Characterizing NTU-Bankruptcy Rules using Bargaining Axioms

Bas Dietzenbacher[*†] Hans Peters[‡]

July 21, 2018

Abstract

This paper takes an axiomatic bargaining approach to bankruptcy problems with nontransferable utility by characterizing bankruptcy rules in terms of properties from bargaining theory. In particular, we derive new axiomatic characterizations of the proportional rule, the truncated proportional rule, and the constrained relative equal awards rule using properties which concern changes in the estate or the claims.

Keywords: NTU-bankruptcy problem, axiomatic analysis, bargaining theory
JEL classification: C78, D74

1 Introduction

In a bankruptcy problem with nontransferable utility, claimants have incompatible claims on an insufficient estate which is expressed in a set of attainable utility allocations. These problems arise when claimants have individual utility functions over their monetary payoffs. Allowing for a rich class of utility functions, the estate can take a very general form. Bankruptcy rules assign to any such bankruptcy problem a feasible utility allocation, i.e. an allocation for which the individual utility payoffs are bounded by the corresponding claims. On the one hand, bankruptcy problems with nontransferable utility generalize monetary bankruptcy problems (cf. O'Neill 1982). On the other hand, they can be considered as an alternative interpretation of bargaining problems with claims (cf. Chun and Thomson 1992).

Recently, Dietzenbacher, Estévez-Fernández, Borm, and Hendrickx (2016) and Dietzenbacher, Borm, and Estévez-Fernández (2017) took an axiomatic approach to bankruptcy problems with nontransferable utility by characterizing bankruptcy rules in terms of adequately generalized properties from bankruptcy theory. To explore the proportional rule, the truncated proportional rule, and the constrained relative equal awards rule, they introduced the relative symmetry axiom, which imposes a relatively equal treatment of relatively equal claimants, and the truncation invariance axiom, which imposes invariance under truncation of the claims by the estate.

Orshan, Valenciano, and Zarzuelo (2003), Estévez-Fernández, Borm, and Fiestras-Janeiro (2014) and Dietzenbacher (2018) took a game theoretic approach to bankruptcy problems

[*]E-mail: bas.dietzenbacher@gmail.com
[†]CentER and Department of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands
[‡]Department of Quantitative Economics, Maastricht University, P.O. Box 616, 6200 MD Maastricht, The Netherlands

with nontransferable utility by defining an appropriate coalitional bankruptcy game and focusing on the structure of the core. Besides, Dietzenbacher (2018) characterized the class of game theoretic bankruptcy rules by the truncation invariance property.

This paper takes an axiomatic bargaining approach to bankruptcy problems with nontransferable utility by characterizing bankruptcy rules in terms of properties from bargaining theory. Similar to Chun and Thomson (1992), we interpret the estate of a bankruptcy problem as the feasible set of a bargaining problem as introduced by Nash (1950) which is enriched by a claims vector. However, we adopt the standard assumption from bankruptcy theory that individual utility is normalized in such a way that allocating nothing corresponds to a utility level of zero. Therefore, it is convenient to consider the zero vector as a natural benchmark for allocations instead of an exogenous disagreement point as within bargaining problems. Although not addressed in this paper, this does still allow for the approach of Herrero (1997), which interprets the vector of minimal rights of a bankruptcy problem, the maximal individual payoffs within the estate when all other claimants are allocated their claims, as the corresponding endogenous disagreement point of a bargaining problem with claims.

We consider the role of the claims vector within bankruptcy problems as being 'dual' to the role of the disagreement point within bargaining problems. Where the disagreement point serves as a lower bound for rational payoff allocations within a bargaining problem, the claims vector serves as an upper bound for feasible payoff allocations within a bankruptcy problem. Following the classical axiomatic theory of bargaining, we formulate several properties which concern changes in the estate or the claims, where the latter ones are based on axioms concerning changes in the disagreement point, and study their implications. In particular, we translate several axioms from bargaining theory to the domain of bankruptcy problems with nontransferable utility, study their relations, and combine them with the axioms relative symmetry and truncation invariance from bankruptcy theory to derive new axiomatic characterizations of the proportional rule, the truncated proportional rule, and the constrained relative equal awards rule.

This paper is organized in the following way. Section 2 provides an overview of notions for bankruptcy problems with nontransferable utility. In Section 3, we introduce and study the implications of axioms concerning changes in the estate. In Section 4, we introduce and study the implications of axioms concerning changes in the claims. Section 5 concludes and formulates some suggestions for future research.

2 Preliminaries

Let N be a nonempty and finite set of *claimants*. For any $x, y \in \mathbb{R}^N_+$, $x \leq y$ denotes $x_i \leq y_i$ for all $i \in N$, and $x < y$ denotes $x_i < y_i$ for all $i \in N$. For any set of payoff allocations $E \subseteq \mathbb{R}^N_+$,

- the *comprehensive hull* is given by $\text{comp}(E) = \{x \in \mathbb{R}^N_+ \mid \exists_{y \in E} : y \geq x\}$;
- the *weak upper contour set* is given by $\text{WUC}(E) = \{x \in \mathbb{R}^N_+ \mid \neg\exists_{y \in E} : y > x\}$;
- the *weak Pareto set* is given by $\text{WP}(E) = \{x \in E \mid \neg\exists_{y \in E} : y > x\}$;
- the *strong Pareto set* is given by $\text{SP}(E) = \{x \in E \mid \neg\exists_{y \in E, y \neq x} : y \geq x\}$.

Note that $\text{SP}(E) \subseteq \text{WP}(E) \subseteq \text{WUC}(E)$. A set of payoff allocations $E \subseteq \mathbb{R}^N_+$ is called *comprehensive* if $E = \text{comp}(E)$, and *nonleveled* if $\text{SP}(E) = \text{WP}(E)$.

A *bankruptcy problem with nontransferable utility* (cf. Orshan et al. 2003) is a triple (N, E, c) in which $E \subseteq \mathbb{R}^N_+$ is a nonempty, closed, bounded, comprehensive, and nonleveled *estate*, and $c \in \mathrm{WUC}(E)$ is a vector of *claims*.[1] Let BR^N denote the class of all bankruptcy problems with claimant set N. For convenience, an NTU-bankruptcy problem is denoted by $(E, c) \in \mathrm{BR}^N$. Note that $0_N \in E$ and $\mathrm{WUC}(E)$ is closed for all $(E, c) \in \mathrm{BR}^N$. Moreover, both $(E \cup E', c) \in \mathrm{BR}^N$ and $(E \cap E', c) \in \mathrm{BR}^N$ for all $(E, c), (E', c) \in \mathrm{BR}^N$.

Note that the estate is not required to be convex. This means that we allow for utility functions which are not necessarily of the Von Neumann-Morgenstern type. However, most of our results do not rely on the admission of nonconvex estates and can therefore be reformulated on the domain of bankruptcy problems with convex estate.

Let $(E, c) \in \mathrm{BR}^N$. The set of *positive claimants* is given by

$$N^c_+ = \{i \in N \mid c_i > 0\}.$$

The *truncated estate* $\hat{E}_c \subseteq \mathbb{R}^N_+$ is given by

$$\hat{E}_c = \{x \in E \mid x \leq c\}.$$

The vector of *utopia values* $u^E \in \mathbb{R}^N_+$ is given by

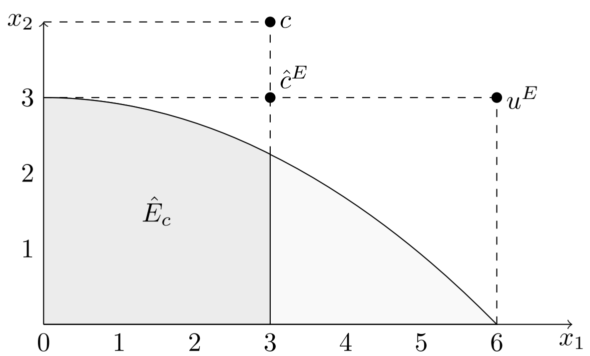
$$u^E = (\max\{x_i \mid x \in E\})_{i \in N}.$$

The vector of *truncated claims* $\hat{c}^E \in \mathbb{R}^N_+$ is given by

$$\hat{c}^E = \left(\min\{c_i, u^E_i\}\right)_{i \in N}.$$

Note that $u^{\hat{E}_c} = \hat{c}^E$ and $\hat{E}_c = \hat{E}_{\hat{c}^E}$.

Example 1
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3, 4)$. Then $N^c_+ = \{1, 2\}$, $\hat{E}_c = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36, x_1 \leq 3\}$, $u^E = (6, 3)$, and $\hat{c}^E = (3, 3)$. This is illustrated as follows.



△

A *bankruptcy rule* $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ assigns to any $(E, c) \in \mathrm{BR}^N$ a payoff allocation $f(E, c) \in \mathrm{WP}(E)$ for which $f(E, c) \leq c$.[2]

[1] Chun and Thomson (1992) studied a different domain in which the estate is convex but not necessarily nonleveled, and individual claims cannot exceed the corresponding utopia values.

[2] Alternatively, we could describe these conditions as the axioms *efficiency* and *claims boundedness*. However, we follow the standard bankruptcy approach in which these conditions are directly included in the definition of a rule.

A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies

– *relative symmetry* if $\frac{f_i(E,c)}{u_i^E} = \frac{f_j(E,c)}{u_j^E}$ for all $(E, c) \in \mathrm{BR}^N$ for which $E \neq \{0_N\}$ and any $i, j \in N$ for which $\frac{c_i}{u_i^E} = \frac{c_j}{u_j^E}$;

– *truncation invariance* if $f(E, c) = f(E, \hat{c}^E)$ for all $(E, c) \in \mathrm{BR}^N$.

Relative symmetry imposes a relative equal treatment of claimants with relative equal claims, i.e. equal proportions of their claims with respect to their utopia values. Truncation invariance states that it is not relevant to claim more than your utopia value, supported by the fact that claimants are not allocated more than their utopia values in any feasible estate allocation. The following three bankruptcy rules all satisfy relative symmetry.

The *proportional rule* $\mathrm{Prop} : \mathrm{BR}^N \to \mathbb{R}^N_+$ (cf. Dietzenbacher et al. 2016) assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathrm{Prop}(E, c) = \lambda^{E,c} c,$$

where $\lambda^{E,c} = \max\{t \in [0, 1] \mid tc \in E\}$. The proportional rule satisfies relative symmetry, but does not satisfy truncation invariance.

The *truncated proportional rule* $\mathrm{TProp} : \mathrm{BR}^N \to \mathbb{R}^N_+$ (cf. Dietzenbacher et al. 2017) assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathrm{TProp}(E, c) = \mathrm{Prop}(E, \hat{c}^E).$$

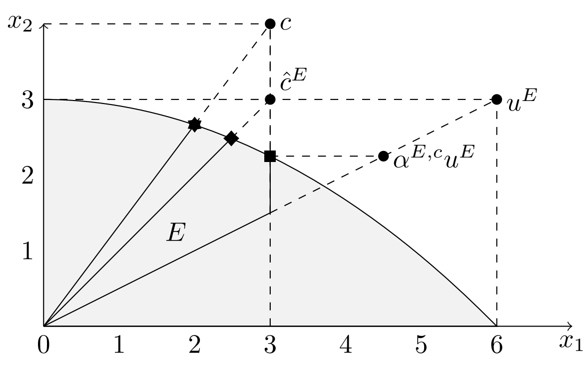
The truncated proportional rule satisfies both relative symmetry and truncation invariance.

The *constrained relative equal awards rule* $\mathrm{CREA} : \mathrm{BR}^N \to \mathbb{R}^N_+$ (cf. Dietzenbacher et al. 2016) assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathrm{CREA}(E, c) = \left(\min\{c_i, \alpha^{E,c} u_i^E\}\right)_{i \in N},$$

where $\alpha^{E,c} = \max\{t \in [0, 1] \mid (\min\{c_i, t u_i^E\})_{i \in N} \in E\}$. The constrained relative equal awards rule satisfies both relative symmetry and truncation invariance.

Example 2
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3, 4)$ as in Example 1. Then $\mathrm{Prop}(E, c) = (2, 2\frac{2}{3})$ (✸), $\mathrm{TProp}(E, c) = (6\sqrt{2} - 6, 6\sqrt{2} - 6)$ (◆), and $\mathrm{CREA}(E, c) = (3, 2\frac{1}{4})$ (■). This is illustrated as follows.



△

3 Estate Axioms

In this section, we introduce and study the implications of axioms concerning changes in the estate. Starting from the well-known independence of irrelevant alternatives axiom introduced by Nash (1950), several axioms concerning changes in the feasible set of bargaining problems have been proposed in the literature. Kalai (1977) introduced a strong monotonicity axiom and the axiom of step-by-step negotiations, which were further studied by Roth (1979). Thomson and Myerson (1980) introduced the axioms domination and independence of undominating alternatives. Peters (2010) introduced the independence of nonindividually rational outcomes axiom to describe solutions for bargaining problems which only depend on the rational payoff allocations within the feasible set.

As exploited by Roth (1977) for the independence of irrelevant alternatives axiom, in the formulation of these properties the disagreement point is required to be fixed. We translate these properties to the domain of bankruptcy problems with nontransferable utility in such a way that the vector of claims is required to be fixed.

Definition 3.1 (Estate Axioms)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies

- *step-by-step negotiations* if $f(E,c) = f(E',c) + f((E - \{f(E',c)\})_+, c - f(E',c))$[3] for all $(E,c),(E',c) \in \mathrm{BR}^N$ for which $E' \subseteq E$;
- *estate monotonicity* if $f(E,c) \geq f(E',c)$ for all $(E,c),(E',c) \in \mathrm{BR}^N$ for which $E' \subseteq E$;
- *domination* if $f(E,c) \leq f(E',c)$ or $f(E,c) \geq f(E',c)$ for all $(E,c),(E',c) \in \mathrm{BR}^N$;
- *independence of irrelevant alternatives* if $f(E,c) = f(E',c)$ for all $(E,c),(E',c) \in \mathrm{BR}^N$ for which $E' \subseteq E$ and $f(E,c) \in \mathrm{WP}(E')$;
- *independence of undominating alternatives* if $f(E,c) = f(E',c)$ for all $(E,c),(E',c) \in \mathrm{BR}^N$ for which $E' \subseteq E$ and $f(E',c) \in \mathrm{WP}(E)$;
- *independence of unclaimed alternatives* if $f(E,c) = f(E',c)$ for all $(E,c),(E',c) \in \mathrm{BR}^N$ for which $\hat{E}_c = \hat{E}'_c$.

Suppose that the estate of a bankruptcy problem turns out to be larger than initially thought. Then the step-by-step negotiations axiom says that solving the larger problem by cancelling the initial allocation leads to the same payoffs as solving the larger problem from the initial allocation onwards. Estate monotonicity requires that no claimant can be worse off if the estate turns out to be larger. Domination implies that any two estates are comparable in terms of their corresponding solutions. Independence of irrelevant alternatives states that the solution should not change when the estate becomes smaller while the solution remains feasible. Independence of undominating alternatives states that the solution should not change when the estate becomes larger while none of the new payoff allocations is better for all claimants. The independence of unclaimed alternatives, also invoked by Chun and Thomson (1992), describes bankruptcy rules which only depend on the feasible payoff allocations within the estate.

Note that the proportional rule satisfies all these properties. The following lemma studies the relations between the estate axioms. Some of these relations bear some similarities with the results of Thomson and Myerson (1980). The proof is provided in the appendix.

[3] Here, $(E - \{f(E',c)\})_+ = \{x \in \mathbb{R}^N_+ \mid x + f(E',c) \in E\}$.

Lemma 3.1
Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule.

(i) If f satisfies step-by-step negotiations, then f satisfies estate monotonicity.

(ii) Then f satisfies estate monotonicity if and only if f satisfies domination.

(iii) If f satisfies estate monotonicity, then f satisfies independence of irrelevant alternatives.

(iv) If f satisfies estate monotonicity, then f satisfies independence of undominating alternatives.

(v) If f satisfies independence of irrelevant alternatives, then f satisfies independence of unclaimed alternatives.

(vi) If f satisfies independence of undominating alternatives, then f satisfies independence of unclaimed alternatives.

As shown by the following two rules, the axioms independence of irrelevant alternatives and independence of undominating alternatives are independent.

The bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ which assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$f(E,c) = \begin{cases} (\hat{c}^E_1, \max\{x \mid (\hat{c}^E_1, x) \in E\}) & \text{if } N = \{1,2\} \text{ and } \hat{c}^E_1 \geq \hat{c}^E_2; \\ (\max\{x \mid (x, \hat{c}^E_2) \in E\}, \hat{c}^E_2) & \text{if } N = \{1,2\} \text{ and } \hat{c}^E_1 < \hat{c}^E_2; \\ \mathrm{Prop}(E,c) & \text{otherwise} \end{cases}$$

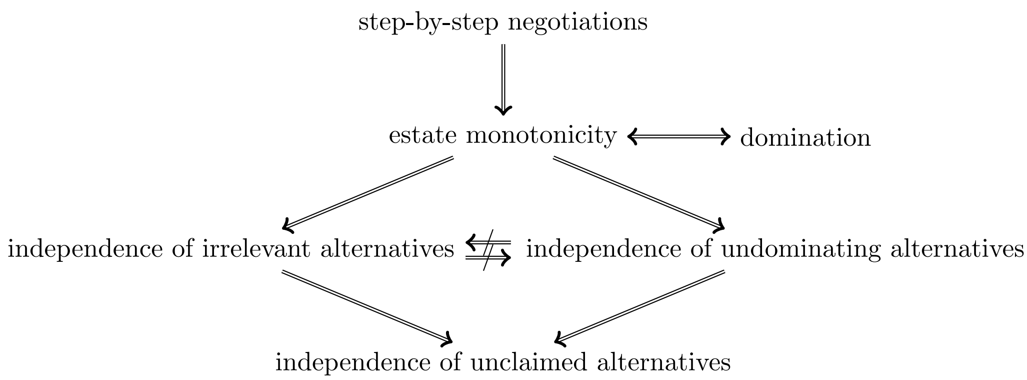
satisfies independence of irrelevant alternatives, but does not satisfy independence of undominating alternatives.

The bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ which assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$f(E,c) = \begin{cases} (\hat{c}^E_1, \max\{x \mid (\hat{c}^E_1, x) \in E\}) & \text{if } N = \{1,2\} \text{ and } \hat{c}^E_1 \leq \hat{c}^E_2; \\ (\max\{x \mid (x, \hat{c}^E_2) \in E\}, \hat{c}^E_2) & \text{if } N = \{1,2\} \text{ and } \hat{c}^E_1 > \hat{c}^E_2; \\ \mathrm{Prop}(E,c) & \text{otherwise} \end{cases}$$

satisfies independence of undominating alternatives, but does not satisfy independence of irrelevant alternatives.

The relations of all estate axioms can be summarized by the following diagram.



The axioms independence of irrelevant alternatives and independence of undominating alternatives are independent. However, if relative symmetry is required, then the two properties become equivalent and are only satisfied by the proportional rule.

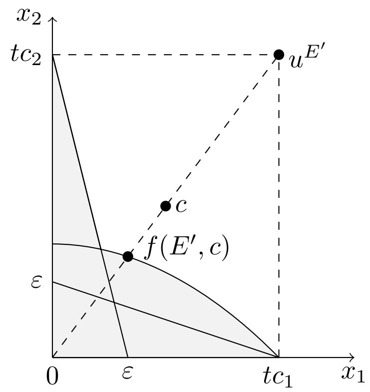
Theorem 3.2
The proportional rule is the unique bankruptcy rule satisfying relative symmetry and independence of irrelevant alternatives.

Proof. The proportional rule satisfies relative symmetry. By Lemma 3.1 and Lemma A.1[4], or by direct inspection, the proportional rule satisfies independence of irrelevant alternatives. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying relative symmetry and independence of irrelevant alternatives. Let $(E, c) \in \mathrm{BR}^N$. If $E = \{0_N\}$, then $f(E, c) = 0_N = \mathrm{Prop}(E, c)$. Suppose that $E \neq \{0_N\}$. Then $u^E \in \mathbb{R}^N_{++}$ and $N^c_+ \neq \emptyset$. Denote

$$t = \max_{i \in N^c_+} \left\{ \frac{u^E_i}{c_i} \right\} \text{ and } \varepsilon = \min_{i \in N^c_+} \{\mathrm{Prop}_i(E, c)\}.$$

Define

$$E' = \bigcup_{i \in N^c_+} \mathrm{comp}\left(\mathrm{conv}\left(\left\{\left(tc_i, 0_{N\setminus\{i\}}\right)\right\} \cup \left\{\left(\varepsilon, 0_{N\setminus\{j\}}\right) \mid j \in N \setminus \{i\}\right\}\right)\right) \cup E.$$



Then $(E', c) \in \mathrm{BR}^N$ and $E \subseteq E'$. Moreover, $u^{E'}_{N^c_+} = tc_{N^c_+}$ and $\lambda^{E',c} = \lambda^{E,c}$. We have $c_i u^{E'}_j = tc_i c_j = c_j u^{E'}_i$ for all $i, j \in N^c_+$. Since f satisfies relative symmetry, this means that $f(E', c) = \lambda^{E',c} c = \lambda^{E,c} c = \mathrm{Prop}(E, c)$. Since f satisfies independence of irrelevant alternatives, this implies that $f(E, c) = f(E', c) = \mathrm{Prop}(E, c)$. $\square$

In contrast to all further results, Theorem 3.2 exploits the fact that the estate of a bankruptcy problem is not required to be convex. All further results do not rely on the admission of nonconvex estates and can therefore be reformulated on the domain of bankruptcy problems with convex estate.

[4]Throughout this paper, we refer to the appendix for the derivations of properties satisfied by specific bankruptcy rules.

Theorem 3.3
The proportional rule is the unique bankruptcy rule satisfying relative symmetry and independence of undominating alternatives.

Proof. The proportional rule satisfies relative symmetry. By Lemma 3.1 and Lemma A.1, or by direct inspection, the proportional rule satisfies independence of undominating alternatives. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying relative symmetry and independence of undominating alternatives. Let $(E, c) \in \mathrm{BR}^N$. If $E = \{0_N\}$, then $f(E, c) = 0_N = \mathrm{Prop}(E, c)$. Suppose that $E \neq \{0_N\}$. Then $u^E \in \mathbb{R}^N_{++}$ and $N^c_+ \neq \emptyset$. If $|N^c_+| = 1$, then $f(E, c) = (u^E_{N^c_+}, 0_{N \setminus N^c_+}) = \mathrm{Prop}(E, c)$. Suppose that $|N^c_+| \geq 2$. Denote

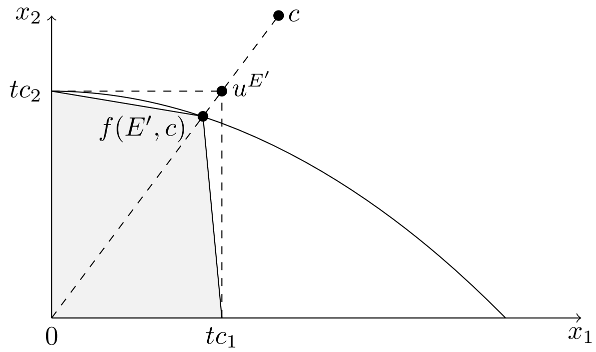
$$t = \min_{i \in N^c_+} \left\{ \frac{u^E_i}{c_i} \right\}.$$

Let $\varepsilon \in \mathbb{R}^N_{++}$ be defined by

$$\varepsilon_i = \begin{cases} \frac{1}{|N^c_+|-1} \left(tc_i - \mathrm{Prop}_i(E, c) \right) & \text{for all } i \in N^c_+; \\ u^E_i & \text{for all } i \in N \setminus N^c_+. \end{cases}$$

Define $E' = \mathrm{comp}(\mathrm{conv}(A_1 \cup A_2)) \cap E$, where

$$A_1 = \left\{ \left(\left(\mathrm{Prop}_i(E, c) + |N^c_+ \setminus S| \varepsilon_i \right)_{i \in S}, 0_{N \setminus S} \right) \;\middle|\; S \in 2^{N^c_+} \setminus \{\emptyset\} \right\}$$
$$\text{and } A_2 = \left\{ \left(\varepsilon_i, 0_{N \setminus \{i\}} \right) \;\middle|\; i \in N \setminus N^c_+ \right\}.$$



Then $(E', c) \in \mathrm{BR}^N$ and $E' \subseteq E$. Moreover, $u^{E'}_{N^c_+} = tc_{N^c_+}$ and $\lambda^{E', c} = \lambda^{E, c}$. We have $c_i u^{E'}_j = tc_i c_j = c_j u^{E'}_i$ for all $i, j \in N^c_+$. Since f satisfies relative symmetry, this means that $f(E', c) = \lambda^{E', c} c = \lambda^{E, c} c = \mathrm{Prop}(E, c)$. Since f satisfies independence of undominating alternatives, this implies that $f(E, c) = f(E', c) = \mathrm{Prop}(E, c)$. □

To show that relative symmetry is independent of any estate axiom, we introduce the constrained equal awards rule. The *constrained equal awards rule* $\mathrm{CEA} : \mathrm{BR}^N \to \mathbb{R}^N_+$ assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathrm{CEA}(E, c) = \left(\min\{c_i, a\} \right)_{i \in N},$$

where $a \in \mathbb{R}_+$ is such that $\mathrm{CEA}(E, c) \in \mathrm{WP}(E)$.

Where the constrained *relative* equal awards rule aims to allocate payoffs relatively equal among the claimants, the constrained equal awards rule aims to allocate payoffs absolutely equal among the claimants. The constrained relative equal awards rule satisfies relative symmetry, but does not satisfy independence of unclaimed alternatives. The constrained equal awards rule satisfies step-by-step negotiations, but does not satisfy relative symmetry. This is summarized in the following table.

	Prop	CREA	CEA
relative symmetry	$+$	$+$	$-$
step-by-step negotiations	$+$	$-$	$+$
estate monotonicity	$+$	$-$	$+$
domination	$+$	$-$	$+$
independence of irrelevant alternatives	$+$	$-$	$+$
independence of undominating alternatives	$+$	$-$	$+$
independence of unclaimed alternatives	$+$	$-$	$+$

This means that relative symmetry is independent of any estate axiom, which implies that the properties in an axiomatic characterization of the proportional rule remain independent if independence of irrelevant alternatives in Theorem 3.2 or independence of undominating alternatives in Theorem 3.3 is strengthened to domination, estate monotonicity, or step-by-step negotiations.

The proportional rule is not the unique bankruptcy rule satisfying relative symmetry and independence of unclaimed alternatives, since the truncated proportional rule also satisfies these two properties. Nevertheless, these two properties lead to the proportional rule for a large class of problems.

Lemma 3.4
Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule. If f satisfies relative symmetry and independence of unclaimed alternatives, then $f(E,c) = \mathrm{Prop}(E,c)$ for all $(E,c) \in \mathrm{BR}^N$ for which $c < u^E$.

Proof. Assume that f satisfies relative symmetry and independence of unclaimed alternatives. Let $(E,c) \in \mathrm{BR}^N$ be such that $c < u^E$. Then $u^E \in \mathbb{R}^N_{++}$ and $|N^c_+| \geq 2$. Denote

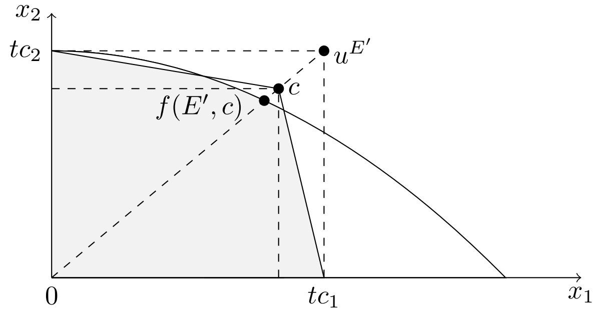
$$t = \min_{i \in N^c_+} \left\{ \frac{u^E_i}{c_i} \right\}.$$

Let $\varepsilon \in \mathbb{R}^N_{++}$ be defined by

$$\varepsilon_i = \begin{cases} \frac{t-1}{|N^c_+|-1} c_i & \text{for all } i \in N^c_+; \\ u^E_i & \text{for all } i \in N \setminus N^c_+. \end{cases}$$

Define $E' = \mathrm{comp}(\mathrm{conv}(A_1 \cup A_2)) \cap E$, where

$$A_1 = \left\{ \left(\left(c_i + |N^c_+ \setminus S| \varepsilon_i \right)_{i \in S}, 0_{N \setminus S} \right) \;\middle|\; S \in 2^{N^c_+} \setminus \{\emptyset\} \right\}$$
$$\text{and } A_2 = \left\{ \left(\varepsilon_i, 0_{N \setminus \{i\}} \right) \;\middle|\; i \in N \setminus N^c_+ \right\}.$$



Then $(E',c) \in \mathrm{BR}^N$ and $\hat{E}_c = \hat{E}'_c$. Moreover, $u^{E'}_{N^c_+} = tc_{N^c_+}$ and $\lambda^{E',c} = \lambda^{E,c}$. We have $c_i u^{E'}_j = tc_i c_j = c_j u^{E'}_i$ for all $i, j \in N^c_+$. Since f satisfies relative symmetry, this means that $f(E',c) = \lambda^{E',c} c = \lambda^{E,c} c = \mathrm{Prop}(E,c)$. Since f satisfies independence of unclaimed alternatives, this implies that $f(E,c) = f(E',c) = \mathrm{Prop}(E,c)$. □

If we combine independence of unclaimed alternatives with the bankruptcy axioms relative symmetry and truncation invariance, and the weak technical requirement claims continuity, we derive an axiomatic characterization of the truncated proportional rule by using Lemma 3.4.

Definition 3.2 (Claims Continuity)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies *claims continuity* if $f(E,c)$ is continuous in c for all $(E,c) \in \mathrm{BR}^N$.

Theorem 3.5
The truncated proportional rule is the unique bankruptcy rule satisfying relative symmetry, truncation invariance, independence of unclaimed alternatives, and claims continuity.

Proof. The truncated proportional rule satisfies relative symmetry and truncation invariance. By Lemma A.2 and Lemma A.3, the truncated proportional rule satisfies independence of unclaimed alternatives and claims continuity. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying relative symmetry, truncation invariance, independence of unclaimed alternatives, and claims continuity. Let $(E,c) \in \mathrm{BR}^N$. If $E = \{0_N\}$, then $f(E,c) = 0_N = \mathrm{TProp}(E,c)$. Suppose that $E \neq \{0_N\}$. Then $u^E \in \mathbb{R}^N_{++}$ and $N^c_+ \neq \emptyset$. If $|N^c_+| = 1$, then $f(E,c) = (u^E_{N^c_+}, 0_{N \setminus N^c_+}) = \mathrm{TProp}(E,c)$. Suppose that $|N^c_+| \geq 2$. Let $\{x_k\}_{k \in \mathbb{N}}$ be a sequence in $\mathrm{WUC}(E)$ defined by $x_k = \frac{1}{k}\mathrm{Prop}(E, \hat{c}^E) + (1 - \frac{1}{k})\hat{c}^E$ for all $k \in \mathbb{N}$. Then $x_k < u^E$ for all $k \in \mathbb{N}$ and $\lim_{k \to \infty} x_k = \hat{c}^E$. Since f satisfies relative symmetry and independence of unclaimed alternatives, Lemma 3.4 implies that $f(E, x_k) = \mathrm{Prop}(E, x_k) = \mathrm{Prop}(E, \hat{c}^E) = \mathrm{TProp}(E,c)$ for all $k \in \mathbb{N}$. Since f satisfies claims continuity, this means that $f(E, \hat{c}^E) = \lim_{k \to \infty} f(E, x_k) = \mathrm{TProp}(E,c)$. Since f satisfies truncation invariance, this implies that $f(E,c) = f(E, \hat{c}^E) = \mathrm{TProp}(E,c)$. □

To show that the properties in Theorem 3.5 are independent, we introduce the restricted truncated proportional rule. The *restricted truncated proportional rule* $\mathrm{RTProp} : \mathrm{BR}^N \to \mathbb{R}^N_+$ assigns to any $(E,c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathrm{RTProp}(E,c) = \begin{cases} \mathrm{Prop}(E,c) & \text{if } c < u^E; \\ (tu^E_S, 0_{N \setminus S}) & \text{otherwise,} \end{cases}$$

where $S = \{i \in N \mid c_i \geq u^E_i\}$ and $t \in [0,1]$ is such that $\mathrm{RTProp}(E,c) \in \mathrm{WP}(E)$.

The restricted truncated proportional rule satisfies relative symmetry, truncation invariance, and independence of unclaimed alternatives, but does not satisfy claims continuity. The constrained relative equal awards rule satisfies relative symmetry, truncation invariance, and claims continuity, but does not satisfy independence of unclaimed alternatives. The proportional rule satisfies relative symmetry, independence of unclaimed alternatives, and claims continuity, but does not satisfy truncation invariance. The constrained equal awards rule satisfies truncation invariance, independence of unclaimed alternatives, and claims continuity, but does not satisfy relative symmetry. This is summarized in the following table.

	TProp	RTProp	CREA	Prop	CEA
relative symmetry	+	+	+	+	−
truncation invariance	+	+	+	−	+
independence of unclaimed alternatives	+	+	−	+	+
claims continuity	+	−	+	+	+

This means that the properties in Theorem 3.5 are independent.

4 Claims Axioms

In this section, we introduce and study the implications of axioms concerning changes in the claims. Several axioms concerning changes in the disagreement point of bargaining problems have been proposed in the literature. We translate these properties to the domain of bankruptcy problems with nontransferable utility in such a way that they concern similar changes in the vector of claims while the estate is required to be fixed.

Definition 4.1 (Claims Axioms)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies

- *claims linearity* if $f(E,c) = f(E, \theta c + (1-\theta)c')$ for all $(E,c), (E,c') \in \mathrm{BR}^N$ for which $f(E,c) = f(E,c')$ and any $\theta \in \mathbb{R}$ for which $(E, \theta c + (1-\theta)c') \in \mathrm{BR}^N$;
- *weak claims linearity* if $f(E,c) = f(E, \theta c + (1-\theta) f(E,c))$ for all $(E,c) \in \mathrm{BR}^N$ and any $\theta \in \mathbb{R}_+$;
- *claims convexity* if $f(E,c) = f(E, \theta c + (1-\theta)c')$ for all $(E,c), (E,c') \in \mathrm{BR}^N$ for which $f(E,c) = f(E,c')$ and any $\theta \in [0,1]$;
- *weak claims convexity* if $f(E,c) = f(E, \theta c + (1-\theta) f(E,c))$ for all $(E,c) \in \mathrm{BR}^N$ and any $\theta \in [0,1]$.

Note that the proportional rule satisfies all these properties. The claims linearity axiom describes bankruptcy rules for which all claim vectors on the line connecting two claim vectors with equal outcomes lead to the same payoff allocation. The claims convexity axiom is based on a bargaining axiom of Livne (1988) and Chun and Thomson (1990). If there is uncertainty about which of the two claim vectors with equal outcomes applies, then any expected value leads to the same payoff allocation. The corresponding weaker axioms of claims linearity and claims convexity, which only require that linear or convex combinations of the claim vector and its outcome lead to the same payoff allocation, are based on bargaining axioms of Peters and Van Damme (1991) and Peters (2010).

The following lemma presents the relations between the claims axioms.

Lemma 4.1
Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ *be a bankruptcy rule.*

(i) If f *satisfies claims linearity, then* f *satisfies claims convexity.*

(ii) If f *satisfies claims linearity, then* f *satisfies weak claims linearity.*

(iii) If f *satisfies claims convexity, then* f *satisfies weak claims convexity.*

(iv) If f *satisfies weak claims linearity, then* f *satisfies weak claims convexity.*

As shown by the following two rules, the axioms claims convexity and weak claims linearity are independent.

The *restricted constrained relative equal awards rule* $\mathrm{RCREA} : \mathrm{BR}^N \to \mathbb{R}^N_+$, which assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathrm{RCREA}(E,c) = \begin{cases} \mathrm{CREA}(E,c) & \text{if } c < u^E \text{ or } c \geq \lambda^{E,u^E} u^E; \\ (tu^E_S, 0_{N \setminus S}) & \text{otherwise,} \end{cases}$$

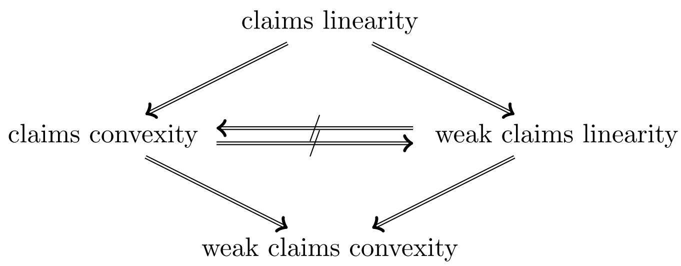
where $S = \{i \in N \mid c_i \geq u^E_i\}$ and $t \in [0,1]$ is such that $\mathrm{RCREA}(E,c) \in \mathrm{WP}(E)$, satisfies claims convexity, but does not satisfy weak claims linearity.

The bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ which assigns to any $(E,c) \in \mathrm{BR}^N$ the payoff allocation

$$f(E,c) = \begin{cases} (\hat{c}^E_1, \min\{c_2, tu^E_2\}, \min\{c_3, tu^E_3\}) & \text{if } N = \{1,2,3\} \text{ and } c_2 u^E_3 = c_3 u^E_2; \\ \mathrm{CREA}(E,c) & \text{otherwise,} \end{cases}$$

where $t \in [0,1]$ is such that $f(E,c) \in \mathrm{WP}(E)$, satisfies weak claims linearity, but does not satisfy claims convexity.

The relations of all claims axioms can be summarized by the following diagram.



The constrained relative equal awards rule is not the unique rule satisfying relative symmetry, truncation invariance, and claims convexity, since the restricted constrained relative equal awards rule also satisfies these three properties. However, the constrained relative equal awards rule is the only bankruptcy rule satisfying relative symmetry, truncation invariance, and weak claims linearity.

Theorem 4.2
The constrained relative equal awards rule is the unique bankruptcy rule satisfying relative symmetry, truncation invariance, and weak claims linearity.

Proof. The constrained relative equal awards rule satisfies relative symmetry and truncation invariance. By Lemma A.4, the constrained relative equal awards rule satisfies weak claims linearity. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying relative symmetry, truncation invariance, and weak claims linearity. Let $(E,c) \in \mathrm{BR}^N$. If $c \in E$, then $f(E,c) = c = \mathrm{CREA}(E,c)$. Suppose that $c \notin E$. Denote $S = \{i \in N \mid f_i(E,c) < c_i\}$. Then $S \neq \emptyset$. Let $x \in \mathbb{R}^N_+$ with $x = \theta c + (1-\theta)f(E,c)$ for some $\theta \in \mathbb{R}_+$ be such that $x_S \geq u^E_S$. Then $\hat{x}^E_S = (\min\{x_i, u^E_i\})_{i \in S} = u^E_S$ and $\hat{x}^E_i u^E_j = u^E_i u^E_j = \hat{x}^E_j u^E_i$ for all $i,j \in S$. Since f satisfies relative symmetry, this means that $f_S(E, \hat{x}^E) = t u^E_S$ for some $t \in [0,1]$. Since f satisfies truncation invariance, this implies that $f_S(E,x) = f_S(E,\hat{x}^E) = t u^E_S$. Since f satisfies weak claims linearity, $f_S(E,c) = f_S(E,x) = t u^E_S$. Then $f_S(E,c) \leq \alpha^{E,c} u^E_S$, since otherwise $f(E,c) \geq \mathrm{CREA}(E,c)$ and $f(E,c) \neq \mathrm{CREA}(E,c)$, which contradicts that E is nonleveled.

Suppose that there exists an $i \in N \setminus S$ such that $f_i(E,c) > \alpha^{E,c} u^E_i$. Then $f_j(E,c) u^E_i \leq \alpha^{E,c} u^E_j u^E_i < f_i(E,c) u^E_j$ for all $j \in S$. Let $y \in \mathbb{R}^N_+$ be such that $y = \theta c + (1-\theta) f(E,c)$ for some $\theta \in \mathbb{R}_{++}$ and $y_j u^E_i = f_i(E,c) u^E_j$ for some $j \in S$. Then $y_i u^E_j = f_i(E,c) u^E_j = y_j u^E_i$. Since f satisfies relative symmetry, this means that $f_i(E,y) u^E_j = f_j(E,y) u^E_i$. Since f satisfies weak claims linearity, this implies that $f_i(E,c) u^E_j = f_j(E,c) u^E_i$. This is a contradiction. Hence, $f_i(E,c) \leq \min\{c_i, \alpha^{E,c} u^E_i\} = \mathrm{CREA}_i(E,c)$ for all $i \in N$. Since E is nonleveled, this implies that $f(E,c) = \mathrm{CREA}(E,c)$. □

The truncated proportional rule satisfies relative symmetry and truncation invariance, but does not satisfy weak claims linearity. The proportional rule satisfies relative symmetry and weak claims linearity, but does not satisfy truncation invariance. The constrained equal awards rule satisfies truncation invariance and weak claims linearity, but does not satisfy relative symmetry. This is summarized in the following table.

	CREA	TProp	Prop	CEA
relative symmetry	+	+	+	−
truncation invariance	+	+	−	+
weak claims linearity	+	−	+	+

This means that the properties in Theorem 4.2 are independent.

The axioms concerning changes in the claims can also be combined with the axioms concerning changes in the estate. The proportional rule is not the unique bankruptcy rule satisfying relative symmetry and independence of unclaimed alternatives, since the truncated proportional rule also satisfies these two properties. However, if weak claims linearity is required in addition, then these properties are only satisfied by the proportional rule.

Theorem 4.3
The proportional rule is the unique bankruptcy rule satisfying relative symmetry, independence of unclaimed alternatives, and weak claims linearity.

Proof. The proportional rule satisfies relative symmetry. By Lemma 3.1, Lemma 4.1, Lemma A.1, and Lemma A.5, the proportional rule satisfies independence of unclaimed alternatives and weak claims linearity. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying relative symmetry, independence of unclaimed alternatives, and weak claims linearity. Let $(E,c) \in \mathrm{BR}^N$. If $E = \{0_N\}$, then $f(E,c) = 0_N = \mathrm{Prop}(E,c)$. Suppose that $E \neq \{0_N\}$.

Then $u^E \in \mathbb{R}^N_{++}$ and $N^c_+ \neq \emptyset$. If $|N^c_+| = 1$, then $f(E,c) = (u^E_{N^c_+}, 0_{N\setminus N^c_+}) = \text{Prop}(E,c)$. Suppose that $|N^c_+| \geq 2$. Let $x \in \mathbb{R}^N_+$ with $x = \theta c + (1-\theta)\text{Prop}(E,c)$ for some $\theta \in (0,1]$ be such that $x < u^E$. Since f satisfies relative symmetry and independence of unclaimed alternatives, Lemma 3.4 implies that $f(E,x) = \text{Prop}(E,x) = \text{Prop}(E,c)$. Since f satisfies weak claims linearity, this implies that $f(E,c) = f(E, \frac{1}{\theta}x + (1-\frac{1}{\theta})f(E,x)) = f(E,x) = \text{Prop}(E,c)$. □

To show that relative symmetry and independence of unclaimed alternatives are independent of any claims axiom, we introduce two classes of bankruptcy rules.

First, let Ψ denote the class of all continuous functions $\psi : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ for which

- $\lim_{x_1 \to 0} \psi(x_1, x_2) = \infty$ for all $x_2 \in \mathbb{R}_{++}$;
- $\lim_{x_2 \to 0} \psi(x_1, x_2) = 0$ for all $x_1 \in \mathbb{R}_{++}$;
- $\psi(x) \geq \psi(y)$ for all $x, y \in \mathbb{R}^2_{++}$ for which $x_1 < y_1$ and $x_2 > y_2$.

For any $\psi \in \Psi$ and any $(E,c) \in \text{BR}^N$ for which $N = N^c_+ = \{1,2\}$, $E \neq \{0_N\}$, and $c \notin E$, let $\xi \in \text{WP}(E)$ be defined such that $\psi(\xi) = \frac{c_2 - \xi_2}{c_1 - \xi_1}$. Note that ξ exists and is uniquely defined.

For any $\psi \in \Psi$ for which $\psi(\frac{1}{2}, \frac{1}{2}) = 1$, the bankruptcy rule $f^\psi_1 : \text{BR}^N \to \mathbb{R}^N_+$ assigns to any $(E,c) \in \text{BR}^N$ the payoff allocation

$$f^\psi_1(E,c) = \begin{cases} \xi & \text{if } N = N^c_+ = \{1,2\},\ E = \{x \in \mathbb{R}^N_+ \mid x_1 + x_2 \leq 1\}, \text{ and } c \notin E; \\ \text{Prop}(E,c) & \text{otherwise.} \end{cases}$$

Note that $f^\psi_1 = \text{Prop}$ if and only if $\psi(x) = \frac{x_2}{x_1}$ for all $x \in \mathbb{R}^2_{++}$.

For any $\psi \in \Psi$, the bankruptcy rule $f^\psi_2 : \text{BR}^N \to \mathbb{R}^N_+$ assigns to any $(E,c) \in \text{BR}^N$ the payoff allocation

$$f^\psi_2(E,c) = \begin{cases} \xi & \text{if } N = N^c_+ = \{1,2\},\ E \neq \{0_N\}, \text{ and } c \notin E; \\ \text{Prop}(E,c) & \text{otherwise.} \end{cases}$$

Note that $f^\psi_2 = \text{Prop}$ if and only if $\psi(x) = \frac{x_2}{x_1}$ for all $x \in \mathbb{R}^2_{++}$.

The truncated proportional rule satisfies relative symmetry and independence of unclaimed alternatives, but does not satisfy weak claims convexity. Any bankruptcy rule $f^\psi_1 \neq \text{Prop}$ for which $\psi \in \Psi$ and $\psi(\frac{1}{2}, \frac{1}{2}) = 1$ satisfies relative symmetry and claims linearity, but does not satisfy independence of unclaimed alternatives. Any bankruptcy rule $f^\psi_2 \neq \text{Prop}$ for which $\psi \in \Psi$ satisfies independence of unclaimed alternatives and claims linearity, but does not satisfy relative symmetry.

	Prop	TProp	$f^\psi_1 \neq$ Prop	$f^\psi_2 \neq$ Prop
relative symmetry	+	+	+	−
independence of unclaimed alternatives	+	+	−	+
claims linearity	+	−	+	+
weak claims linearity	+	−	+	+
claims convexity	+	−	+	+
weak claims convexity	+	−	+	+

This means that relative symmetry and independence of unclaimed alternatives are independent of any claims axiom. In particular, the properties in Theorem 4.3 are independent. Moreover, the properties in the axiomatic characterization of the proportional rule remain independent if weak claims linearity in Theorem 4.3 is strengthened to claims linearity.

The proportional rule is not the unique bankruptcy rule satisfying relative symmetry, independence of unclaimed alternatives, and claims convexity. The *restricted proportional rule* $\text{RProp} : \text{BR}^N \to \mathbb{R}^N_+$, which assigns to any $(E, c) \in \text{BR}^N$ the payoff allocation

$$\text{RProp}(E, c) = \begin{cases} \text{Prop}(E, c) & \text{if } c < u^E; \\ (tc_S, 0_{N \setminus S}) & \text{otherwise,} \end{cases}$$

where $S = \{i \in N \mid \forall_{j \in N} : c_j u^E_i \leq c_i u^E_j\}$ and $t \in [0, 1]$ is such that $\text{RProp}(E, c) \in \text{WP}(E)$, also satisfies relative symmetry, independence of unclaimed alternatives, and claims convexity. However, if positive claimants are required to get positive awards, then these properties are only satisfied by the proportional rule.

Definition 4.2 (Positive Awards)
A bankruptcy rule $f : \text{BR}^N \to \mathbb{R}^N_+$ satisfies *positive awards* if $f_{N^c_+}(E, c) > 0_{N^c_+}$ for all $(E, c) \in \text{BR}^N$ for which $E \neq \{0_N\}$.

Theorem 4.4
The proportional rule is the unique bankruptcy rule satisfying relative symmetry, independence of unclaimed alternatives, weak claims convexity, and positive awards.

Proof. The proportional rule satisfies relative symmetry. By Lemma 3.1, Lemma 4.1, Lemma A.1, Lemma A.5, and Lemma A.6, the proportional rule satisfies independence of unclaimed alternatives, weak claims convexity, and positive awards. Let $f : \text{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying relative symmetry, independence of unclaimed alternatives, weak claims convexity, and positive awards. Let $(E, c) \in \text{BR}^N$. If $E = \{0_N\}$, then $f(E, c) = 0_N = \text{Prop}(E, c)$. Suppose that $E \neq \{0_N\}$. Then $u^E \in \mathbb{R}^N_{++}$ and $N^c_+ \neq \emptyset$. If $|N^c_+| = 1$, then $f(E, c) = (u^E_{N^c_+}, 0_{N \setminus N^c_+}) = \text{Prop}(E, c)$. Suppose that $|N^c_+| \geq 2$. Since f satisfies positive awards, there exists an $x \in \mathbb{R}^N_+$ with $x = \theta c + (1 - \theta) f(E, c)$ for some $\theta \in (0, 1]$ such that $x < u^E$. Since f satisfies relative symmetry and independence of unclaimed alternatives, Lemma 3.4 implies that $f(E, x) = \text{Prop}(E, x)$. Since f satisfies weak claims convexity, this implies that $f(E, c) = f(E, x) = \text{Prop}(E, x) = \text{Prop}(E, c)$. $\square$

The restricted proportional rule satisfies relative symmetry, independence of unclaimed alternatives, and claims convexity, but does not satisfy positive awards. The truncated proportional rule satisfies relative symmetry, independence of unclaimed alternatives, and positive awards, but does not satisfy weak claims convexity. The constrained relative equal awards rule satisfies relative symmetry, claims convexity, and positive awards, but does not satisfy independence of unclaimed alternatives. The constrained equal awards rule satisfies independence of unclaimed alternatives, claims convexity, and positive awards, but does not satisfy relative symmetry. This is summarized in the following table.

	Prop	RProp	TProp	CREA	CEA
relative symmetry	+	+	+	+	−
independence of unclaimed alternatives	+	+	+	−	+
claims convexity	+	+	−	+	+
weak claims convexity	+	+	−	+	+
positive awards	+	−	+	+	+

This means that the properties in Theorem 4.4 are independent. Moreover, the properties in the axiomatic characterization of the proportional rule remain independent if weak claims convexity in Theorem 4.4 is strengthened to claims convexity.

5 Concluding Remarks

In this paper, we derived new axiomatic characterizations of the proportional rule, the truncated proportional rule, and the constrained relative equal awards rule for bankruptcy problems with nontransferable utility using axioms from bargaining theory. An overview of the corresponding properties, including the bankruptcy axioms relative symmetry and truncation invariance, the axioms concerning changes in the estate, the axioms concerning changes in the claims, and the weak technical requirements claims continuity and positive awards, is provided in the following table. The constrained equal awards rule is included for illustrative purposes.

	Prop	TProp	CREA	CEA
relative symmetry	+	+	+	−
truncation invariance	−	+	+	+
step-by-step negotiations	+	−	−	+
estate monotonicity	+	−	−	+
domination	+	−	−	+
independence of irrelevant alternatives	+	−	−	+
independence of undominating alternatives	+	−	−	+
independence of unclaimed alternatives	+	+	−	+
claims continuity	+	+	+	+
claims linearity	+	−	−	−
weak claims linearity	+	−	+	+
claims convexity	+	−	+	+
weak claims convexity	+	−	+	+
positive awards	+	+	+	+

The following table provides an overview of the axiomatic characterizations derived in this paper.

	Prop	Prop	Prop	Prop	TProp	CREA
relative symmetry	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
truncation invariance					$\star$	$\star$
indep. of irrelevant alternatives	$\star$					
indep. of undominating alternatives		$\star$				
indep. of unclaimed alternatives			$\star$	$\star$	$\star$	
claims continuity					$\star$	
weak claims linearity			$\star$			$\star$
weak claims convexity				$\star$		
positive awards				$\star$		

Alternatively, one could also interpret solutions for bargaining problems as new rules for bankruptcy problems, in line with the work of Dagan and Volij (1993) for bankruptcy problems with transferable utility. Future research allows to formalize this reverse approach in order to further connect bankruptcy problems with bargaining problems.

A bankruptcy rule based on the solution of Nash (1950) could maximize the product of the utility payoffs of all positive claimants over the truncated estate. On the domain of bankruptcy problems with convex estate, such a Nash bankruptcy rule bears some similarities with the constrained equal awards rule, since it satisfies truncation invariance, independence of irrelevant alternatives, and weak claims linearity, but does not satisfy relative symmetry.

The proportional rule for bankruptcy problems corresponds to a specific proportional solution for bargaining problems as studied by Kalai (1977) and Roth (1979).

Where the solution of Kalai and Smorodinsky (1975) let the utopia values of the rational payoff allocations determine an outcome direction, the solution of Kalai and Rosenthal (1978) let the utopia values of all payoff allocations within the feasible set determine an outcome direction for bargaining problems. For bankruptcy problems, this translates to the utopia values of the truncated estate and the utopia values of the estate, respectively. In this way, the solution of Kalai and Smorodinsky (1975) induces the truncated proportional rule and the solution of Kalai and Rosenthal (1978), when explicitly bounded by the claims, induces the constrained relative equal awards rule.

Other solutions for bargaining problems were introduced by Freimer and Yu (1976), which would prescribe the feasible payoff allocation with minimal distance to the vector of claims or the vector of utopia values in the context of bankruptcy problems. A similar solution is also studied by Mariotti and Villar (2005).

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Appendix

Lemma 3.1

Proof. (i) Assume that f satisfies step-by-step negotiations. Let $(E, c), (E', c) \in \mathrm{BR}^N$ be such that $E' \subseteq E$. Then $f(E, c) = f(E', c) + f((E - \{f(E', c)\})_+, c - f(E', c)) \geq f(E', c)$. Hence, f satisfies estate monotonicity.

(ii) Assume that f satisfies estate monotonicity. Let $(E, c), (E', c) \in \mathrm{BR}^N$. Suppose that $f(E, c) \in E'$. Then $f(E, c) \in \mathrm{WP}(E \cap E')$, $f(E \cap E', c) \leq f(E, c)$, and $f(E \cap E', c) \leq f(E', c)$. Since E is nonleveled, this implies that $f(E, c) = f(E \cap E', c) \leq f(E', c)$.

Now suppose that $f(E, c) \notin E'$. Since E is comprehensive, this means that $f(E, c) \in \mathrm{WP}(E \cup E')$, $f(E, c) \leq f(E \cup E', c)$, and $f(E', c) \leq f(E \cup E', c)$. Since E is nonleveled, this implies that $f(E, c) = f(E \cup E', c) \geq f(E', c)$. Hence, f satisfies domination.

Assume that f satisfies domination. Let $(E, c), (E', c) \in \mathrm{BR}^N$ be such that $E' \subseteq E$. Then $f(E, c) \leq f(E', c)$ or $f(E, c) \geq f(E', c)$. Since E is nonleveled, this implies that $f(E', c) \leq f(E, c)$. Hence, f satisfies estate monotonicity.

(iii) Assume that f satisfies estate monotonicity. Let $(E, c), (E', c) \in \mathrm{BR}^N$ be such that $E' \subseteq E$ and $f(E, c) \in \mathrm{WP}(E')$. Then $f(E, c) \geq f(E', c)$. Since E is nonleveled, this implies that $f(E, c) = f(E', c)$. Hence, f satisfies independence of irrelevant alternatives.

(iv) Assume that f satisfies estate monotonicity. Let $(E,c),(E',c) \in \mathrm{BR}^N$ be such that $E' \subseteq E$ and $f(E',c) \in \mathrm{WP}(E)$. Then $f(E,c) \geq f(E',c)$. Since E is nonleveled, this implies that $f(E,c) = f(E',c)$. Hence, f satisfies independence of undominating alternatives.

(v) Assume that f satisfies independence of irrelevant alternatives. Let $(E,c),(E',c) \in \mathrm{BR}^N$ be such that $\hat{E}_c = \hat{E}'_c$. Then $f(E,c), f(E',c) \in \mathrm{WP}(E \cap E')$. This implies that $f(E,c) = f(E \cap E', c) = f(E',c)$. Hence, f satisfies independence of unclaimed alternatives.

(vi) Assume that f satisfies independence of undominating alternatives. Let $(E,c) \in \mathrm{BR}^N$ and $(E',c) \in \mathrm{BR}^N$ be such that $\hat{E}_c = \hat{E}'_c$. Then $f(E,c), f(E',c) \in \mathrm{WP}(E \cup E')$. This implies that $f(E,c) = f(E \cup E', c) = f(E',c)$. Hence, f satisfies independence of unclaimed alternatives. □

Lemma A.1
The proportional rule satisfies step-by-step negotiations.

Proof. Let $(E,c),(E',c) \in \mathrm{BR}^N$ be such that $E' \subseteq E$. Then $\mathrm{Prop}(E,c) = \lambda^{E,c}c$ and

$$\begin{aligned}
&\mathrm{Prop}(E',c) + \mathrm{Prop}((E - \{\mathrm{Prop}(E',c)\})_+, c - \mathrm{Prop}(E',c)) \\
&= \lambda^{E',c}c + \mathrm{Prop}((E - \{\lambda^{E',c}c\})_+, c - \lambda^{E',c}c) \\
&= \lambda^{E',c}c + \lambda^{(E-\{\lambda^{E',c}c\})_+,(1-\lambda^{E',c})c}(1-\lambda^{E',c})c \\
&= \left(\lambda^{E',c} + \lambda^{(E-\{\lambda^{E',c}c\})_+,(1-\lambda^{E',c})c}(1-\lambda^{E',c})\right)c.
\end{aligned}$$

Moreover, $\mathrm{Prop}(E',c) + \mathrm{Prop}((E - \{\mathrm{Prop}(E',c)\})_+, c - \mathrm{Prop}(E',c)) \in \mathrm{WP}(E)$. Since E is nonleveled, this implies that $\mathrm{Prop}(E',c) + \mathrm{Prop}((E - \{\mathrm{Prop}(E',c)\})_+, c - \mathrm{Prop}(E',c)) = \mathrm{Prop}(E,c)$. Hence, the proportional rule satisfies step-by-step negotiations. □

Lemma A.2
The truncated proportional rule satisfies independence of unclaimed alternatives.

Proof. Let $(E,c),(E',c) \in \mathrm{BR}^N$ be such that $\hat{E}_c = \hat{E}'_c$. Then $\hat{E}_{\hat{c}^E} = \hat{E}_c = \hat{E}'_c = \hat{E}'_{\hat{c}^{E'}}$ and $\hat{c}^E = u^{\hat{E}_c} = u^{\hat{E}'_c} = \hat{c}^{E'}$. By Lemma 3.1 and Lemma A.1, the proportional rule satisfies independence of unclaimed alternatives. Then

$$\mathrm{TProp}(E,c) = \mathrm{Prop}(E,\hat{c}^E) = \mathrm{Prop}(E',\hat{c}^{E'}) = \mathrm{TProp}(E',c).$$

Hence, the truncated proportional rule satisfies independence of unclaimed alternatives. □

Lemma A.3
The truncated proportional rule satisfies claims continuity.

Proof. Let $(E,c) \in \mathrm{BR}^N$. Then $\lim_{x\to c} \hat{x}^E = \hat{c}^E$, $\lim_{x \to c} \lambda^{E,\hat{x}^E} = \lambda^{E,\hat{c}^E}$, and

$$\begin{aligned}
\lim_{x\to c} \mathrm{TProp}(E,x) &= \lim_{x\to c} \mathrm{Prop}(E,\hat{x}^E) = \lim_{x\to c} \lambda^{E,\hat{x}^E}\hat{x}^E = \lim_{x\to c}\lambda^{E,\hat{x}^E} \lim_{x\to c}\hat{x}^E \\
&= \lambda^{E,\hat{c}^E}\hat{c}^E = \mathrm{Prop}(E,\hat{c}^E) = \mathrm{TProp}(E,c).
\end{aligned}$$

Hence, the truncated proportional rule satisfies claims continuity. □

Lemma A.4
The constrained relative equal awards rule satisfies weak claims linearity.

Proof. Let $(E, c) \in \mathrm{BR}^N$ and let $\theta \in \mathbb{R}_+$. Suppose that $\alpha^{E,\theta c+(1-\theta)\mathrm{CREA}(E,c)} \geq \alpha^{E,c}$. For all $i \in N$,

$$\begin{aligned} &\mathrm{CREA}_i(E, \theta c + (1-\theta)\mathrm{CREA}(E, c)) \\ =\ &\min\{\theta c_i + (1-\theta)\mathrm{CREA}_i(E, c), \alpha^{E,\theta c+(1-\theta)\mathrm{CREA}(E,c)} u_i^E\} \\ =\ &\min\{\mathrm{CREA}_i(E, c) + \theta(c_i - \mathrm{CREA}_i(E, c)), \alpha^{E,\theta c+(1-\theta)\mathrm{CREA}(E,c)} u_i^E\} \\ \geq\ &\min\{\mathrm{CREA}_i(E, c), \alpha^{E,c} u_i^E\} \\ =\ &\mathrm{CREA}_i(E, c). \end{aligned}$$

Since E is nonleveled, this implies that $\mathrm{CREA}(E, c) = \mathrm{CREA}(E, \theta c + (1-\theta)\mathrm{CREA}(E, c))$. Now, suppose that $\alpha^{E,\theta c+(1-\theta)\mathrm{CREA}(E,c)} \leq \alpha^{E,c}$. For all $i \in N$,

$$\begin{aligned} &\mathrm{CREA}_i(E, \theta c + (1-\theta)\mathrm{CREA}(E, c)) \\ =\ &\min\{\theta c_i + (1-\theta)\mathrm{CREA}_i(E, c), \alpha^{E,\theta c+(1-\theta)\mathrm{CREA}(E,c)} u_i^E\} \\ =\ &\min\{\mathrm{CREA}_i(E, c) + \theta(c_i - \mathrm{CREA}_i(E, c)), \alpha^{E,\theta c+(1-\theta)\mathrm{CREA}(E,c)} u_i^E\} \\ \leq\ &\min\{\mathrm{CREA}_i(E, c) + \theta(c_i - \mathrm{CREA}_i(E, c)), \alpha^{E,c} u_i^E\} \\ =\ &\mathrm{CREA}_i(E, c). \end{aligned}$$

Since E is nonleveled, this implies that $\mathrm{CREA}(E, c) = \mathrm{CREA}(E, \theta c + (1-\theta)\mathrm{CREA}(E, c))$. Hence, the constrained relative equal awards rule satisfies weak claims linearity. □

Lemma A.5
The proportional rule satisfies claims linearity.

Proof. Let $(E, c), (E, c') \in \mathrm{BR}^N$ be such that $\mathrm{Prop}(E, c) = \mathrm{Prop}(E, c')$, and let $\theta \in \mathbb{R}$ be such that $(E, \theta c + (1-\theta)c') \in \mathrm{BR}^N$. If $E = \{0_N\}$, then $f(E, c) = 0_N = \mathrm{Prop}(E, \theta c + (1-\theta)c')$. Suppose that $E \neq \{0_N\}$. Then $\lambda^{E,c} c = \lambda^{E,c'} c'$ and

$$\mathrm{Prop}(E, \theta c + (1-\theta)c') = \lambda^{E,\theta c+(1-\theta)c'}(\theta c + (1-\theta)c') = \lambda^{E,\theta c+(1-\theta)c'} \left(\theta + (1-\theta)\frac{\lambda^{E,c}}{\lambda^{E,c'}}\right) c.$$

Since E is nonleveled, this implies that $\mathrm{Prop}(E, c) = \mathrm{Prop}(E, \theta c + (1-\theta)c')$. Hence, the proportional rule satisfies claims linearity. □

Lemma A.6
The proportional rule satisfies positive awards.

Proof. Let $(E, c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$ and let $i \in N_+^c$. Then $\mathrm{Prop}_i(E, c) = \lambda^{E,c} c_i > 0$. Hence, the proportional rule satisfies positive awards. □