Hans Peters

Power and the Shapley Value

0.1 Introduction

The Shapley value [19] can be used not only as a rule to divide the gains from cooperation in a game with transferable utility, but also as a measure of power in simple games, that is, games in which the worth of a coalition is zero if it is loosing and one if it is winning. Think, for instance, of a parliament where the winning coalitions are those that have a majority. In this case, the Shapley value is also called the Shapley-Shubik power index [20], and it measures, in a specific way, the number of times that a player, for instance a political party, is pivotal – turns a loosing into a winning coalition by joining it. A closely related power index is the Banzhaf index or Banzhaf-Coleman index [1], but there are many other indices as well (see [2] for a recent overview).

A drawback of this use of the Shapley value and, for that matter, also of other power indices, is that it takes into account neither the issues at hand nor special relations and structures that may exist among the players. For instance, the political position of a political party – left, right – nor the content of issues on which parliamentary voting takes place, are taken into account when computing the power of a party according to the Shapley-Shubik index. As a remedy, the political science literature considers spatial models, where political parties are positioned with respect to a number, say k, of criteria, and a power index, besides the simple game, takes this constellation in $\mathbb{R}^k$ into account. An example of this is the Owen-Shapley spatial power index [15] which has been axiomatically characterized in [18]. See also [21] for a partial overview of this literature.

The first objective of this chapter is to present a model that generalizes both simple games and spatial models by specifying exactly which issues (alternatives) can be controlled by which players and coalitions. See Section 0.3, which is based on [12]. In particular, we develop a class of power indices that extend the Shapley value.

The second objective is to review and link together a few models in which relations that may exist between the players and that influence their power are taken into account. In Section 0.4 we consider the case where players and coalitions may be controlled by other coalitions, a typical example being provided by firms and investors in a network determined by share holdings. This work was preceded by [5] and [9, 10]; the power indices that will be discussed were developed in [11]. A refinement of this model to directed graphs, based on [17], is discussed in Section 0.5. Section 0.6 concludes.

0.2 Preliminaries

We start with some notations. For a set D we denote by $P(D)$ the set of all subsets of D, and by $P_0(D)$ the set of all nonempty subsets of D. By $|D|$ we denote the number of elements of D.

Throughout, $N = \{1, \ldots, n\}$ ($n \in \mathbb{N}$) is the set of *players*. Subsets of N are also called *coalitions*. A game with transferable utility or *TU-game* is a pair (N, v), where $v : P(N) \to \mathbb{R}$ with $v(\emptyset) = 0$. The number $v(S)$ is the *worth* of coalition S. The TU-game (N, v) is *simple* if $v(S) \in \{0, 1\}$ for all $S \in P(N)$, $v(N) = 1$, and $S \subseteq T \Rightarrow v(S) \leq v(T)$ for all $S, T \in P(N)$. If $v(S) = 1$ coalition S is *winning*, otherwise it is *loosing*. For $T \subseteq N$ the *unanimity game* (N, u_T) is defined by $u_T(S) = 1$ whenever $T \subseteq S$, and $u_T(S) = 0$ otherwise. Instead of (N, v) we also often write v.

The Shapley value of a game (N, v) for player i is given by the expression

$$Sh_i(N, v) = \sum_{S \subseteq N : i \notin S} \frac{|S|!(n - |S| - 1)!}{n!} [v(S \cup \{i\}) - v(S)].$$

An alternative expression using so-called dividends will be introduced later in the chapter.

0.3 Effectivity and Power

In order to illustrate the aim of this section, which is based on [12], we start with a few examples.

Example 0.1 *Two men m_1 and m_2 and a woman f have the following options (alternatives): each of them stays single, denoted by s; f marries m_1, denoted by w_1; or f marries m_2, denoted by w_2. Each person has the right to stay single, and for a marriage the consent of both involved persons is required. In this situation, can we say anything about the power of each person? (The example is adapted from [7]; it appears as Example 2.2.3 in [16].)*

Example 0.2 *Consider the following 'game form':*

$$\begin{array}{cc} & \begin{array}{ccc} L & M & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} a & d & c \\ c & b & d \end{pmatrix} \end{array}$$

Here, $N = \{1, 2\}$, player 1 chooses rows, player 2 chooses columns, and

$\{a, b, c, d\}$ is a set of alternatives. (We obtain a bimatrix game if we add utilities of the players over the set of alternatives.) Again the question is: what can we say about the power of the players?

We will answer the questions raised in these examples by developing a class of power measures for 'effectivity functions'. Let A denote the set of *alternatives*. We fix a set $\mathcal{T} \subseteq P_0(A)$, where $\mathcal{T} = P_0(A)$ if A is a finite set; if A is infinite, endowed with a topology, then $\mathcal{T}$ will be the collection of nonempty closed subsets of A.

Definition 0.1 An *effectivity function* (for $\mathcal{T}$) is a map $E : P(N) \to P(\mathcal{T})$ such that (i) $P(\emptyset) = \emptyset$, (ii) $A \in E(S)$ for every $S \in P_0(N)$, (iii) $E(N) = \mathcal{T}$, and (iv) $B \in E(S)$ implies $B' \in E(T)$ for all $B, B' \in \mathcal{T}$ and $S, T \in P_0(N)$ such that $B \subseteq B'$ and $S \subseteq T$. The set of all effectivity functions is denoted by $\mathcal{E}$.[1]

If $B \in E(S)$, then we say that S is effective for B, and this is interpreted as coalition S being able to guarantee that the 'final' alternative is in B, or is entitled to this alternative being in B. Condition (i) in Definition 0.1 means that the empty coalition is not effective for anything. Condition (ii) means that every coalition is effective for the set of all alternatives, which is a trivial condition reflecting the assumption that there has to be some 'final' outcome. Condition (iii) means that the grand coalition of all players is almighty: it is effective for every nonempty set of alternatives. Condition (iv) means that if S is effective for B then every (weakly) larger coalition is effective for every (weakly) larger set of alternatives. The last condition is usually called 'monotonicity'. An additional condition that is usually satisfied by an effectivity function is *superadditivity*, which means that if a coalition S is effective for a set B and T is effective for C and if S and T are disjoint, then $S \cup T$ is effective for $B \cap C$. Here, however, we do not impose this condition on an effectivity function.

Example 0.3 *(i) In Example 0.1 the set of alternatives is $A = \{s, w_1, w_2\}$ and for the associated effectivity function E we have $E(\{m_1\}) = E(\{m_2\}) = \{A\}$, $E(\{f\}) = \{B \in P_0(A) \mid s \in B\}$, $E(\{m_i, f\}) = \{B \in P_0(A) \mid s \in B \text{ or } w_i \in B\}$ for $i = 1, 2$, and $E(N) = P_0(A)$.*

(ii) For Example 0.2 we have $E(\{1\}) = \{B \in P_0(A) \mid \{a, d, c\} \subseteq B \text{ or } \{c, b, d\} \subseteq B\}$, $E(\{2\}) = \{B \in P_0(A) \mid \{a, c\} \subseteq B \text{ or } \{b, d\} \subseteq B \text{ or } \{c, d\} \subseteq B\}$, and $E(N) = P_0(A)$.

(iii) As a third example, let (N, v) be a simple game, and let A be some set of alternatives. Then with (N, v) we can associate an effectivity function E by letting $E(S) = \mathcal{T}$ if S is winning and $E(S) = \{A\}$ if $S \neq \emptyset$ is loosing.

Our aim is to find reasonable measures of power for effectivity functions:

[1] The term 'effectivity function' was coined by [14]. For an earlier use of the concept see for instance [6].

Definition 0.2 A *power index* on $\mathcal{E}$ is a map $\varphi : \mathcal{E} \to \mathbb{R}^N$ with $\sum_{i \in N} \varphi_i(E) = 1$.

We will impose three basic axioms on a power index for effectivity functions. First a few pieces of notation: for $E, F \in \mathcal{E}$, $E \cup F$ and $E \cap F$ are defined by $E \cup F(S) = E(S) \cup F(S)$ and $E \cap F(S) = E(S) \cap F(S)$ for all $S \in P(N)$. It is straightforward to verify that $E \cup F, E \cap F \in \mathcal{E}$.

The main axiom that we will impose on a power index φ is the Transfer Property, which was first formulated for a value on simple games by [4].

Transfer Property For all $E, F \in \mathcal{E}$,

$$\varphi(E \cup F) + \varphi(E \cap F) = \varphi(E) + \varphi(F).$$

Throughout, we will also impose anonymity. For a permutation π of N and an effectivity function $E \in \mathcal{E}$, let $\pi E \in \mathcal{E}$ be defined by $(\pi E)(\pi(S)) = E(S)$ for all $S \in P(N)$.

Anonymity $\varphi_i(E) = \varphi_{\pi(i)}(\pi E)$ for every $E \in \mathcal{E}$, every permutation π of N, and every $i \in N$.

The third axiom is a monotonicity condition.

Monotonicity $\varphi_i(E) \leq \varphi_i(F)$ for all $E, F \in \mathcal{E}$ and every $i \in N$ such that $E(S) \setminus E(S \setminus \{i\}) \subseteq F(S) \setminus F(S \setminus \{i\})$ for all $S \in P(N)$.

The Transfer Property replaces the usual additivity or linearity condition for values of TU-games. Anonymity is clear, and Monotonicity requires that a player whose contributions in an effectivity function F are larger than in E, should also be assigned more power in F than in E. Monotonicity is somewhat similar to the monotonicity condition of [25] used to characterize the Shapley value and replacing the additivity condition. In our present richer context, both conditions – that is, the Transfer Property and Monotonicity – are complementary and will be used in one and the same characterization.

In order to formulate the results below, for an effectivity function E and a set of alternatives $B \in \mathcal{T}$ we define the simple game v_B^E by $v_B^E(S) = 1$ if $B \in E(S)$ and $v_B^E(S) = 0$ otherwise. In other words, the winning coalitions in v_B^E are exactly those that are effective for B.

0.3.1 Finitely Many Alternatives

The first result is for a finite set of alternatives. A *weight system* is a collection $\omega = \left(\omega^B\right)_{B \in P_0(A)}$ of nonnegative real numbers such that $\sum_{B \in P_0(A)} \omega^B = 1$. For a weight system ω we define the power index Φ^ω by

$$\Phi^\omega(E) = \sum_{B \in P_0(A)} \omega^B Sh\left(v_B^E\right)$$

for every $E \in \mathcal{E}$.

The following theorem is Theorem 4.10 in [12], to which we refer the reader for a proof.[2]

Theorem 0.1 *A power index φ satisfies the Transfer Property, Anonymity, and Monotonicity if and only if there is a weight system ω such that $\varphi = \Phi^\omega$.*

Example 0.4 *We consider the effectivity functions in Example 0.3.*
(i) In this case, we have, for (m_1, m_2, f):

$$Sh(N, v_B^E) = \begin{cases} (\frac{1}{2}, 0, \frac{1}{2}) & \text{if } B = \{w_1\} \\ (0, \frac{1}{2}, \frac{1}{2}) & \text{if } B = \{w_2\} \\ (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) & \text{if } B = \{w_1, w_2, s\} \\ (\frac{1}{6}, \frac{1}{6}, \frac{2}{3}) & \text{in all other cases.} \end{cases}$$

Hence,

$$\begin{aligned} \Phi^\omega(E) &= \omega^{\{w_1\}}(\frac{1}{2}, 0, \frac{1}{2}) + \omega^{\{w_2\}}(0, \frac{1}{2}, \frac{1}{2}) + \omega^A(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \\ &\quad + (1 - \omega^{\{w_1\}} - \omega^{\{w_2\}} - \omega^A)(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}). \end{aligned}$$

For instance, a plausible choice of weights could be $\omega^{\{w_1\}} = \omega^{\{w_2\}} = \omega^{\{s\}} = \frac{1}{3}$ and $\omega^B = 0$ otherwise, and then $\Phi^\omega(E) = (\frac{2}{9}, \frac{2}{9}, \frac{5}{9})$.
(ii) In this case,

$$Sh(N, v_B^E) = \begin{cases} (0, 1) & \text{if } B \in \{\{a,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}\} \\ (\frac{1}{2}, \frac{1}{2}) & \text{in all other cases.} \end{cases}$$

Hence, $\Phi^\omega(E) = \alpha(0,1) + (1-\alpha)(\frac{1}{2}, \frac{1}{2})$, where $\alpha = \omega^{\{a,c\}} + \omega^{\{b,d\}} + \omega^{\{c,d\}} + \omega^{\{a,b,c\}} + \omega^{\{a,b,d\}}$.
(iii) We assume that A is finite. Now $(N, v_B^E) = (N, v)$ for every $B \in P_0(A) \setminus \{A\}$, and $v_A^E(S) = 1$ for all $S \in P_0(N)$. Consequently, $\Phi^\omega(E) = (1 - \omega^A)Sh(N, v) + \omega^A(\frac{1}{n}, \ldots, \frac{1}{n})$. For the (plausible) case where $\omega^A = 0$ we therefore have that $\Phi^\omega(E) = Sh(N, v)$, i.e., Φ^ω is just the Shapley value.

In [12] further axioms are added, which refine the (large) class of power indices characterized in Theorem 0.1. For instance, in most applications one would expect the weight of a subset of alternatives to decrease with the size of the subset, since being effective for a set implies being effective for every superset.

In the next subsection we consider the case of an infinite set of alternatives, and at the same time impose further axiomatic restrictions.

[2]The proof in [12] holds for superadditive effectivity functions, but it can be checked that it still holds and even simplifies without the superadditivity condition.

0.3.2 Infinitely Many Alternatives

We now assume that A is a possibly infinite set, endowed with a topology. More precisely, we assume that for every $a \in A$ the set $\{a\}$ is closed.[3]

Call a player $i \in N$ a *null player* in $E \in \mathcal{E}$ if $E(S) \setminus E(S \setminus \{i\}) = \emptyset$ for all $S \in P(N)$.

We consider the following further axioms for a power index φ.

Strong Monotonicity $\varphi_i(E) \leq \varphi_i(F)$ for all $E, F \in \mathcal{E}$ and every $i \in N$ such that $\{a\} \in E(S) \setminus E(S \setminus \{i\})$ implies $\{a\} \in F(S) \setminus F(S \setminus \{i\})$ for all $S \in P(N)$ and all $a \in A$.

Null Player $\varphi_i(E) = 0$ for every $E \in \mathcal{E}$ and every null player i in E.

Continuity For every sequence $(E_k)_{k \in \mathbb{N}}$ of effectivity functions with $E_1 \subseteq E_2 \subseteq \ldots$ it holds that

$$\varphi\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k \to \infty} \varphi\left(E_k\right).$$

Clearly the premiss in the definition of Strong Monotonicity is weaker than the one in the definition of Monotonicity. Jointly with other conditions Strong Monotonicity will imply that only single alternatives matter for a power index. In the definition of Continuity the union at the left-hand side of the equality is a well-defined effectivity function: see Lemma 5.1 in [12].

For a probability measure μ on the σ-field of Borel sets generated by the topology on A we define the map $\Phi^\mu : \mathcal{E} \to \mathbb{R}^N$ by

$$\Phi_i^\mu(E) = \int_A Sh_i(v_{\{a\}}^E) d\mu(a)$$

for every $E \in \mathcal{E}$ and $i \in N$. Clearly, $\sum_{i \in N} \Phi_i^\mu(E) = 1$ for every $E \in \mathcal{E}$, so that Φ^μ is a power index. See [12] for a proof of the following result.

Theorem 0.2 *Let φ be a power index. Then φ satisfies the Transfer Property, Anonymity, Strong Monotonicity, Continuity, and the Null Player Property if and only if there is a probability measure μ such that $\varphi = \Phi^\mu$.*

Due to in particular the strong monotonicity requirement, compared to Theorem 0.1 now only Shapley values of simple games associated with single alternatives occur, and the weight system is replaced by the probability measure μ.

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[3]I.e., A is a T_1-*space.*

0.3.2.1 An Application: The Owen-Shapley Spatial Power Index

A simple game (N, v) is *proper* if $v(S) = 1$ implies that $v(N \setminus S) = 0$ for each $S \in P(N)$. Let $k \in \mathbb{N}$, $k \geq 2$. A *spatial game* is a pair $g = (v, p)$ where v is a proper simple game and $p = (p^1, \ldots, p^n) \in (\mathbb{R}^k)^N$ with $p^i \neq p^j$ for all $i, j \in N$ with $i \neq j$. Here, $p^i \in \mathbb{R}^k$ is the *position* of player i. For instance, $k = 2$, i is a political party, p^i_1 reflects i's position with respect to public spending on defense, and p^i_2 reflects i's position with respect to public spending on education.

Following [15] we let the set of *issues* A be represented by the unit sphere in $\mathbb{R}^k$, i.e.,

$$A = \{a \in \mathbb{R}^k : ||a|| = 1\} ,$$

where $|| \cdot ||$ is the Euclidean distance, and we interpret the inner product $p^i \cdot a$ as a measure of the attractiveness of issue $a \in A$ for a player with position p^i. More precisely, we interpret the inequality $p^i \cdot a \leq p^j \cdot a$ as player i being more in favor of issue $a \in A$ than player j.[4] For a spatial game $g = (v, p)$ and an issue $a \in A$, we say that player i is *pivotal for* a if $\{j \in N \mid p^j \cdot a \leq p^i \cdot a\}$ is a winning coalition but $\{j \in N \mid p^j \cdot a \leq p^i \cdot a\} \setminus \{i\}$ is loosing. Here, one should think of a coalition being formed in favor of an issue a: the players join the coalition in order of their enthusiasm for a, and the pivotal player is the player who upon joining the coalition turns this from a loosing into a winning coalition.

We assume that A is endowed with the relative topology induced by the Euclidean topology on $\mathbb{R}^k$. It is not difficult to see that for almost all $a \in A$ there is a unique pivotal player. Let λ be the Lebesgue measure on A, and define the probability measure ν on A by $\nu(B) = \lambda(B)/\lambda(A)$ for every Borel set $B \subseteq A$. Then the *Owen-Shapley spatial power index* OS assigns to each player i the number $OS_i(v, p) = \nu(B)$ if B is the set of issues for which player i is pivotal.

Figure 0.1 illustrates the Owen-Shapley spatial power index for a spatial game in which the simple game is a three-person unanimity game, that is, a spatial game $(\{1, 2, 3\}, u_{\{1,2,3\}}, (p_1, p_2, p_3))$: in the simple game $u_{\{1,2,3\}}$ only the grand coalition $\{1, 2, 3\}$ is winning. The issues on the arc of the circle containing the point p^i are those for which player i is pivotal, for each $i = 1, 2, 3$.

Now for a spatial game (v, p) we construct an effectivity function F by letting player $i \in N$ be effective for the singleton $\{a\}$ if i is pivotal for a. Formally, for every $S \in P_0(N)$,

$$F(S) = \{B \subseteq A \mid B = A \text{ or } i \text{ is pivotal for } b \text{ for some } i \in S \text{ and } b \in B\}.$$

If i alone is pivotal for a then $Sh_i(v^F_{\{a\}}) = 1$ and $Sh_j(v^F_{\{a\}}) = 0$ for all

[4] Of course, this is just a matter of choice: without loss of generality one could also take the reverse inequality. Further, one can interpret $p^i \cdot a$ as representing the 'utility' of an issue a for player i with position p^i – thus, implicitly linear 'utility' is assumed.

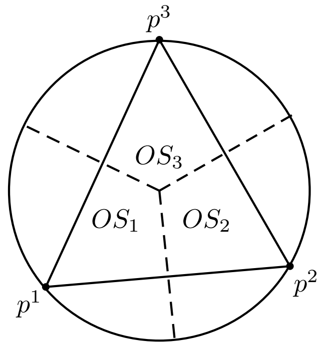

Figure 0.1 *The Owen-Shapley spatial power index in a three-person spatial unanimity game. The dashed lines are perpendicular to the edges of the triangle, and the powers of the players are proportional to the three pieces of the disk.*

$j \in N \setminus \{i\}$. Therefore, for every player i we have

$$OS_i(v, p) = \Phi_i^{\nu}(F) = \int_A Sh_i(v_{\{a\}}^F) d\nu(a).$$

Thus, the Owen-Shapley spatial power index is a special case of the power indices characterized in Theorem 0.2. See [18] for another axiomatic characterization of OS, and [13] for a generalization.

0.4 Control and Power

The results in this section are based on [11]. The leading example in [11] is depicted in Figure 0.2. The diagram describes the Porsche and VW voting rights by the end of 2012, based on the annual reports 2012 of Volkswagen AG and Porsche Automobil Holding SE GmbH. The players are Porsche Families (1), Qatar (2), Lower Saxony (3), Porsche SE (4), Volkswagen AG (5), Porsche AG (6), and other (small) stockholders (7).

Based on this diagram and some further restrictions and laws, for which we refer to [11], one can describe the 'control structure': for each coalition of players, which players are controlled by this coalition? One way[5] in which this can be done is by simple games or the zero game: for each player i, the winning coalitions in a simple game w_i are those that control player i, which means: have the required percentage of votes over that player. If a player is not controlled by any coalition then we take for w_i the zero game (N, z), i.e.,

[5]See [11] for a different but equivalent way.

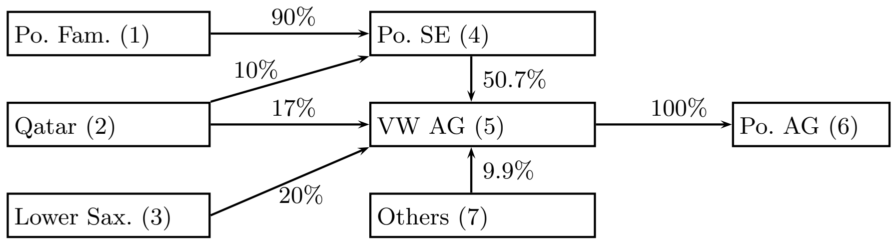

Figure 0.2 *The Porsche-VW case.*

$z(S) = 0$ for every $S \in P(N)$. For the example in Figure 0.2, we arrive at the following games: for $S \in P(N)$,

$$\begin{aligned} w_1 = w_2 = w_3 = w_7 &= z, \\ w_4(S) = 1 &\Leftrightarrow \{1\} \subseteq S, \\ w_5(S) = 1 &\Leftrightarrow \{2,3,4\} \subseteq S, \\ w_6(S) = 1 &\Leftrightarrow \{5\} \subseteq S. \end{aligned}$$

These simple games express *direct* control. For instance, any coalition containing players 2, 3, and 4, controls player 5. Note, however, that player 4 is controlled by any coalition containing player 1, and therefore player 5 is also *indirectly* controlled by any coalition containing players 1, 2, and 3. By incorporating all such indirect control relations as well we obtain the simple games w_i^* given by, for each $S \in P(N)$:

$$\begin{aligned} w_1^* = w_2^* = w_3^* = w_7^* &= z, \\ w_4^*(S) = 1 &\Leftrightarrow \{1\} \subseteq S, \\ w_5^*(S) = 1 &\Leftrightarrow \{2,3,4\} \subseteq S \text{ or } \{1,2,3\} \subseteq S, \\ w_6^*(S) = 1 &\Leftrightarrow \{5\} \subseteq S \text{ or } \{1,2,3\} \subseteq S \text{ or } \{2,3,4\} \subseteq S. \end{aligned}$$

We will develop a class of power indices for situations like this. Formally, a *control structure* is an n-tuple $\bar{w} = (w_1, \ldots, w_n)$ where, for each $i \in N$, w_i is either the zero game z or a simple game satisfying: for all $j \in N$ and all $S, T \subseteq N$, if $w_i(S) = w_j(T) = 1$, then $w_i((S \setminus \{j\}) \cup T) = 1$. The last condition means that $\bar{w}$ also captures indirect control: coalition S controls i, and if j is a member of S but at the same time controlled by T, then j's position in S can be replaced by the coalition T. Let $\mathcal{W}$ denote the set of all control structures.[6]

Definition 0.3 A *power index* on $\mathcal{W}$ is a map $\varphi : \mathcal{W} \to \mathbb{R}^N$.

[6]Control structures are equivalent to so-called command games in [9, 10]. For an earlier approach see [5].

Call player i a *null-player* in $\bar{w} \in \mathcal{W}$ if (i) $w_i = z$, and (ii) for all $j \in N \setminus \{i\}$ and $S \in P(N)$, $w_j(S) = w_j(S \setminus \{i\})$. Hence, a null-player is a player who is neither controlled nor adds anything to controlling other players. An example is player 7 in the Porsche-VW case. The first axiom we impose on a power index φ is as follows.

Null Player $\varphi_i(\bar{w}) = 0$ for every $\bar{w} \in \mathcal{W}$ and every null player i in $\bar{w}$.

A player who does not add anything to control but is controlled by some coalition of players, has in some sense even less power than a null-player. For this reason it is natural to assume that a power index can assign negative numbers in this framework. Actually, we impose the following axiom.

Zerosum $\sum_{i \in N} \varphi_i(\bar{w}) = 0$ for every $\bar{w} \in \mathcal{W}$.

This axiom implies that, in general, there will be players with positive power as well as players with negative power.

For a permutation π of N and a control structure $\bar{w}$ we denote by $\pi\bar{w}$ the control structure with $(\pi\bar{w})_{\pi(i)}(\pi S) = w_i(S)$ for every $i \in N$ and $S \in P(N)$.

Anonymity $\varphi_i(\bar{w}) = \varphi_{\pi(i)}(\pi\bar{w})$ for every $\bar{w} \in \mathcal{W}$, every permutation π of N, and every $i \in N$.

Finally, the transfer property takes the following form.

Transfer Property $\varphi_i(\bar{w}) - \varphi_i(\bar{w}') = \varphi_i(\bar{v}) - \varphi_i(\bar{v}')$ for every $i \in N$ and all $\bar{w}, \bar{w}', \bar{v}, \bar{v}' \in \mathcal{W}$ such that, for all $S \in P(N)$, (i) $w_i'(S) = 1 \Rightarrow w_i(S) = 1$ and $v_i'(S) = 1 \Rightarrow v_i(S) = 1$, (ii) $[w_i'(S) = 0$ and $w_i(S) = 1 \Leftrightarrow v_i'(S) = 0$ and $v_i(S) = 1]$.

In words, if $\bar{w}$ arises from $\bar{w}'$ and $\bar{v}$ arises from $\bar{v}'$ by adding the same winning coalitions, then for each player the change in power when going from $\bar{w}'$ to $\bar{w}$ should be equal to the change in power when going from $\bar{v}'$ to $\bar{v}$.

In fact, it can be shown that the Transfer Property is equivalent to the following: for all $\bar{w}, \bar{v} \in \mathcal{W}$,

$$\varphi(\bar{w}) + \varphi(\bar{v}) = \varphi(\bar{w} \vee \bar{v}) + \varphi(\bar{w} \wedge \bar{v})$$

where $\bar{w} \vee \bar{v} = (\max(w_1, v_1), \ldots, \max(w_n, v_n))$ and $\bar{w} \wedge \bar{v} = (\min(w_1, v_1), \ldots, \min(w_n, v_n))$, with the maxima and minima defined coalition-wise. This is similar to the original formula in [4], but the formulation of the Transfer Property above has a more intuitive interpretation.

These four conditions determine a family of power indices. In order to formulate this result, recall that the *dividends* $d(S)$ [8] of a TU-game v with player set N are defined, recursively, by

$$d(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ v(S) - \sum_{T \subsetneq S} d(T) & \text{otherwise} \end{cases}$$

for all $S \subseteq N$. For a control structure $\bar{w} = (w_1, \ldots, w_n)$ and $i \in N$, we write $d_i^{\bar{w}}$ for the dividends of w_i. Also recall that the Shapley value of a TU-game v is alternatively given by

$$Sh_i(v) = \sum_{S: i \in S} \frac{d(S)}{|S|}$$

for every $i \in N$.

For every *weight vector* $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2}$, we now define the power index Φ^ω by

$$\begin{aligned} \Phi_i^\omega(\bar{w}) \quad = \quad & \sum_{k \in N \setminus \{i\}} \left(\sum_{S: i \in S, k \notin S} \frac{d_k^{\bar{w}}(S)}{|S|} \alpha_{|S|} + \sum_{S: i \in S, k \in S} \frac{d_k^{\bar{w}}(S)}{|S|} \beta_{|S|} \right) \\ & - \sum_{k \in N \setminus \{i\}} \left(\sum_{S: i \notin S, k \in S} \frac{d_i^{\bar{w}}(S)}{|S|} \alpha_{|S|} + \sum_{S: i \in S, k \in S} \frac{d_i^{\bar{w}}(S)}{|S|} \beta_{|S|} \right) \quad (1) \end{aligned}$$

for all $\bar{w} \in \mathcal{W}$ and $i \in N$. This formula looks quite complicated but it nevertheless has a clear interpretation, as follows. The expression in brackets in the first line of (1) says that player i receives a weighted sum of dividends in the game w_k; this expresses the power player i derives from his role in controlling player k. The weights depend, both on the size of the coalition of whose dividend player i receives a share, and on whether or not the controlled player k is a member of that coalition. Thus, the first line in (1) represents the total power player i acquires from his role in controlling the other players. In the second line, the total (similarly weighted) power that all other players acquire from controlling player i, is subtracted.

See [11] for a proof of the following theorem.

Theorem 0.3 *Let $\varphi : \mathcal{W} \to \mathbb{R}^N$ be a power index. Then φ satisfies Null Player, Zerosum, Anonymity, and the Transfer Property if and only if there is a weight vector $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2}$ such that $\varphi = \Phi^\omega$.*

In Theorem 0.3 the weights ω are completely free and can be any real numbers. In [11], several conditions are considered that result in a refinement of this class of power indices. Here, we restrict our attention to the following 'scaling' condition.

Controlled Player For all $\bar{w} \in \mathcal{W}$, $j \in N$ with $w_j \neq z$, and $i \in N$ with $w_i = z$,

$$\varphi_j(\bar{w}) = \begin{cases} -1 & \text{if } w_k(S) = w_k(S \setminus \{j\}) \text{ for all } S \subseteq N \text{ and } k \in N \\ \varphi_i(\bar{w}) - 1 & \text{if } w_k(S \setminus \{i\}) = w_k(S \setminus \{j\}) \text{ for all } S \subseteq N \\ & \quad \text{such that } i, j \in S \text{ and all } k \in N. \end{cases}$$

The first line in the Controlled Player condition says that if j is a 'controlled player', i.e., controlled by at least one coalition and, thus, by N, but does not exercise any control himself, then the power of j is fixed at -1. Hence, the power of a least powerful player is fixed at -1. Further, if i is an uncontrolled player, i.e., controlled by no coalition at all, but i and j exercise the same marginal control with respect to any coalition and player, then their difference in power is fixed at 1, that is, i gets assigned 1 more than j. We now have the following corollary (see [11]).

Corollary 0.1 *There is a unique power index satisfying Null Player, Zero-sum, Anonymity, the Transfer Property, and Controlled Player, namely the power index* Φ^ω *with* $\omega = (1, \ldots, 1) \in \mathbb{R}^{2n-2}$.

We apply this unique power index to the Porsche-Volkswagen case.

Example 0.5 *For the Porsche-VW case and* $\omega = (1, \ldots, 1) \in \mathbb{R}^{2n-2}$ *we obtain* $\Phi_1^\omega(\bar{w}) = \frac{67}{60}$, $\Phi_2^\omega(\bar{w}) = \Phi_3^\omega(\bar{w}) = \frac{32}{60}$, $\Phi_4^\omega(\bar{w}) = -\frac{53}{60}$, $\Phi_5^\omega(\bar{w}) = -\frac{18}{60}$, $\Phi_6^\omega(\bar{w}) = -1$, *and* $\Phi_7^\omega(\bar{w}) = 0$. *It is interesting to compare the power of Porsche Families with its power at the end of 2007. Figure 0.3 depicts the control structure between the same companies at the end of 2007. At that time, Volkswagen was not controlled by any group of main investors. Although Porsche SE has veto power in the game on Volkswagen AG, we ignore this fact, as it is not clear how this power can be exercised. This situation results in a control structure* $\bar{v}$ *with coalition* S *winning in* v_4 *if and only if* $1 \in S$, *and coalition* S *winning in* v_6 *if and only if* $1 \in S$. *Thus, even while ignoring the power of Porsche Families on Volkswagen, we still have* $\Phi_1^\omega(\bar{v}) = 2 > \frac{67}{60} = \Phi_1^\omega(\bar{w})$. *Hence, according to this power index it had more power in 2007 than it had in the situation described by Figure 0.2 (end 2012).*

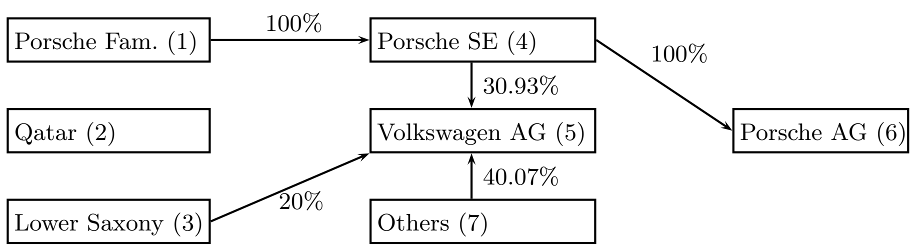

Figure 0.3 *Porsche and VW voting rights by the end of 2007, based on the 2007 annual report of Volkswagen AG and the 2007/2008 annual report of Porsche Automobil Holding SE GmbH.*

0.5 Power on Digraphs

In this section we closely follow [17]. The model in [17] is a special case of a control structure as defined in the previous section. More precisely, [17] considers control structures $\bar{w} = (w_1, \ldots, w_n)$ such that each w_i is uniquely determined by the winning singleton coalitions, i.e., for each coalition S we have $w_i(S) = 1$ if and only if there is a $k \in S$ with $w_i(\{k\}) = 1$. Such a control structure can be identified with a directed graph or *digraph* with N as the set of *nodes* and a *link* (edge) from i to j if and only if $w_j(\{i\}) = 1$, i.e., player j is controlled by player i. Let $\mathcal{D} \subseteq \mathcal{W}$ denote the set of all such control structures or digraphs.

On a power index $\varphi : \mathcal{D} \to \mathbb{R}^N$ we impose the same axioms as in the preceding section. Since the definitions of a null player and of the axioms of Null Player, Zerosum, Anonymity, and the Transfer Property do not change for power indices on $\mathcal{D}$, we do not repeat these definitions here.

For $M \in P_0(N)$ and $j \in N$ let the control structure $\bar{u}^{M,j} \in \mathcal{D}$ be defined by $u_i^{M,j} = z$ for all $i \neq j$, and $u_j^{M,j}(\{i\}) = 1$ if and only $i \in M$ for all $i \in N$. Hence, $\bar{u}^{M,j}$ can be identified with the digraph that has (only) links from every player $i \in M$ to player j. The following lemma (Lemma 4.3 in [17]) follows directly from the definitions.

Lemma 0.1 *Let the power index φ on $\mathcal{D}$ satisfy Null Player, Zerosum, and Anonymity. Then there are $\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n \in \mathbb{R}$ such that for every $M \in P_0(N)$ and $j \in N$, with $m = |M|$:*

(a) *if $j \notin M$ then for every $i \in N$*

$$\varphi_i(\bar{u}^{M,j}) = \begin{cases} 0 & \text{if } i \notin M \cup j \\ \alpha_m/m & \text{if } i \in M \\ -\alpha_m & \text{if } i = j \end{cases}$$

(b) *if $j \in M$ then for every $i \in N$*

$$\varphi_i(\bar{u}^{M,j}) = \begin{cases} 0 & \text{if } i \notin M \\ \beta_m/m & \text{if } i \in M \setminus j \\ \beta_m/m - \beta_m & \text{if } i = j \end{cases}$$

where $\alpha_0 = \beta_1 = 0$.

The digraph $\bar{u}^{M,j}$ plays a role similar to that of a unanimity TU-game. By adding the Transfer Property we obtain the following result (Theorem 4.4 in [17]). Here, for $\bar{w} \in \mathcal{D}$ and $j \in N$, $M_j^{\bar{w}} = \{i \in N \mid w_j(\{i\}) = 1\}$ is the set of players who have a link to player j, i.e., who control player j.

Theorem 0.4 *A power index φ on $\mathcal{D}$ satisfies Null Player, Zero-sum, Anonymity, and the Transfer Property if and only if there are $\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n \in \mathbb{R}$ such that for each $\bar{w} \in \mathcal{D}$ we have $\varphi(\bar{w}) = \sum_{j \in N} \varphi(u^{M_j^{\bar{w}}, j})$, with $\varphi(u^{M_j^{\bar{w}}, j})$ as defined in* (a) *and* (b) *of Lemma 0.1.*

Denote a power index φ as in Theorem 0.4 with parameters $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ and $\beta = (\beta_2, \ldots, \beta_n)$ by $\varphi^{\alpha,\beta}$. It can be checked that the weights in Theorem 0.4 coincide with those in Theorem 0.3: more precisely, on $\mathcal{D}$, for $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n)$, Φ^ω coincides with the power index $\varphi^{\alpha,\beta}$.

Theorem 0.4 does not put any restrictions on the parameters α, β. We next present three possibly plausible further conditions which have to do with adding additional links.

The first condition says that if we add a link to a player j from some player i then this should not change the power of the players who already have a link to j. In the control parlance: if player j gets additionally controlled by some player i then this should not change the power of the players who were already controlling j.

Link Addition 1 Let $i, j \in N$ and let $\bar{w}, \bar{w}' \in \mathcal{D}$ differ only in that $w_j(i) = 0$ whereas $w'_j(i) = 1$. Then $\varphi_h(\bar{w}) = \varphi_h(\bar{w}')$ for all $h \in N \setminus \{j\}$ such that $w_j(h) = 1$.

Corollary 0.2 *Let $\varphi = \varphi^{\alpha,\beta}$. Then φ satisfies LA1 if and only if there is a $c \in \mathbb{R}$ such that $\alpha_k = kc$ for all $k = 1, \ldots, n-1$ and $\beta_k = kc$ for all $k = 2, \ldots, n$.*

Thus, under LA1 we obtain a one-parameter family of power indices of the form

$$\varphi_i^c(\bar{w}) = c\left(|\{j \in N \mid w_j(i) = 1\}| - |\{j \in N \mid w_i(j) = 1\}|\right),$$

where $c \in \mathbb{R}$. For instance, for $c = 1$ and in the control terminology, the power of player i is equal to the number of players controlled by player i minus the number of players controlling player i. This is similar to the Copeland score in social choice theory [3], under the interpretation that a link from i to j means that i is preferred to j.

The second condition requires that it is player j whose power does not change if a link is added from some player i to j, provided that there was already a link from some other player to j. In terms of control: if a player j becomes additionally controlled by some player i then this should not change the power of player j.

Link Addition 2 Let $i, j \in N$ and let $\bar{w}, \bar{w}' \in \mathcal{D}$ differ only in that $w_j(i) = 0$ whereas $w'_j(i) = 1$. Also, let $w_j(\{k\}) = 1$ for some $k \in N \setminus \{j\}$. Then $\varphi_j(\bar{w}) = \varphi_j(\bar{w}')$.

Corollary 0.3 *Let* $\varphi = \varphi^{\alpha,\beta}$. *Then* φ *satisfies LA2 if and only if there is a* $c \in \mathbb{R}$ *such that* $\alpha_k = c$ *for all* $k = 1, \ldots, n-1$ *and* $\beta_k = \frac{k}{k-1} c$ *for all* $k = 2, \ldots, n$.

The power indices characterized in Corollary 0.3 take the form

$$\bar{\varphi}_i^c(\bar{w}) = \sum_{j \in N \setminus \{i\} \,:\, w_j(\{i\})=1} \frac{c}{|\{k \in N \setminus \{j\} \mid w_j(k) = 1\}|} - c\, 1_{\{M_i^{\bar{w}} \setminus \{i\} \neq \emptyset\}}$$

where $1_{\{P\}} = 1$ if statement P is true and $1_{\{P\}} = 0$ otherwise. According to a power index $\bar{\varphi}^c$, if a player i has a link to some other player j, then he equally shares the amount of power c with the other players having a link to j, except possibly j. If player i is controlled by some one else than himself, then he looses an amount c of power. This power index is similar to the idea of the β-measure as in [24] or its reflexive variant in [23].

The final condition we consider says that if we add a link from a player i to a player j but player j was already controlled by some other player (possibly by himself) then both have the same gain (or loss) in power. This condition may make sense, perhaps not so much in the control setting, but rather in a setting where players i and j have some common interests – for instance, they work in the same department of a university.

Link Addition 3 Let $i, j \in N$ and let $\bar{w}, \bar{w}' \in \mathcal{D}$ differ only in that $w_j(i) = 0$ whereas $w_j'(i) = 1$. Also, let $w_j(\{k\}) = 1$ for some $k \in N$. Then $\varphi_i(\bar{w}) - \varphi_i(\bar{w}') = \varphi_j(\bar{w}) - \varphi_j(\bar{w}')$.

Corollary 0.4 *Let* $\varphi = \varphi^{\alpha,\beta}$. *Then* φ *satisfies LA3 if and only if there is a* $c \in \mathbb{R}$ *such that* $\alpha_k = \frac{2}{k+1} c$ *for all* $k = 1, \ldots, n-1$, *and* $\beta_k = 0$ *for all* $k = 2, \ldots, n$.

The power indices characterized in Corollary 0.4 take the form:

$$\widetilde{\varphi}_i^c(\bar{w}) = \sum_{j \in N \,:\, w_j(\{i\})=1, w_j(\{j\})=0} \frac{\alpha_{|M_j^{\bar{w}}|}}{|M_j^{\bar{w}}|} - \alpha_{|M_i^{\bar{w}}|}\, 1_{\{M_i^{\bar{w}} \neq \emptyset,\ i \notin M_i^{\bar{w}}\}} \ ,$$

with α_k as in Corollary 0.4. In control terms, according to a power index $\widetilde{\varphi}^c$, if player j controls himself, then no player, including player j, derives (positive or negative) power from controlling j. Further, the power (negative, if $c > 0$) from being controlled decreases as the number of controlling players increases. We note that $\widetilde{\varphi}^c$ is related to the apex power index in [22]).

0.6 Conclusions

In this chapter we have reviewed some of our works on power indices for situations that go beyond simple games, namely power indices for effectivity functions and power indices for control structures. We have also shown that a spatial power index like that of Owen and Shapley [15] is a special case of the former, while some power indices for digraphs, like the Copeland score [3], are special cases of the latter. Our power indices for effectivity functions are clearly based on the Shapley value: see Theorems 0.1 and 0.2. The general formula for power indices for control structures in Theorem 0.3 is based on dividends and therefore indirectly related to the Shapley value. For instance, if we take $\alpha_1 = \ldots = \alpha_{n-1} = \beta_2 = \ldots = \beta_n = \gamma \in \mathbb{R}$, then the resulting power index Φ^ω is given by

$$\Phi_i^\omega(\bar{w}) = \gamma \left(\sum_{k \in N} Sh_i(w_k) - 1 \right)$$

for every $\bar{w} \in \mathcal{W}$ and $i \in N$. Also, for $\varphi(u^{M_j^{\bar{w}},j})$ in Theorem 0.4 we can write

$$\varphi(u^{M_j^{\bar{w}},j}) = \begin{cases} \alpha_{|M_j^{\bar{w}}|} \, Sh\left(u_{M_j^{\bar{w}}} - u_{\{j\}}\right) & \text{if } j \notin M_j^{\bar{w}} \\ \beta_{|M_j^{\bar{w}}|} \, Sh\left(u_{M_j^{\bar{w}}} - u_{\{j\}}\right) & \text{if } j \in M_j^{\bar{w}} \end{cases}$$

where u_T for $T \subseteq N$ is the unanimity TU-game.

The two approaches, i.e. to effectivity functions and control structures, cannot be directly linked. Total power according to a power index for control structures (Theorem 0.3) is equal to zero: in this case, it is natural to assign both negative and positive power and to have the sum equal to zero. Total power according to a power index for effectivity functions (Theorems 0.1 and 0.2) is equal to one, and the power of every player is nonnegative; which is as in the standard case. Moreover, in the control case, if player i controls player j, then in some sense player i is 'effective' for $\{j\}$, but this is not quite consistent with the usual conditions on an effectivity function: for instance, if player i controls player j then this does not imply that i controls every set of players containing j, but if i is effective for $\{j\}$ in the formal meaning of this expression as defined above, then i is effective for every set of players containing j. Hence, it is not obvious how to construct a general model encompassing the two approaches reviewed in this chapter.

0.7 Acknowledgements

I would like to thank the editors of this volume for their constructive comments.

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