\sigma_1^2$ and $\sigma_2^2$ respectively. In the first stage each traders chooses the variance of their signal $\sigma_1^2$ and $\sigma_2^2$. This is observed my the market. In the second stage trade occurs as in the model. Fix the quadratic cost of trade at 0.01.
Figure 3 plots the best response function for each trader. Trader 1’s signal precision is on the y-axis and trader 2’s signal precision is on the x-axis. The best response functions of the two traders cross just once, so there is a unique Nash equilibrium in this game. From this we can see that traders may choose not to get accurate signals of the true value of the asset, even when this information is costless. By getting imperfect signals they are able to show other traders that they cannot steals all the profits. Note that this hinges crucially on the accuracy of information being observable. If traders could cheat and get a better signal without being observed, then this would be a profitable deviation. Thus, a form of the prisoners dilemma would apply. It also hinges on having only a fixed number of traders, as information becomes available to more traders then the market will become efficient.

This example has served to highlight some of the effects, and the benefits, of imperfect correlation among signals. These effects play an important role in dynamics of information revelation over time.

### 3.2 Market manipulation: chasing small signal

In many ways, market manipulation is far more general than symmetric asset pricing models give credit. A key to this is the asymmetry of information. In the market there is the potential for many forms of private information, some more informative than others, some known by a larger number of traders. It is these kinds of asymmetries that can give rise to informed traders’ willingness to trade against their perceived value of the asset. This idea was highlighted by Foster and Viswanathan (1994).

Foster and Viswanathan (1994) use a discrete time model with two informed traders and with two pieces of private information. One piece of private information is common to both informed traders, the other is held by a single trader. They show that early on both traders trade heavily on the common information, and in doing so the better informed trader may trade against is true information about the
asset value. Results from this paper suggest that when the cost of trading becomes small, the common piece of information is revealed almost instantaneously and the quick adjusts to the new information. However, the common information is revealed more slowly when traders only have imperfect signals. This leads to a stage where both traders first chase the common signal, after which the better informed trader begins to reveal his non-common private information.

Consider two informed traders, a better informed trader and a lesser informed trader. The lesser informed trader receives a signal about the asset \( s \), while the better informed trader learns the signal \( s \) and the true value of the asset, \( v \). They show that the better informed trader is willing to trade positively on the signal, \( s \), even though he has perfect information about the asset value. In this way the better informed trader is able to slow the learning of the lesser informed trader, and take any profits when the signal is on the right side of the market.

To illustrate this model further and better understand these effects, consider the following example. The value of the asset, \( v \), is normally distributed with mean \( \mu_0 \) and variance \( \sigma_v^2 \). Additionally to the value of the asset, there is a signal \( s \), where

\[
s = v + \vartheta
\]

and \( \vartheta \) is independent of \( v \) and normally distributed with mean 0 and variance \( \sigma_\vartheta^2 \).

There are two informed traders labeled 1 and 2, trader 1 will be called the better informed trader, and trader 2 the lesser informed trader. The signals of these traders are given by

\[
X_1 = \begin{pmatrix} v \\ s_1 \end{pmatrix} \\
X_2 = s_2
\]

where \( s_1 = s + \theta \) and \( s_2 = s - \theta \), and \( \theta \) is jointly independent of \( v \) and \( \vartheta \) and normally distributed with mean 0 and variance \( \sigma_\theta^2 \). The addition of \( \theta \) represents some small additional noise on traders common information, when \( \sigma_\theta^2 \) is zero then the two informed traders have common knowledge of signal \( s \).

In this model we find that both trader initially compete heavily on the signal \( s \). If \( s \) is commonly known to both traders then this trading becomes increasingly intense as the quadratic cost becomes small. As the quadratic cost gets close to zero the signal \( s \) is almost completely revealed at time zero. Once the common knowledge of \( s \) is broken, \( \sigma_\theta^2 \) is greater than zero then, the information of the lesser informed trader is released more slowly.

To look at how this works let us consider two examples. For both examples set the variance of \( v \) equal to one, \( \sigma_v^2 = 1 \) and the variance of the signal \( s \) equal to two, \( \sigma_s^2 = 1 \).

- Example 1: Both traders have perfect information about \( s \), \( \sigma_\vartheta^2 = 0 \).
- Example 2: Both traders have an imperfect signal about \( s \), \( \sigma_\vartheta^2 = 0.5 \).
Let $\mathcal{G}_2(t)$ be the filtration of the lesser informed trader and let $\hat{v}_2(t) = E[v|\mathcal{G}_2(t), U_1]$. A measure of the lesser informed trader’s private information is given by $E \left[ (\hat{v}_2(t) - p(t))^2 | \mathcal{F}(t), U_1, U_2 \right]$. This is the variance in the lesser informed traders esitamte of the asset value, $v$, conditional on the markets information.

Figure 4: Variance of lesser informed trader’s private information about the asset value

Figure 4 show the private information of the lessor informed trader over time in the two examples. In example 1, when both informed traders have perfect information about the signal $s$, both informed traders compete heavily on this information. As the quadratic cost becomes small, this information is almost completely revealed at the start of trading. In example 2, once common information is broken, the competition is decreased as the better informed trader’s information about $s$ cannot be completely competed away.

Now let us consider the trading strategy of the better informed trader in Example 2. The linear strategy takes the form

$$u_1(t) = \beta_v(t) (v - p(t)) + \beta_s(t) (s_1 - \bar{s}_1(t))$$

where $\beta_v(t)$ and $\beta_s(t)$ are the better informed trader’s trading intensities on the value of the asset and his signal respectively.
Figure 5: The better informed trader’s trading intensities: Example 2

Figure 5 show the better informed traders trading intensities on the value of the asset (left) and on their signal (right). As can be seen on the right, the informed trader trades very intensely on his signal near the beginning of trade. This serves to reveal the lessor informed traders information, and stop the lessor informed trader competing away the better informed traders profit.

Oddly enough, the better informed trader’s trading intensity on the value of the asset starts off negative. This is because initially the marker makers are updating very quickly about the asset value due to the heavy trading on the signal, $s$. The market maker’s updates of price are so responsive to the order flow that the informed trader finds it optimal to act on a contraian manner, pushing the price away from the true value of the asset early so as to profit from a greater mispricing of the asset later in the trading window. An outcome of this is that if the signal and the value are on the wrong side of the market initially, then the informed trader manipulates the market by trading with the price distortion.

The key to this result is not the asymmetry among informed traders, but rather the asymmetry between the forms of information. The more correlated informed trader’s signals are about a piece of information, the more heavily it will be traded on. Thus, informed traders will be willing to trade heavily on less informative information when that know other trader’s have similar information to themselves.

3.3 Market manipulation and the demand shock

This section examines how an informed trader trades to use their private information on different dimensions. Sadzik and Woolnough (2014) show that when there is a single informed trader with information about both the asset value, $v$, and the demand shock, $m$, the informed trader will trade positively on both dimensions of private information. In this way the informed trader exacerbates the affect of the persistence in liquidity trade.
Information about the demand shock benefits an informed trader in two ways. Firstly, it allows an informed trader to better learn from market information. Secondly, an informed trader is able to exploit this persistent demand by selling directly to the liquidity traders. This form of trade is beneficial when the demand shock is on the wrong side of the market; liquidity traders are buying more than expected and the asset is overpriced, or selling more and the asset is under priced. Market makers belief about the demand shock and asset value are both positively related to the order flow. By trading with the demand shock, an informed trader is able to increase the negative correlation between the market makers belief about the value of the asset and their belief about the demand shock. This has the effect of increasing the profit an informed trader receives by selling to the liquidity traders. Results indicate that an informed traders may be willing to amplify the demand shock early in the trading window, before offsetting the demand to make a profit. This effect is stronger when uncertainty about the demand shock is small ex-ante.

Consider the case of a single informed trader who gets a perfect signal about both $v$ and $m$, and let $p(t)$ and $\bar{m}(t)$ be the market makers estimates of $v$ and $m$ respectively at time $t$. The informed trader uses a strategy of the form

$$u(t) = \beta(t)(v - p(t)) + \delta(t)(m - \bar{m}(t))$$

Sadzik and Woolnough show that, in the continuous time setting, the informed trader will use a strategy where $\delta(t)$ is positive for all $t \in [0,1)$. This is in contrast to a single period model where it is optimal to set delta equal to negative one half. When $m - \bar{m}(t)$ is greater than zero, the market underestimates the average amount liquidity traders are buying. By using a strategy with $\delta(t) > 0$, the informed trader chooses to also buy more when $m - \bar{m}(t)$ is greater than zero. This further increases the aggregate order flow. So, rather than offsetting the effect that persistence in liquidity trade is having on the market, the informed trader re-enforces the effect making it larger.

Now consider the case where there are two informed traders, the first informed trader gets a perfect signal about $v$ and the second gets a perfect signal about $m$. In the model of Sadzik and Woolnough, the single informed trader is a monopolist on both dimensions of information. In this case, each informed trader is a monopolist, but only on single dimension of information. Over the course of trading, each informed trader will learn about the other traders’ signals. This will lead to an element of competition not present in the single trader case.

Denote the trader informed about $v$ (value informed trader) and the trader informed about $m$ (liquidity informed trader) traders 1 and 2 respectively. Again, let $\bar{v}(t)$ and $\bar{m}(t)$ be the market makers estimates of $v$ and $m$ respectively at time $t$. The value informed trader will use a strategy of the form $u_1(t) = \beta(t)(v - p(t))$, while the liquidity informed trader will use a strategy of the form $u_2(t) = \delta(t)(m - \bar{m}(t))$. The value informed trader makes a profit by trading positively on $v - p(t)$, that is $\beta(t) > 0$.

The strategy of the liquidity informed trader is less straight forward. Knowing $m$ allows this trader to better filter the information about value being provided by the value informed trader. The liquidity
informed trader can use a strategy of the form

$$u_2(t) = \delta_v(t)E[v - p(t)|G_2(t), U_1] + \delta_m(t)(m - \bar{m}(t))$$  \hspace{1cm} (10)$$

By setting $\delta_v(t) > 0$, the liquidity informed trader’s profits derive directly from the asset mispricing. Additionally, the liquidity informed trader has similar incentives, to those found by Sadzik and Woolnough, when trading on $m - \bar{m}(t)$. The liquidity informed trader can boost future profits by trading positively on $m - \bar{m}(t)$, that is $\delta_m(t) > 0$. However, as compared to the case of the single informed trader, the benefits of this strategy are tempered by the value informed trader (Trader 1) who knows $v$ and will erode these profits.

In equilibrium $E[v - p(t)|G_2(t), U_1]$ is perfectly collinear with $m - \bar{m}(t)$, so the liquidity informed trader’s strategy can be written as $u_2(t) = \delta(t)(m - \bar{m}(t))$, where

$$\delta(t) = \delta_m(t) + \frac{\text{cov}(v, m|F(t), U_1, U_2)}{\text{var}(m|F(t), U_1, U_2)} \delta_v(t)$$

The trading of liquidity traders naturally creates a negative correlation between $v - p(t)$ and $m - \bar{m}(t)$. Thus, by trading negatively on $m - \bar{m}(t)$, the liquidity informed trader can exploit information about $m$ to better evaluate the value of the asset $v$ and make a profit from trade. These characteristics are displayed in Figure 6.
Figure 6: Liquidity informed trader's strategy (knowledge of $m$)

This figure shows $\delta_{m}$, $\delta_{v}$ and $\delta$ for initial covariance matrix $\Sigma(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and the cost of trade, $\epsilon = 0.1, 0.01, 0.001$.

The strategy, $\delta$ is the net effect of these competing incentives. As time one approaches delta will become negative, since the incentives to make a profit today become greater relative to the incentives to increase future profits. However, early on in the trading window, the liquidity informed trader may be willing to accentuate the trade of the liquidity traders, leading to a positive delta. The incentives become stronger when the initial variance of $m$ is small. The impact of a small variance is that the rate at which the market makers update $m$ is lower and the natural negative correlation generated by liquidity trade is smaller.
Figure 7 shows $\delta$ for different levels of initial variances of $m$ to illustrate how the optimal delta evolves over time. Delta tends to start higher, then as more negative correlation builds up, the informed trader will begin to offset increasing levels of the liquidity traders’ mean trade. When the variance of $m$ is small, relative to the white noise, the informed trader will initially set delta positive. This initial amplification of $m$ helps push liquidity trade to the wrong side of the market, which the liquidity informed trader profits from. As the variance of $m$ becomes large, relative to the white noise, market makers update more quickly on $m$. This reduces the informed traders incentives to trade positively on $m - \hat{m}(t)$, as this trade reveals a greater amount of the liquidity informed traders’ private information.

### 3.4 Welfare of liquidity traders

Another aspect to consider is that of welfare. In the model, the profit of the informed traders comes, on average, at the expense of the behavioral liquidity traders. In expectation, market makers in the model make zero profit. However, liquidity traders are not modeled explicitly, so it is difficult to properly
define a welfare criteria for these traders. However, a reasonable approximation to this could be the aggregate expected profit of the informed traders.

To examine this we consider the following four models.

- Model 1: A single informed trader who is perfectly informed about $v$ and $m$.
- Model 2: Two informed traders, one perfectly informed about $v$ (value informed trader) and the other perfectly informed about $m$ (liquidity informed trader).
- Model 3: A single informed trade who is perfectly informed about $v$, but has no information about $m$.
- Model 4: The market is perfectly informed about $m$, and there is a single informed trader informed about $v$.

Figure 8: Aggregate profit of informed traders

Figure 8 shows the aggregate profits for the informed traders at different levels of variance in $m$. The variance of $v$ is set to be 1, and $v$ and $m$ are independent. The graph has been made using an epsilon of 0.001, however this has little effect on the qualitative features of the figure. For model 4 the variance
of $m$ is always zero, as the exogenous demand shock is known by the market. This model is exactly
the model considered by Kyle (1985) and is included as a point of reference.

It can be seen from Figure 8 that the best case scenario, from the point of view of liquidity traders, is
when $m$ is known by the market. In this case there is only a single dimension of private information
and the variance of $m$ is effectively zero. This results in a flat curve as shown in the figure. The worst
case scenario is model 1, when there is a single informed trader who is perfectly informed about $v$ and
$m$. In this case the informed trader is able to fully exploit both dimensions of information to make
profit.

The more interesting comparison comes between models 2 and 3. In model 3 there is only a single
informed trader. This trader does not know $m$, but by knowing $v$ is able to filter information about $m$
better than the market makers. This means that over time the informed trader is able to learn about
$m$ and exploit this information. In model 3 this is an additional informed trader who knows about $m$.
This increased competition reduces the total profit made by the informed traders and decreases the
expected losses made by the liquidity traders.

4 Example of bubbles

Empirical evidence shows that in periods of inflated prices institutional investors may be willing to buy
inflated stock, riding the price bubble, before selling the stock later when the prices have gone even
higher. Two empirical studies that document this phenomena are Brunnermeier and Nagel (2004) and
Temin and Voth (2004). Brunnermeier and Nagel study Hedge funds during the technology bubble of
the late nineties. They show that Hedge funds were heavily invested in technology stocks during the
run up of the bubble, holding more than the market portfolio, and holding peaked about six months
before the bubble peaked. Temin and Voth show that Hoare’s bank, a private investment bank in
England, exhibit a similar trading strategy during the South Sea bubble of 1720. Hoare’s bank was
buying South Sea stock during the run up of the price bubble, then started to decrease its holdings
before the crash. Temin and Voth also find that, during the bubble, Hoare’s bank showed signs that
they believed the stock was overpriced, meaning they were aware of the existence of the bubble.

In standard models of insider trading, informed traders act, on average, to decrease the mispricing of
an asset. Bubbles may still form in these models, driven by the variance in liquidity trade. However,
these sort of trade patterns, where investor buy as the bubble forms and sell at the peak, are absent.
Once an informed investor believes the stock is overvalued they will start to apply selling pressure,
meaning that their holdings will decline if a stock is overpriced.

In this model, informed traders’ desires to exacerbate the the exogenous demand shock can lead to
exactly this sort of trading behavior during price bubbles. To give an example of this, consider the
case where there are a set of $n$ trader who are ex-ante symmetric. Each informed trader gets an
imperfect signal about both about both $v$ and $m$. To simplify the analysis it will be assumed that,
taken together, informed traders signals perfectly reveal $v$ and $m$. In the case of $v$ this is without loss of generality.\footnote{Informed traders and market makers are risk neutral. This means that only the best estimate of $v$, based on all privately held information, is sufficient for the value of the asset. Let $v^* = E[v|X_1, ..., X_n]$. Informed traders’ signals could always be written as signals on $v^*$ with some correlated white noise.} For $m$ this assumption is with some loss of generality, but relaxing the assumption does not significantly alter result.

By ex-ante symmetric I mean that traders’ signals are equally informativeness, and for any two informed traders the correlation between their signals is the same. More formally, for all $i, j \in \{1, ..., n\}$ and $k \in \{0, 1, ..., n\}$, such that $i, j \neq k$

$$\text{Var} (X_i) = \text{Var} (X_j)$$
$$\text{Cov} (X_i, X_k) = \text{Cov} (X_j, X_k)$$

and, $\alpha' H_i = \alpha' H_j$ and $\mu' H_i = \mu' H_j$.

Now, let $v$ and $m$ be independent and normally distributed with means $\mu(0)$ and $0$ and variance $\sigma_v^2$ and $\sigma_m^2$. Assume that each informed trader gets two signals, one signal about $v$ and one signal about $m$. These signal take the form

$$v_i = v + \vartheta_i$$
$$m_i = m + \theta_i$$

where $v_i$ and $m_i$ are the signals of informed trader $i \in \{1, ..., n\}$. For all $i \in \{1, ..., n\}$, $\vartheta_i$ and $\theta_j$ are assumed to be normally distributed with mean $0$ and variance $\sigma_\vartheta^2$ and $\sigma_\theta^2$. It is assumed that all signals are joint normally distributed and assume that $v$, $m$, $\vartheta_i$ and $\theta_j$, for $i, j \in \{1, ..., n\}$, are all jointly independent of each other. Further

$$v = \frac{1}{n} \sum_{i=1}^{n} v_i$$
$$m = \frac{1}{n} \sum_{i=1}^{n} m_i$$

This means that together the informed traders have a perfect signal about $v$ and $m$. So, for each $i, j \in \{1, ..., n\}$ it will be the case that $\vartheta_i$ and $\vartheta_j$ are correlated, and $\theta_i$ and $\theta_j$ are correlated.

In the equilibrium considered, each informed trader uses a symmetric strategy, where $G_i(t) = G_j(t)$. An implication of this is that for the market makers

$$E [v_i|U_1, ..., U_n, F_t] = E [v|U_1, ..., U_n, F_t], \text{ and}$$
$$E [m_i|U_1, ..., U_n, F_t] = E [m|U_1, ..., U_n, F_t]$$

This means that informed traders strategies can be written in the form

$$u_i(t) = \beta(t) (v_i - p(t)) + \delta(t) (m_i - m(t))$$
where \( p(t) \) is the price, \( p(t) = E[v|U_1, ..., U_n, F_t] \), and \( \bar{m}(t) \) is the market makers estimate of the demand shock, \( \bar{m}(t) = E[m|U_1, ..., U_n, F_t] \).

**Example**

In the following numerical example; the number of informed traders is, \( n = 10 \), \( \sigma^2_v \) and \( \sigma^2_m \) are given by 1 and 0.5. Further let the variances of \( \vartheta \) and \( \theta \) be 20 and 10 respectively, and \( \epsilon \) is set at 0.005.\(^{10}\)

**Figure 9: Trading intensities**

Figure 9 shows how an informed trader trades on the different dimension of their information. The graph in the top left, \( \beta(t) \), shows their trading intensity on their signal of the asset value. As can be seen by the positive \( \beta(t) \), the informed traders acts to correct mispricing on this dimension. The graph on the lower left shows how informed traders are acting on their expect value of the asset, rather than on their signal.

The graphs on the right show how the informed traders trade on their information about the demand shock, \( m \). Again, the top graphs shows how they trade on their signal of the demand shock, and the bottom graph shows how they trade on their expectation of the demand shock. Both these graph show the same basic shape, informed trade first trade positively on the demand before later trading negatively on it. This first positive trade will have the affect of exacerbating the demand shock, and this is what will cause phenomena similar to price bubbles. For example, consider the case where the demand shock is initially positive, \( m - \bar{m}(0) > 0 \). This will increase informed trader’s buying, and for a small over pricing the informed trader will be trading on the wrong side of the market.

\(^{10}\)In this case the value of epsilon is small enough that it does not have a large effect on the equilibrium.
In this equilibrium, all informed traders are using the same strategy. This means that the aggregate trade flow does not depend on the exact distribution of signals, only on the realizations of \( v \) and \( m \), and the realized liquidity trade. This can be seen by taking the sum of informed traders’ order flows at each time \( t \).

\[
\sum_{i=1}^{n} u_i(t) = \sum_{i=1}^{n} \beta(t) (v_i - p(t)) + \delta(t) (m_i - \bar{m}(t)) = n [\beta(t) (v - p(t)) + \delta(t) (m - \bar{m}(t))] \]

This means that, conditional on the realizations of \( v \) and \( m \), it is possible to map out the expected price path and expected average holdings of the informed trader over time. Consider the case where initially the asset is correctly priced, \( v = p_0 = 0 \), but there is a large positive exogenous demand shock, \( m = 2.5\sigma_m^2 \).

In figure 10, the graph on the left shows the expected price path in this situation, and the expectation of the value for a trader who received a set of signals equal to the mean signals, \( v \) and \( m \). This graph shows how the price is expected to rise over time. It also shows how, on average, informed traders believe the asset is overprice. However, informed traders’ beliefs rise with the price bubble. At all point along the path the informed traders believe the asset is over value, but each informed trader does not realize how much it is over valued. By trading with the demand shock, each informed trader is not just fooling the market makers into thinking that the asset is more valuable than it is, they are also fooling the other informed traders. Eventually the gap between the price and the expected value of informed traders becomes to large, informed traders start to sell and this selling pressure crashes the bubble.

In the graph on the right in figure 10, the average holding of informed traders is shown. This shows how informed trader are willing to ride the bubble. Informed traders initially buy with the exogenous demand shock, helping to push the price up. They then start to sell out before the peak which enables them to make a profit. This sort of behavior is missing from standard models of informed trading. In these models informed traders are always trading against the mispricing of the asset. It would be possible for bubbles to form in these models, as they can be driven solely by the white noise in
liquidity trade. The key difference between these models is the ex-post trading patterns of the informed traders. In the standard model informed traders would always be selling if the asset is overpriced. This means that if we were to look back on period of bubbles, informed traders should have been applying correcting forces to the bubbles and show falling holdings of the stock.

As can be seen in this example, this model can provide justifications for the empirical trader patterns seen during episodes considered to be price bubbles. Informed trader's rationally ride the bubble, buying as the price initially begins to rise before selling out before the peak.

5 Concluding Remarks

This paper presents a general framework to consider the presence of multiple informed traders with private information. The framework allows the analysis of a broad range of information structures, including asymmetric information. Further, this addition allows the analysis of asymmetric signal structures within the model leading to a much richer analysis of strategic incentives. This provides the opportunity to study many aspects of trading which are lost when traders are symmetrically informed. Finally, this paper also adds to the theoretical literature of price bubbles by providing a model of price distortions with an endogenous price path, and no short sales constraints.

Appendix

Proof of Proposition 1: The market makers observe \(dQ_t\), where

\[
dQ_t = (G(t)' - \mu') (X - \bar{X}(t)) dt + \mu' X dt + dB_t
\]

Let \(d\bar{Q}_t = dQ_t + (G(t)' - \mu') \bar{X}_t dt\). Using this, the market makers filtering problem can be written as

\[
d\bar{Q}_t = G(t)'X dt + dB_t
\]

where \(X\) is the hidden state variable. This forms a standard Kalman-Bucy filter. The updating of \(\bar{X}(t)\) and \(\Sigma(t)\) are given by

\[
d\bar{X} = L(t) (d\bar{Q}_t - G(t)'\bar{X}(t) dt)
\]

\[
= L(t) (dQ_t - \mu' \bar{X}(t) dt)
\]

\[
d\Sigma(t)/dt = -L(t)L(t)'
\]

where

\[
L(t) = \Sigma(t)G(t)
\]
with the initial conditions $X(0)$ and $\Sigma(0)$. See \textellipsis .

**Proof of Proposition 2:** This proposition has two parts, the first step consider how an informed trader updates based on their initial signals. The second step considers how an informed trader updates over time.

Initially, $X$ is joint normally distributed with mean $\bar{X}(0)$ and variance $\Sigma(0)$. Informed trade $i$ receives the signals $X_i$. The possible correlation between $X_i$ and other signals in $X$. The informed traders set of signals are given by

$$X_i = H_i'X$$

Apply the Kalman filter, the informed updates of the mean and variance of $X$ are given by

$$\tilde{X}_i(0) = \bar{X}(0) + \Sigma(0)H_i(H_i'\Sigma(0)H_i)^{-1}(X_i - \bar{X}_i(0))$$

$$\tilde{\Sigma}_i(0) = \Sigma(0) - \Sigma(0)H_i(H_i'\Sigma(0)H_i)^{-1}H_i'\Sigma(0)$$

From time 0 onwards, the informed trader observes $dQ_t$ and $u_i(t)$, the informed trader is also able to infer the market makers estimates of $\bar{X}(0)$. When all other informed traders play their equilibrium strategy,

$$dQ_t = \tilde{G}_i(t)' (X - \bar{X}(t)) dt + u_i(t)dt + dB_t$$

Let $d\tilde{Q}_i(t)$ be given by

$$d\tilde{Q}_i(t) = dQ_t + \tilde{G}_i(t)' \bar{X}(t) dt - u_i(t)dt$$

then

$$d\tilde{Q}_i(t) = \tilde{G}_i(t)'X dt + dB_t$$

The problem now becomes a standard Kalman-Bucy filter. The updating by the informed trader is given by

$$d\tilde{X}_i(t) = K_i(t) \left( d\tilde{Q}_i(t) - \tilde{G}_i(t)'\bar{X}_i(t) dt \right)$$

$$d\tilde{\Sigma}_i(t) = -K_i(t)K_i(t)' dt$$

where

$$K_i(t) = \tilde{\Sigma}_i(t)\tilde{G}_i(t)$$

Further, define $\tilde{B}_i(t)$ as

$$\tilde{B}_i(t) = \int_0^t d\tilde{Q}_i(s) - \int_0^t \tilde{G}_i(t)'\bar{X}_i(t) dt$$
$\tilde{B}_i(t)$ is standard Brownian motion.

**Proof of Proposition 3:** If informed trader $i$ plays their equilibrium strategy

$$E[X|U_1, \ldots, U_n, F_t, X_i] = E[X|U_1, \ldots, U_n, \mathcal{G}_i(t)]$$

As the filtration generated by $(F_t, X_t, (u_i(s))_{s \leq t})$ is equivalent to $\mathcal{G}_i(t)$.

By proposition 1, $X$ conditional on $(U_1, \ldots, U_n, F_t, \mathcal{F}_t)$ is multivariate normal with mean $\tilde{X}(t)$ and variance $\Sigma(t)$. Updating the conditional expectation based on the signal $X_i = H_i'X$, gives

$$E[X|U_1, \ldots, U_n, F_t, X_i] = \tilde{X}(t) + \eta_i(t) (X_i - H_i'\tilde{X}(t))$$

where

$$\eta_i(t) = \Sigma(t)H_i (H_i'\Sigma(t)H_i)^{-1}$$

This, if informed trader $i$ plays the equilibrium strategy $\tilde{X}_i(t) - \tilde{X}(t) = \eta_i(t) (X_i - \tilde{X}_i(t))$.

**Proof of Proposition 4:** Informed trader $i$ take as given $(L_i)_{t \in [0,1]}$ and $(\tilde{G}_i)_{t \in [0,1]}$, which are both bounded continuous functions. Informed trader $i$’s problem is to maximize

$$J(U_i) = E\left[\int_0^1 \left(u_i(t)\alpha^T (X - \tilde{X}(t)) - \frac{\epsilon}{2}u_i(t)^2\right) dt \bigg| P, \{U_j\}_{j \neq i}\right]$$

subject to $u_i(t)$ being $\mathcal{G}_i(t)$ measurable.

The informed trader’s optimal control problem involves the control imperfectly observed state variable $X$. However, using the separation principle this can be reduced to the control of a system with observed state variable.

$$J(U_i) = E\left[\int_0^1 \left(u_i(t)\alpha^T (\tilde{X}(t) - X(t)) - \frac{\epsilon}{2}u_i(t)^2\right) \bigg| \mathcal{G}_i(t), P, \{U_j\}_{j \neq i}\right]$$

Letting $Z^i(t) = (\tilde{X}_i(t) - \tilde{X}_i)$, the informed traders problem can be written as the maximization of

$$\tilde{J}(u) = E\left[\int_0^1 u_i(t)\alpha'Z^i(t) - 0.5\epsilon (u_i(t))^2 dt \bigg| \mathcal{G}_i(t), P, \{U_j\}_{j \neq i}\right]$$
subject to the law of motion

\[dZ^i(t) = -L_i \tilde{G}_i(t) Z^i(t) dt - L_i u_i(t) dt + (K_i(t) - L_i) \tilde{B}_t\]

where this law of motion follows from proposition 2.

Now the problem can be written as quadratic linear regulator problem. Let \(u_i(t) = \frac{1}{\epsilon} \alpha Z^i(t) + \tilde{u}_i(t)\) and \(M(t) = \alpha^2 \frac{1}{2} \alpha\), then

\[\tilde{J}(u) = E \left[ \int_0^1 Z^i(t)' M(t) Z^i(t) - 0.5 \epsilon (\tilde{u}_i(t))^2 dt \right]\]

subject to the law of motion

\[dZ^i(t) = -L_i \left( \tilde{G}_i(t) + \frac{1}{\epsilon} \alpha \right) Z^i(t) dt - L_i \tilde{u}_i(t) dt + (K_i(t) - L_i) \tilde{B}_t\]

and the condition that \(u_i(t)\) being \(G_i(t)\) measurable. This forms the standard version of the stochastic linear regulator problem.

Let

\[J(s, Z^i, \tilde{u}) = \left[ \int_s^1 Z^i(t)' M(t) Z^i(t) - 0.5 \epsilon (\tilde{u}_i(t))^2 dt \right]\]

and

\[W(s, Z^i) = \max_{\tilde{u}} J(s, Z^i, \tilde{u})\]

The profit function is then given by

\[W(s, Z^i(s)) = Z^i(s)' A_i(s) Z^i(s) + b(s)\]

The optimal control is given by

\[u_i(t) = B_i(s) Z^i(s)\]

with

\[B_i(s) = \frac{1}{\epsilon} (\alpha' - 2L_i' A_i(s))\]

and the profit function is the unique solution to the differential equation

\[\dot{A}_i(s) = -\frac{\epsilon}{2} B_i(s)' B_i(s) + \tilde{G}_i(s)L(s)' A_i(s) + A_i(s)L(s)\tilde{G}_i(s)'

with terminal condition \(A_i(s) = 0\). (See Fleming and Rishel1975, page 165)
Using proposition 3, if informed trader \( i \) has played an equilibrium strategy up until time \( s \), then

\[
Z_i'(s) = \eta_i(s) (X_i - \bar{X}_i(s))
\]

Letting \( G_i(s)' = B_i(s)\eta_i(s) \), then the informed trader's optimal strategy can then be written as

\[
u_i(s) = G_i(s)' (X_i - \bar{X}_i(s))
\]

**Proof of Corollary 1:** Let \( g(t) \) be given by

\[
g(t) = \frac{(\epsilon + 2A_1(t)\Sigma(t))^2}{\Sigma(t)}
\]

First, \( g(t) \geq 0 \) for all \( t \). Using equations (7)-(?) we have

\[
\dot{g}(t) = -1
\]

and

\[
\dot{\Sigma}(t) = -\Sigma(t)/g(t)
\]

We can use \( g(t) \) to replace \( A_1(t) \) in the ODEs. This reduces the problem to a more simple boundary value problem in \( \Sigma(t) \) and \( g(t) \) subject to the (7) and (8) with the boundary conditions

\[
\Sigma(0) = \sigma_v^2
\]

\[
\sqrt{\Sigma(1)g(1)} - \epsilon = 0
\]

Let \( g = g(1) \) then

\[
g(t) = 1 - t + g
\]

From this we can find a explicit solution to \( \Sigma(t) \) using (7) and the initial condition (8).

\[
\Sigma(t) = \sigma_v^2 \frac{1 - t + g}{1 + g}
\]

Using (10) and (11), plus the terminal condition (9), the unique positive solution for \( g \) is

\[
g = \frac{\epsilon}{2\sigma_v^2} \left( \epsilon + \sqrt{\epsilon^2 + 4\sigma_v^2} \right)
\]
The formula for $A_1(t)$ can then be solved using equation (5).■
References


