Set and revealed preference axioms for multi-valued choice*

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March 2019

Abstract

We consider choice correspondences for arbitrary sets of alternatives, and focus on the condition of independence of irrelevant alternatives and on two weaker versions of it, as well as on the weak axiom of revealed preference for sets, in contrast to revealed preference for singletons. We establish the connection between the condition of independence of irrelevant alternatives and so-called strong sets.

Keywords Revealed preference axioms, multi-valued choice, independence of irrelevant alternatives
JEL-classification D01, D71

1 Introduction

1.1 Background

This paper contributes to a question with a long history. For (single-valued) choice functions the condition of Independence of Irrelevant Alternatives (IIA) requires that if the alternative chosen from a choice set is still available in a subset, then it should be chosen in that subset. This condition already occurs as condition no. 7 in Nash's axiomatic bargaining model (Nash, 1950). For choice functions, and collections of choice sets that are closed under nonempty intersection – as is also the case in the present paper – IIA is equivalent to the Weak Axiom of Revealed Preference. The latter condition excludes cycles of length two in the revealed preference relation. The questions we consider in this paper are, first, how to extend the IIA condition for choice functions to conditions for (multi-valued) choice correspondences and, second, how these extensions relate to revealed preference for sets, rather than singletons. We also relate the strongest of these extensions to the collection of so-called strong (choice) sets.

The perhaps most natural extension was again proposed by Nash in an informal note in 1950 (see Shubik, 1982, p. 420): if F is the choice correspondence, X is a choice set, and $F(X)$ has a nonempty intersection with a subset Y of X, then $F(Y)$ is equal to this

*The authors thank Bettina Klaus, Flip Klijn, Jordi Massó, Luis Pedro Santos-Pinto, and Dries Vermeulen for helpful suggestions.

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intersection. This condition also appears as Postulate 6 in Chernoff (1954) and Condition C4 in Arrow (1959). The most common and obvious interpretation is that the set of alternatives chosen by a choice correspondence should be viewed as the set of best alternatives (in some sense or another) among the available ones: consequently, each of these alternatives is also best in any subset of available alternatives to which it belongs. We refer to this extension simply as Independence of Irrelevant Alternatives (IIA).

A second extension is to require that, in this situation, $F(Y)$ be contained in the intersection of $F(X)$ and Y. In other words, F still chooses among the best elements, but not necessarily all available ones. Think of choosing a committee within a society: for a subset of the society one may need to choose a strictly smaller committee, even if more members of the original committee are still available. Or, in terms of a restaurant's menu choice, the lunch menu may be a subset of the dinner menu, but also lunch itself may be lighter than dinner: one may want to consume wine of just one brand instead of several, even if more brands are still available. This extension appears as condition W2 in Schwartz (1976). Clearly, it is a weakening of the first extension, and we call it Weak IIA (WIIA).

A third extension says the following. If Y is contained in X and contains $F(X)$, then $F(Y)$ is equal to $F(X)$. Its interpretation is clear: if we regard $F(X)$ as the set of best choices from X, and all these choices are still available in a smaller set Y, then they are also the best choices in Y. In terms of the restaurant's menu: if the collection of available dishes decreases but the courses that we chose earlier are still available, then we choose them again. Clearly, also this extension is a weakening of IIA, and we call it Restricted IIA (RIIA). As far as we are aware this extension is new.

For a discussion of other possible extensions the reader is referred to Section 1.3.1.

1.2 Our focus

We assume no structure on the set of alternatives – it can be any finite or infinite set, and we study IIA, WIIA, and RIIA choice correspondences with respect to two closely related questions.

The first question is that of rationalizability. This concerns the existence of a binary relation on the collection of choice sets, thus, sets of alternatives rather than only single alternatives, which rationalizes a given choice correspondence. The usual approach in the literature is to consider revealed preference relations on alternatives (singletons); the reader is referred to Section 1.3.2 for a brief overview of this literature. Here, we consider revealed preferences on sets, an approach also (and, as far as we know, only) followed in Brandt and Harrenstein (2011). Specifically, they consider a property called Condition $\hat{\alpha}$ and obtain a characterization (their Theorem 2) of 'set-rationalizable' choice correspondences. Their set-rationalizability condition is what we call WARP (Weak Axiom of Revealed Preference): it excludes cycles of length two in the revealed preference relation on choice sets. We show that WARP is equivalent to RIIA, and that RIIA is indeed equivalent to Condition $\hat{\alpha}$ – see Theorem 1 and Lemma 1. We further show that WARP is implied by WIIA if the choice correspondence F is a projection, i.e., $F \circ F = F$; the latter, in turn, is implied by WARP, as also observed in Brandt and Harrenstein (2011). In Theorem 3 we characterize IIA by WARP combined with another axiom on the revealed preference relation; this axiom is called 'Preference Axiom' (PA) and to the best of our knowledge is also new. We further show (Theorem 4) that IIA implies that the revealed preference relation on choice sets is transitive and (therefore, in view of WARP) acyclic.

The second question is that of identifying a collection of choice sets that determines a choice correspondence satisfying IIA. A choice set S is called a 'strong set' if $F(X)$ is equal to the intersection of X and S whenever $F(X)$ has a nonempty intersection with S. We show that if F satisfies IIA, then these strong sets partition the sets of alternatives and the restriction of the revealed preference relation to the strong sets is complete and acyclic (Theorem 5). In fact, the strong sets determine a unique IIA choice correspondence.

1.3 Further literature

1.3.1 Other IIA extensions

Apart from IIA, WIIA, and RIIA there are a few other possibilities to extend IIA from choice functions to choice correspondences.

A fourth possible extension says that if $Y \subseteq X$ and $F(X) \cap Y \neq \emptyset$, then $F(Y)$ should at least contain the intersection of $F(X)$ and Y. As an interpretation, it could be that additional best alternatives become available in the smaller set. For instance, a first preferred choice of wine from a restaurant's menu is no longer available, making a second preferred choice a best alternative (additional to the still available best menu choices). This condition was first proposed as Postulate 4 in Chernoff (1954), and has consequently been referred to as the Chernoff property (e.g., Moulin 1985, 1988). It also appears as Property α in Sen (1971).

Fifth, a still weaker version of the first three conditions is the following (e.g., Fishburn, 1973): if $F(X)$ is contained in Y, then $F(Y)$ should be contained in $F(X)$. This condition, studied in Aizerman and Malishevski (1981), is referred to as the Aizerman condition; it is implied by Condition W3 in Schwartz (1976).

1.3.2 Revealed preference

Following a tradition initiated for consumer theory by Samuelson (1938) and Houthakker (1950), and continued for general choice problems by – among others – Arrow (1959) and Richter (1966), most of the literature focuses on rationalizability: when does a choice correspondence pick the set of those alternatives that are maximal for some binary relation on the set of alternatives? Arrow (1959) shows that a choice correspondence is rationalizable by a complete and transitive binary relation if and only if it satisfies IIA. Sen (1971) shows that a choice correspondence is rationalizable by a binary relation if and only if it satisfies the Chernoff property and a condition proposed as Property γ but later also referred to as the Expansion condition (e.g., Moulin, 1985).[1] Adding to this the Aizerman condition results in the choice correspondence being rationalizable by an ordering which is complete and has a transitive strict part (Schwartz, 1976; Moulin, 1988). Finally, Aizerman and Malishevski (1981) show that a choice correspondence satisfies both the Chernoff property and the Aizerman condition if and only if it is pseudo-rationalizable by a collection of single-valued, complete, and transitive orderings; that is, if in each choice set, the choice correspondence picks the maximal elements of all the orderings in this collection.

1.4 Organization of the paper

In Section 2, we introduce the model and the main conditions on a choice correspondence that we consider (WARP, IIA, WIIA, and RIIA). Section 3 studies the relations between

[1]Property γ: for all choice sets X and Y, the intersection of $F(X)$ and $F(Y)$ is contained in $F(X \cup Y)$.

these properties. Section 4 introduces the collection of strong sets and studies its relation with IIA. Section 5 concludes with a summary of the main results of the paper and a few open questions.

2 Model and properties

Let A be a finite or infinite *set of alternatives* and let $\mathcal{A}$ denote the set of its non-empty subsets. A *choice correspondence* is a map $F : \mathcal{A} \to \mathcal{A}$ such that $F(X) \subseteq X$ for every $X \in \mathcal{A}$. A choice correspondence F induces an irreflexive binary relation $R_F \subseteq \mathcal{A} \times \mathcal{A}$ by

$$(X, Y) \in R_F \Leftrightarrow \text{there exist } Z \in \mathcal{A} \text{ with } X = F(Z) \text{ and } Y \subseteq Z$$

for all distinct $X, Y \in \mathcal{A}$. In this case we say that X *is revealed preferred to* Y by F and call R_F the *revealed preference relation of* F.

Later on we also use the following definitions and notations. A binary relation R on a set Ω is *transitive* if $(\omega^1, \omega^2), (\omega^2, \omega^3) \in R$ implies $(\omega^1, \omega^3) \in R$ for all distinct $\omega^1, \omega^2, \omega^3 \in \Omega$. The binary relation R has a *cycle* of length n, where $n \in \mathbb{N} \setminus \{1\}$, if there are distinct $\omega^1, \dots, \omega^n \in \Omega$ such that $(\omega^i, \omega^{i+1}) \in R$ for all $i \in \{1, \dots, n-1\}$ and $(\omega^n, \omega^1) \in R$; R is *acyclic* if, for every $n \in \mathbb{N} \setminus \{1\}$, it has no cycles length n.

For a choice correspondence F, we use the notation $F^n(X)$ as shorthand for $F \circ (F \circ (\dots (F(X))))$, that is, the n-fold composition of F with itself.

In the sequel, we denote a generic choice correspondence by F and consequently, its revealed preference relation by R_F.

We now introduce four properties of choice correspondences that we study in this paper. The first property is the standard notion of revealed preference adapted to the present context. In Brandt and Harrenstein (2011) it appears under the name 'set-rationalizability'.

Weak Axiom of Revealed Preference (WARP) For all $X, Y \in \mathcal{A}$, if $(X, Y) \in R_F$, then $(Y, X) \notin R_F$.

In conformity with the literature, in the revealed preference relation, WARP excludes cycles of length two but does not exclude longer cycles (among others, Rose, 1958; Peters and Wakker, 1994; Bossert and Peters, 2009). For completeness, we provide the following example of a choice correspondence satisfying WARP, which contains a cycle of length three, but cycles of arbitrary length can be easily constructed in similar examples.

Example 1. Let $A = \{a, b, c\}$ and define F by

$$F(X) = \begin{cases} \{a, b, c\} & \text{if } X = \{a, b, c\} \\ \{a\} & \text{if } X \in \{\{a, b\}, \{a\}\} \\ \{b\} & \text{if } X \in \{\{b, c\}, \{b\}\} \\ \{c\} & \text{if } X \in \{\{a, c\}, \{c\}\}. \end{cases}$$

It can be easily checked that F satisfies WARP. However, R_F contains a cycle of length three since $(\{a\}, \{b\}), (\{b\}, \{c\}), (\{c\}, \{a\}) \in R_F$.

The other three properties are extensions of the IIA condition for choice functions to choice correspondences. The first extension was first proposed by Nash (cf. Shubik, 1982), and occurs also in Chernoff (1954) and Arrow (1959).

Independence of Irrelevant Alternatives (IIA) For all $X, Y \in \mathcal{A}$ such that $Y \subseteq X$, if $F(X) \cap Y \neq \emptyset$, then $F(Y) = F(X) \cap Y$.

The second extension requires the following. Given a set X, if $F(X)$ is the set chosen, then from every subset of X that has a nonempty intersection with $F(X)$, only alternatives in $F(X)$ are chosen. This property appears as condition W2 in Schwartz (1976).

Weak Independence of Irrelevant Alternatives (WIIA) For all $X, Y \in \mathcal{A}$ such that $Y \subseteq X$, if $F(X) \cap Y \neq \emptyset$, then $F(Y) \subseteq F(X)$.

The third extension requires that, given a set X, if $F(X)$ is the set chosen, then from every subset of X that contains $F(X)$, exactly the alternatives of $F(X)$ are chosen.

Restricted Independence of Irrelevant Alternatives (RIIA). For all $X, Y \in \mathcal{A}$ such that $F(X) \subseteq Y \subseteq X$, $F(Y) = F(X)$.

3 WARP, IIA, WIIA, and RIIA

In this section we study the four properties introduced in Section 2.

3.1 WARP and RIIA

We start by showing that WARP and RIIA are equivalent. Based on this result one could say that RIIA is the appropriate extension of IIA of choice functions to choice correspondences from the point of view of rationalizability.

Theorem 1. *F satisfies WARP if and only if it satisfies RIIA.*

Proof. Let F satisfy WARP and let $X, Y \in \mathcal{A}$ with $F(X) \subseteq Y \subseteq X$. Suppose that $F(Y) \neq F(X)$. Since $F(Y) \subseteq X$, the definition of R_F implies $(F(X), F(Y)) \in R_F$; and since $F(X) \subseteq Y$, we similarly obtain $(F(Y), F(X)) \in R_F$. This violates WARP, and therefore, $F(Y) = F(X)$. Hence, F satisfies RIIA.

Next, let F satisfy RIIA. Let distinct $V, W \in \mathcal{A}$ such that $(V, W), (W, V) \in R_F$. We derive a contradiction as follows. Since $(V, W) \in R_F$, there exists $Z \in \mathcal{A}$ such that $F(Z) = V$ and $W \subseteq Z$. Similarly, since $(W, V) \in R_F$, there exists $Z' \in \mathcal{A}$, such that $F(Z') = W$ and $V \subseteq Z'$. Therefore, by RIIA, $W \subseteq Z$ and $V \subseteq Z'$ imply $F(Z) = F(W \cup F(Z)) = F(W \cup V) = F(F(Z') \cup V) = F(Z')$; thus $V = W$, a contradiction. □

A simple consequence of Theorem 1 is that if F satisfies WARP then it is a projection, i.e., $F^2 = F \circ F = F$.

Corollary 1. *Let F satisfy WARP. Then, for all $X \in \mathcal{A}$, $F^2(X) = F(X)$.*

Proof. By Theorem 1, for all pairs $X, Y \in \mathcal{A}$ such that $Y \subseteq X$, $F(X) = F(Y \cup F(X))$. Choosing $Y = F(X)$ implies $F^2(X) = F(X)$. □

Corollary 1 has also been established in Brandt and Harrenstein (2011). As mentioned in the Introduction, they use the expression 'set-rationalizablity' instead of WARP and show that this is equivalent to the following condition.

Condition $\hat{\alpha}$. For all $X, Y \in \mathcal{A}$, if $F(X \cup Y) \subseteq X \cap Y$, then $F(X \cup Y) = F(X)$.

Condition $\hat{\alpha}$ is a set-valued version of the Chernoff property, which is Postulate 4 in Chernoff (1954) and appears as Property α in Sen (1971). See also Moulin (1985, 1988) for studies of this property. It follows from the result of Brandt and Harrenstein (2011) and Theorem 1 above that Condition $\hat{\alpha}$ is equivalent to RIIA. This is not hard to show directly, as we do in the following lemma, which together with Theorem 1 provides an alternative proof of Theorem 2 in Brandt and Harrenstein (2011).

Lemma 1. *F satisfies Condition $\hat{\alpha}$ if and only it satisfies RIIA.*

Proof. For the only-if direction, suppose that F satisfies Condition $\hat{\alpha}$ and let $X, Y \in \mathcal{A}$ with $F(X) \subseteq Y \subseteq X$. Since $F(X \cup Y) = F(X) \subseteq Y = X \cap Y$, Condition $\hat{\alpha}$ implies that $F(Y) = F(X \cup Y) = F(X)$. Hence, F satisfies RIIA.

For the if-direction, suppose that F satisfies RIIA and let now $X, Y \in \mathcal{A}$ with $F(X \cup Y) \subseteq X \cap Y$. Let $X' = X \cup Y$ and $Y' = X \setminus F(X \cup Y)$. Then $F(X \cup Y) = F(X') = F(Y' \cup F(X')) = F((X \setminus F(X \cup Y)) \cup F(X \cup Y)) = F(X)$, where the second equality follows from RIIA. Hence, F satisfies Condition $\hat{\alpha}$. □

The converse of Corollary 1 is not true, as is shown by the following example.

Example 2. Let $A = [0, 1]$ and define F as follows.

$$F(X) = \begin{cases} \{1\} & \text{if } X = [0,1] \\ X & \text{otherwise.} \end{cases}$$

Clearly $F^2(X) = F(X)$. Next, consider the sets $[0, 1]$ and $\left[\frac{1}{2}, 1\right]$. Since $\left[\frac{1}{2}, 1\right] \subseteq [0, 1]$ and $F([0, 1]) = \{1\}$, $\left(\{1\}, \left[\frac{1}{2}, 1\right]\right) \in R_F$; and since $\{1\} \subseteq \left[\frac{1}{2}, 1\right]$ and $F\left(\left[\frac{1}{2}, 1\right]\right) = \left[\frac{1}{2}, 1\right]$, $\left(\left[\frac{1}{2}, 1\right], \{1\}\right) \in R_F$. Therefore, F violates WARP. A similar example with a finite set A can easily be constructed.

3.2 WIIA

We first show by two examples that WIIA and WARP (or RIIA) are logically independent.

Example 3. Let $A = \mathbb{N}$ and define F by

$$F(X) = \begin{cases} X & \text{if } |X| = 1 \text{ or } X \text{ is infinite} \\ X \setminus \{\max(X)\} & \text{if } 1 < |X| < \infty. \end{cases}$$

Let $Y \subseteq X$ such that $Y \cap F(X) \neq \emptyset$. If X is infinite, then $Y \cap F(X) = Y \cap X = Y$ and $F(Y) \subseteq Y = Y \cap F(X)$. If X is finite and $|Y| = 1$, then $F(Y) = Y = Y \cap F(X)$. Otherwise, $F(Y) \subseteq Y \cap F(X)$. Hence, F satisfies WIIA. However, let $X = \{1, 2, 3\}$ and $Y = \{1, 2\}$. Then $F(X) = \{1, 2\}$ implies $(\{1, 2\}, \{1\}) \in R_F$, while $F(Y) = \{1\}$ implies $(\{1\}, \{1, 2\}) \in R_F$. Hence, F does not satisfy WARP. Similar statements hold for $A = \{1, 2, \ldots, n\}$ with $n \geq 3$.

Example 4. Let $A = [0, 1]$ and define F by

$$F(X) = \begin{cases} X \setminus \{0\} & \text{if } X \subseteq [0,1] \text{ with } X \cap (\frac{1}{2}, 1] \neq \emptyset \\ X & \text{otherwise.} \end{cases}$$

Then $F([0, 1]) = (0, 1]$ whereas $F([0, \frac{1}{2}]) = [0, \frac{1}{2}]$; hence, F does not satisfy WIIA. By direct inspection or by using Theorem 1 it is straightforward that F satisfies WARP. The example can be easily adapted to finite A.

If we add the condition that F be a projection, then WIIA implies WARP.

Theorem 2. *Let F satisfy WIIA and $F^2 = F$. Then F satisfies WARP.*

Proof. By Theorem 1 is is sufficient to prove that F satisfies RIIA. Let $X, Y \in \mathcal{A}$ with $F(X) \subseteq Y \subseteq X$. We prove that $F(Y) = F(X)$. By WIIA, $F(Y) \subseteq F(X) \cap Y = F(X)$. Again by WIIA, since $F(X) \subseteq Y$ and $F(Y) \cap F(X) = F(Y) \neq \emptyset$ we have $F(F(X)) \subseteq F(Y) \cap F(X) = F(Y)$, hence $F(X) \subseteq F(Y)$ since $F^2 = F$. Thus, $F(Y) = F(X)$. □

The converse of Theorem 2 does not hold. If F satisfies WARP then by Corollary 1 it is a projection, but Example 4 shows that WIIA does not have to hold.

The following result shows that if F satisfies WIIA, then so does every n-fold composition of F with itself.

Lemma 2. *Let F satisfy WIIA and let $n \in \mathbb{N}$ with $n \geq 2$. Then F^n satisfies WIIA.*

Proof. The proof is based on induction: $F^1 = F$ satisfies WIIA, and assume that F^k satisfies WIIA for every $k = 2, \ldots, n-1$. Let $X, Y \in \mathcal{A}$ with $Y \subseteq X$ and $F^n(X) \cap Y \neq \emptyset$. We show that $F^n(Y) \subseteq F^n(X)$.

Note that $F^k(X) \cap Y \neq \emptyset$ and the induction assumption imply that $F^k(Y) \subseteq F^k(X)$ for every $k = 1, \ldots, n-1$ and thus that

$$F^\ell(X) \cap F^m(Y) \neq \emptyset \text{ for all } \ell, m \in \{1, \ldots, n-1\}. \tag{1}$$

We now first prove that

$$F^n(X) \cap F^k(Y) \neq \emptyset \text{ for every } k = 0, \ldots, n-1 \tag{2}$$

where $F^0(Y) = Y$. The proof of (2) is by induction. By assumption, $F^n(X) \cap F^0(Y) = F^n(X) \cap Y \neq \emptyset$. Let $2 \leq \ell \leq n-1$ and assume that $F^n(X) \cap F^k(Y) \neq \emptyset$ for every $k = 1, \ldots, \ell-1$. We show that $F^n(X) \cap F^\ell(Y) \neq \emptyset$. First, since $\emptyset \neq F^{n-1}(X) \cap F^{\ell-1}(Y) \subseteq F^{n-1}(X)$ by (1), and $F^n(X) \cap (F^{n-1}(X) \cap F^{\ell-1}(Y)) = F^n(X) \cap F^{\ell-1}(Y) \neq \emptyset$ by the induction assumption for this part, WIIA of F implies

$$F(F^{n-1}(X) \cap F^{\ell-1}(Y)) \subseteq F^n(X). \tag{3}$$

Second, since since $\emptyset \neq F^{n-1}(X) \cap F^{\ell-1}(Y) \subseteq F^{\ell-1}(Y)$ by (1), and $F^\ell(Y) \cap (F^{n-1}(X) \cap F^{\ell-1}(Y)) = F^\ell(Y) \cap F^{n-1}(X) \neq \emptyset$ by (1), WIIA of F implies

$$F(F^{n-1}(X) \cap F^{\ell-1}(Y)) \subseteq F^\ell(Y). \tag{4}$$

By (3) and (4), $F^n(X) \cap F^\ell(Y) \neq \emptyset$, which completes the proof of (2).

Now, since $F^{n-1}(X) \cap Y \neq \emptyset$, the assumed WIIA of F^{n-1} implies $F^{n-1}(Y) \subseteq F^{n-1}(X)$. Since by (2) we have $F^n(X) \cap F^{n-1}(Y) \neq \emptyset$, WIIA of F implies $F^n(Y) \subseteq F^n(X)$. This completes the proof of the lemma. □

If A is finite, then there exists $n \in \mathbb{N}$ with $n \leq |A| - 1$ such that $F^\ell = F^n$ for all $\ell \geq n$. In that case, we have the following corollary.

Corollary 2. *Let A be finite and let F satisfy WIIA. Then $F^{|A|-1}$ satisfies WARP.*

Proof. Since $F^{|A|-1}$ is a projection, the result follows from Lemma 2 and Theorem 2. □

If A is infinite, then an n as in the finite case does not necessarily exist. However, we may define F^∞ by $F^\infty(X) = \cap_{n\in\mathbb{N}} F^n(X)$ for every $X \in \mathcal{A}$, assuming that this set is nonempty for every $X \in \mathcal{A}$. The following example shows that the last condition is not necessarily satisfied if F satisfies WIIA.

Example 5. Let $A = [0,1]$ and for every $X \in \mathcal{A}$, let $m(X)$ be the maximal number in $\mathbb{N} \cup \{0\}$ such that $X \subseteq [0, \frac{1}{2^{m(X)}}]$. We define F by

$$F(X) = \begin{cases} X \setminus (\frac{1}{2^{m(X)+1}}, \frac{1}{2^{m(X)}}] & \text{if } X \setminus (\frac{1}{2^{m(X)+1}}, \frac{1}{2^{m(X)}}] \neq \emptyset \\ X & \text{otherwise.} \end{cases}$$

It is easy to check that F satisfies WIIA. However, $\cap_{n\in\mathbb{N}} F^n(A \setminus \{0\}) = \cap_{n\in\mathbb{N}}(0, \frac{1}{2^n}] = \emptyset$.

Remark 1. If F satisfies WIIA and F^∞ is well defined, then it follows from Lemma 2 that F^∞ satisfies WIIA. Namely, let $X, Y \in \mathcal{A}$, $Y \subseteq X$, and $F^\infty(X) \cap Y \neq \emptyset$. Then in particular $F^n(X) \cap Y \neq \emptyset$ for every $n \in \mathbb{N}$, so that by Lemma 2 we have $F^\infty(Y) = \cap_{n\in\mathbb{N}} F^n(Y) = \cap_{n\in\mathbb{N}}(F^n(X) \cap Y) = (\cap_{n\in\mathbb{N}} F^n(X)) \cap Y = F^\infty(X) \cap Y$.

Even, however, if F^∞ is well-defined, and F and thus F^∞ satisfy WIIA, F^∞ does not have to be a projection. For instance, let $A = \{-1, 0\} \cup \{\frac{1}{2^n} \mid n \in \mathbb{N}\}$. Define F by

$$F(X) = \begin{cases} X \setminus \{\max\{x : x \in X\}\} & \text{if } |X| > 1 \\ X & \text{otherwise.} \end{cases}$$

Then F^∞ is well-defined, and both F and F^∞ satisfy WIIA. Since $F^\infty(A) = \{-1, 0\}$ and $F^\infty(\{-1, 0\}) = \{-1\}$, we have $F^\infty \circ F^\infty(A) = \{-1\} \neq F^\infty(A)$. Hence, $F^\infty \circ F^\infty \neq F^\infty$. In particular, by Corollary 1, F^∞ does not satisfy WARP.

3.3 IIA

For single-valued choice, IIA is equivalent to WARP as long as the domain of choice sets is closed under nonempty intersection. For choice correspondences RIIA is equivalent to WARP (see Theorem 1), IIA implies WARP, as shown by the lemma below, but not the other way around, as shown by the example that follows.

Lemma 3. *Let F satisfy IIA. Then $F^2 = F$ and F satisfies WARP.*

Proof. That IIA implies $F^2 = F$ is straightforward. Since IIA implies RIIA, the second statement in the lemma follows from Theorem 1. □

Example 6. Let $A = \{a, b, c\}$ and define F by

$$F(X) = \begin{cases} \{a\} & \text{if } X \in \{\{a,b\}, \{a,c\}\} \\ \{b\} & \text{if } X = \{b,c\} \\ X & \text{otherwise.} \end{cases}$$

It can be easily checked that F satisfies WARP. This also follows from Theorem 2 and the fact that F satisfies WIIA and is a projection. However, since $\{a,b\} \subseteq \{a,b,c\}$ and $\{a,b\} \cap F(\{a,b,c\}) \neq \emptyset$, it follows that $F(\{a,b\}) = \{a\}$ violates IIA. By partitioning a set into three nonempty subsets, the example can be easily adapted to an infinite A.

In order to obtain a revealed preference characterization of IIA we introduce the following condition on the revealed preference relation of a choice correspondence: if X is revealed preferred to Y, then every subset of X that includes all alternatives in $X \cap Y$, is also revealed preferred to Y.

Preference Axiom (PA) For all distinct $X, Y, Z \in \mathcal{A}$ such that $X \cap Y \subseteq Z \subseteq X$, if $(X, Y) \in R_F$, then $(Z, Y) \in R_F$.

This axiom can be interpreted as expressing that what really makes X (revealed) preferred to Y is the intersection of X and Y. Indeed, the axiom implies that if X is preferred to Y then $X \cap Y$, if nonempty, is preferred to Y. If X and Y are disjoint then every nonempty subset of X is preferred to Y.

We now have the following characterization of IIA.

Theorem 3. *F satisfies IIA if and only if it satisfies WARP and PA.*

Proof. (a) Let F satisfy IIA. Then F satisfies WARP by Lemma 3. We show that F satisfies PA. Let distinct $X, Y, Z \in \mathcal{A}$ such that $X \cap Y \subseteq Z \subseteq X$ and $(X, Y) \in R_F$. By IIA of F and $(X, Y) \in R_F$, $F(X \cup Y) = X$. Since $Z \cup Y \subseteq X \cup Y$ and $F(X \cup Y) \cap (Z \cup Y) = X \cap (Z \cup Y) = (X \cap Z) \cup (X \cap Y) = Z \neq \emptyset$, IIA of F implies $F(Z \cup Y) = Z$. Therefore, $(Z, Y) \in R_F$.

(b) Let F satisfy WARP and PA. We show that F satisfies IIA. Let $X, Y \in \mathcal{A}$ with $Y \subseteq X$ and $F(X) \cap Y \neq \emptyset$. Suppose that $F(Y) \neq F(X) \cap Y$. Since $F(Y) \subseteq Y \subseteq X$, it follows that $(F(X), F(Y)) \in R_F$. In addition, $(F(X) \cap F(Y)) \subseteq (F(X) \cap Y) \subseteq F(X)$, so that by PA, $(F(X) \cap Y, F(Y)) \in R_F$. But also, $F(Y) \subseteq Y$ and $F(X) \cap Y \subseteq Y$ imply $(F(Y), F(X) \cap Y) \in R_F$. This violates WARP; therefore $F(Y) = F(X) \cap Y$. Thus, F satisfies IIA. $\square$

In Example 6, F satisfies WARP but not IIA. Hence it follows from Theorem 3 that F does not satisfy PA either. This can also be easily established directly. E.g., let $X = \{a, b, c\}$, $Y = \{b\}$, and $Z = \{a, b\}$, then $(X, Y) \in R_F$ but $(Z, Y) \notin R_F$.

The next example shows that PA does not imply IIA or WARP.

Example 7. Let $A = \{a, b, c, d\}$ and define F by

$$F(X) = \begin{cases} \{a, b\} & \text{if } X = \{a, b, c, d\} \\ \{a\} & \text{if } X \subsetneq \{a, b, c, d\} \text{ and } a \in X \\ \{b\} & \text{if } X \subseteq \{b, c, d\} \text{ and } b \in X \\ \{c\} & \text{if } X \subseteq \{c, d\} \text{ and } c \in X \\ \{d\} & \text{if } X = \{d\}. \end{cases}$$

It is straightforward to show that F satisfies PA. Since $\{a, b\} \subseteq \{a, b, c, d\}$, $\{a, b\} \cap F(\{a, b, c, d\}) \neq \emptyset$, and $F(\{a, b\}) = \{a\}$, it follows that F does not satisfy IIA and by Theorem 3, it also does not satisfy WARP.

In Section 3.2 we have already seen that WIIA and WARP are logically independent. The same is true for WIIA and PA. In Example 3, F satisfies WIIA but not PA. The following example shows that PA does not imply WIIA.

Example 8. Let $A = \{a, b, c\}$ and define F by

$$F(X) = \begin{cases} \{a\} & \text{if } X = A \\ X & \text{otherwise.} \end{cases}$$

In this case, F does not satisfy WIIA since $F(\{a,b\}) = \{a,b\} \not\subseteq \{a\} = F(A) \cap \{a,b\}$. To show that F satisfies PA, let X, Y, Z as in the statement of PA and $V \in \mathcal{A}$ such that $F(V) = X$ and $Y \subseteq V$. If $|X| = 1$ then either $V = X$ and then $Z = X = Y$, a contradiction since $Z \neq Y$; or $V = A$, which implies $X = \{a\}$ and therefore $Z = \{a\} = X$, so that $(Z, Y) \in R_F$. If $|X| = 2$, then $V = X$ and $Y \subseteq X$ with $|Y| = 1$; this implies $Z = X$ and thus $(Z, Y) \in R_F$.

For (single-valued) choice functions on a collection of choice sets that is closed under nonempty intersection, IIA and WARP are equivalent, but do not necessarily imply acyclicity of the revealed preference relation (e.g., Gale, 1960; Peters and Wakker, 1994).[2] For choice correspondences on the collection of all nonempty subsets of a set of alternatives, IIA implies WARP (Theorem 3) and, moreover, transitivity and acyclicity of the revealed preference relation, as the next result shows.

Theorem 4. *Let F satisfy IIA. Then R_F is transitive and acyclic.*

Proof. By Theorem 3, F satisfies WARP. It is sufficient to prove that R_F is transitive, since with WARP this implies acyclicity. Let distinct $X_1, X_2, X_3 \in \mathcal{A}$ with $(X_1, X_2), (X_2, X_3) \in R_F$. We prove that $(X_1, X_3) \in R_F$. Let $Z = X_1 \cup X_2 \cup X_3$. We consider two cases for $F(Z)$.

If $F(Z) \cap X_3 = \emptyset$ then $F(Z) \subseteq X_1 \cup X_2$. If $F(Z) \neq X_1$ then $(F(Z), X_1) \in R_F$. On the other hand, since $(X_1, X_2) \in R_F$ there is $Z_1 \in \mathcal{A}$ such that $F(Z_1) = X_1$ whereas $X_2 \subseteq Z_1$; in particular, this implies $(X_1, F(Z)) \in R_F$, so that WARP is violated. Hence, in this case, $F(Z) = X_1$ and therefore $(X_1, X_3) \in R_F$.

If $F(Z) \cap X_3 \neq \emptyset$ then $F(X_2 \cup X_3) = F(Z) \cap (X_2 \cup X_3)$ by IIA, hence $F(X_2 \cup X_3) \cap X_3 \neq \emptyset$. This implies $(F(X_2 \cup X_3), X_2) \in R_F$. On the other hand, by a similar argument as in the first case, $(X_2, X_3) \in R_F$ implies $(X_2, F(X_2 \cup X_3)) \in R_F$, violating WARP. Hence, in this case, $(X_1, X_3) \in R_F$, which concludes the proof of the theorem. □

The converse of Theorem 4 does not hold: the revealed preference relation R_F of the choice correspondence F in Example 7 is transitive and acyclic, but F does not satisfy IIA.

4 Strong sets

In this section we introduce the so-called strong sets and show that such a collection is induced by but also uniquely describes an IIA choice correspondence.

A set $S \in \mathcal{A}$ is a strong set if the following holds. For all sets where some alternatives of S are chosen, all the available alternatives of S are chosen, and only these. Formally we have the following definition.

Definition 1. $S \in \mathcal{A}$ is a *strong set* at F if for all $X \in \mathcal{A}$ for which $F(X) \cap S \neq \emptyset$, we have $F(X) = S \cap X$. The set of strong sets induced by F is denoted by $\mathcal{S}_F$. By $R_{\mathcal{S}_F} = \{(X, Y) \in R_F \mid X, Y \in \mathcal{S}_F\}$ we denote the restriction of R_F to $\mathcal{S}_F$.

We show that the elements of $\mathcal{S}_F$ are pairwise disjoint and that $R_{\mathcal{S}_F}$ is complete and acyclic if F satisfies IIA.

Lemma 4. *Let $S, T \in \mathcal{S}_F$ with $S \neq T$. Then $S \cap T = \emptyset$.*

[2] That is, they do not necessarily imply that F satisfies the so-called Strong Axiom of Revealed Preference.

Proof. Let $Z = S \cup T$. Without loss of generality assume that $F(Z) \cap S \neq \emptyset$. Then $S \in \mathcal{S}_F$ implies $F(Z) = S \cap Z = S$. If $S \cap T \neq \emptyset$, then $F(Z) \cap T \neq \emptyset$; hence $T \in \mathcal{S}_F$ implies $F(Z) = T \cap Z = T$. This contradicts $S \neq T$. Consequently, $S \cap T = \emptyset$. □

Lemma 5. *Let F satisfy IIA. Then $R_{\mathcal{S}_F}$ is complete and acyclic.*

Proof. Let $S, T \in \mathcal{S}_F$ with $S \neq T$. By the definition of $\mathcal{S}_F$, $F(S \cup T) \in \{S, T\}$; hence, $R_{\mathcal{S}_F}$ is complete. Without loss of generality assume that $F(S \cup T) = S$. We show that $(T, S) \notin R_{\mathcal{S}_F}$, which implies that $R_{\mathcal{S}_F}$ has no cycles of length 2. To show this, let $Z \in \mathcal{A}$ with $S \cup T \subseteq Z$. If $(S \cup T) \cap F(Z) \neq \emptyset$, then by IIA, $S = F(S \cup T) = (S \cup T) \cap F(Z)$. Since $S \in \mathcal{S}_F$, this implies that $F(Z) = S$. Since Z was arbitrary, we have $(T, S) \notin R_{\mathcal{S}_F}$.

In order to show that $R_{\mathcal{S}_F}$ has also no cycles of length larger than 2, it is sufficient to prove that it is transitive. Let $(S, T), (T, V) \in R_{\mathcal{S}_F}$ for distinct $S, T, V \in \mathcal{S}_F$. Then for $X = S \cup T \cup V$ we have $S = F(X)$, so that $(S, V) \in R_{\mathcal{S}_F}$. □

The next lemma is needed in the proof of the theorem below.

Lemma 6. *Let choice correspondence F satisfy IIA, and let $\emptyset \neq \mathcal{Z} \subseteq \mathcal{A}$ such that $\cap_{Z \in \mathcal{Z}} F(Z) \neq \emptyset$. Then $\cup_{Z \in \mathcal{Z}} F(Z) = F(\cup_{Z \in \mathcal{Z}} Z)$.*

Proof. Let $x \in \cap_{Z \in \mathcal{Z}} F(Z)$. Since $F(\cup_{Z \in \mathcal{Z}} Z) \subseteq \cup_{Z \in \mathcal{Z}} Z$, there is $Z' \in \mathcal{Z}$ such that $Z' \cap F(\cup_{Z \in \mathcal{Z}} Z) \neq \emptyset$, so that $F(Z') = F(\cup_{Z \in \mathcal{Z}} Z) \cap Z'$ by IIA. Hence, $x \in F(\cup_{Z \in \mathcal{Z}} Z)$, so that $Z' \cap F(\cup_{Z \in \mathcal{Z}} Z) \neq \emptyset$ for all $Z' \in \mathcal{Z}$, and hence $F(Z') = F(\cup_{Z \in \mathcal{Z}} Z) \cap Z'$ for all $Z' \in \mathcal{Z}$ by IIA. The statement in the lemma now follows. □

Theorem 5. *Let F satisfy IIA. Then $\mathcal{S}_F$ is a partition of A and $R_{\mathcal{S}_F}$ is complete and acyclic.*

Proof. Let $x \in A$. In view of Lemmas 4 and 5 we only still have to prove that there is an $S \in \mathcal{S}_F$ such that $x \in S$. Define $\mathcal{Z} = \{Z \in \mathcal{A} \mid x \in F(Z)\}$ and $S = F(\cup_{Z \in \mathcal{Z}} Z)$. By Lemma 6 we have $S = \cup_{Z \in \mathcal{Z}} F(Z) \ni x$, so that it is sufficient to prove that $S \in \mathcal{S}_F$. To this end, let $X \in \mathcal{A}$ such that $F(X) \cap S \neq \emptyset$, then it is sufficient to prove that $F(X) = S \cap X$.

Since $F(X) \cap S \neq \emptyset$, Lemma 6 implies $F(X) \cup S = F(X) \cup F(\cup_{Z \in \mathcal{Z}} Z) = F(X \cup (\cup_{Z \in \mathcal{Z}} Z))$. In particular, this implies that $F(X \cup (\cup_{Z \in \mathcal{Z}} Z)) \cap X \neq \emptyset$ so that by IIA we obtain $F(X) = F(X \cup (\cup_{Z \in \mathcal{Z}} Z)) \cap X \supseteq F(\cup_{Z \in \mathcal{Z}} Z) \cap X$, hence $S \cap X \subseteq F(X)$. Therefore, it is sufficient to prove that $F(X) \subseteq S \cap X$.

By Lemma 6, $F(X) \cap S \neq \emptyset$ implies $F(X) \cap (\cup_{Z \in \mathcal{Z}} F(Z)) \neq \emptyset$. Hence, for some $Z' \in \mathcal{Z}$, $F(X) \cap F(Z') \neq \emptyset$. Thus, by Lemma 6, $F(X \cup Z') = F(X) \cup F(Z')$. It follows that $x \in F(X \cup Z')$ and thus, $X \cup Z' \in \mathcal{Z}$. In addition, since $S = \cup_{Z \in \mathcal{Z}} F(Z)$, $F(X \cup Z') \subseteq S$, and hence, $F(X) \cup F(Z') \subseteq S$. Therefore, $F(X) \subseteq S$ and trivially, $F(X) \subseteq S \cap X$. □

The converse of Theorem 5 does not hold. The following example describes a choice correspondence F of which the strong sets partition A, and are completely and acyclically ordered by R_F, but which does not satisfy IIA.

Example 9. Let $F(A) \subsetneq A$ and for all $X \subseteq \mathcal{A}$ define F by

$$F(X) = \begin{cases} F(A) & \text{if } F(A) \subseteq X \\ X & \text{if } X \subseteq F(A) \\ X \setminus F(A) & \text{otherwise.} \end{cases}$$

It is straightforward to verify that $\mathcal{S}_F = \{F(A), A \setminus F(A)\}$, which is a partition of A. Also, $R_{\mathcal{S}_F} = \{(F(A), A \setminus F(A))\}$. (In fact, it is not difficult to show that F satisfies WARP and is a projection.) Let $X, Y \in \mathcal{A}$ such that $F(A) \subseteq X$, $Y \subseteq X$, $Y \not\subseteq F(A)$, $F(A) \not\subseteq Y$, and $Y \cap F(A) \neq \emptyset$. Then $F(X) = F(A)$ but $F(Y) = Y \setminus F(A) \not\subseteq Y \cap F(A) = Y \cap F(X)$. Hence, F does not satisfy IIA.

Let $S_0 = \emptyset$ and, recursively, $S_i = F(A \setminus \cup_{j=0}^{i-1} S_j)$ for every $i = 1, 2, \ldots$ Clearly, if A is finite then there is an $\ell \geq 1$ such that $S_i = S_\ell$ for all $i > \ell$. In that case, $\{S_1, \ldots, S_\ell\}$ is a partition of A. We have:

Lemma 7. *Let A be finite. If F satisfies IIA, then $\mathcal{S}_F = \{S_1, \ldots, S_\ell\}$ and $S_i R_{\mathcal{S}_F} S_j$ for all $i, j \in \{1, \ldots, \ell\}$ with $i < j$.*

Proof. Let F satisfy IIA. The second statement follows directly from the construction of the sets S_i. For the first statement it is in view of Lemma 4 sufficient to prove that each element of $\{S_1, \ldots, S_\ell\}$ is a strong set. Let $k \in \{1, \ldots, \ell\}$ and $X \in \mathcal{A}$ such that $F(X) \cap S_k \neq \emptyset$. By repeated application of IIA it follows that $X \subseteq A \setminus \cup_{i=1}^{k-1} S_i$. Since $X \cap S_k = X \cap F(A \setminus \cup_{i=1}^{k-1} S_i) \neq \emptyset$, IIA implies $F(X) = X \cap F(A \setminus \cup_{i=1}^{k-1} S_i) = X \cap S_k$. Hence, S_k is a strong set. □

A converse of Lemma 7 for general sets A is the following corollary, which says that every 'well-ordered' partition of $\mathcal{A}$ induces a unique IIA choice correspondence.

Corollary 3. *Let $\mathcal{T}$ be a partition of A and let R be an irreflexive, complete, and acyclic binary relation on $\mathcal{T}$ such that for every $X \in \mathcal{A}$ the collection $\{T \in \mathcal{T} \mid T \cap X \neq \emptyset\}$ has a maximal element T_X with respect to R. Then F defined by $F(X) = X \cap T_X$ for all $X \in \mathcal{A}$ satisfies IIA, $\mathcal{S}_F = \mathcal{T}$, and $R_{\mathcal{S}_F} = R$. Moreover, if G satisfies IIA, $\mathcal{S}_G = \mathcal{T}$, and $R_{\mathcal{S}_G} = R$, then $G = F$.*

Proof. The statements about F are straightforward. Let G be as in the lemma, and $X \in \mathcal{A}$. We prove that $G(X) = X \cap T_X$. For this, by IIA, it is sufficient to prove that $G(T_X \cup (\cup_{T \in \mathcal{T}: T_X R T} T)) = T_X$. If this were not true, then the definition of strong sets would imply $G(T_X \cup (\cup_{T \in \mathcal{T}: T_X R T} T)) = T'$ for some $T' \in \mathcal{T}$ with $T_X R T'$, in contradiction with $R = R_{\mathcal{S}_G}$. □

We conclude with a remark about the construction of all strong sets.

Remark 2. For infinite A the construction above, resulting in the sets $S_1, S_2, \ldots$, does not generally produce all strong sets, or, equivalently, does not generally result in a partition of A. For instance, let $A = \mathbb{N}$, $E = \{2, 4, \ldots\}$ and define F by

$$F(X) = \left\{ \begin{array}{cl} \min X \cap E & \text{if } X \cap E \neq \emptyset \\ \min X & \text{otherwise} \end{array} \right\} \quad \text{for all } X \in \mathcal{A}.$$

Choice correspondence F satisfies IIA and $S_i = \{2i\}$ for every $i = 1, 2, \ldots$ However, the collection of strong sets is $\{S_1, S_2, \ldots, T_1, T_2, \ldots\}$, where $T_i = \{2i-1\}$ for every $i = 1, 2, \ldots$, $S_i R_F S_j$ and $T_i R_F T_j$ for all i, j with $i < j$, and $S_i R_F T_j$ for all i, j. Observe that this collection can be constructed by first determining the sets S_i and then repeating the construction starting with the set $A \setminus \cup_{i=1}^{\infty} S_i = A \setminus E = \{1, 3, \ldots\}$.

More generally, the collection of strong sets is a well-ordered set and therefore it is isomorphic to an ordinal number (e.g., Theorem 2.12 in Jech, 2002).[3] Conversely, with every ordinal number we can associate a collection of strong sets of an IIA choice correspondence.

[3]'Well-ordered' means that there is an ordering R as in Corollary 3. In the example the set of strong sets is isomorphic to the ordinal number $\omega \cdot 2$.

WARP	$\overset{\text{Theorem 1}}{\Longleftrightarrow}$	RIIA
	$\overset{\text{Lemma 1\&Theorem1}}{\Longleftrightarrow}$	Condition $\hat{\alpha}$
	$\overset{\text{Corollary 1}}{\Longrightarrow}$	$F^2 = F$
$F^2 = F$ & WIIA	$\overset{\text{Theorem 2}}{\Longrightarrow}$	WARP
IIA	$\overset{\text{Theorem 3}}{\Longleftrightarrow}$	WARP & PA
	$\overset{\text{Theorem 4}}{\Longrightarrow}$	R_F transitive and acyclic
	$\overset{\text{Theorem 5}}{\Longrightarrow}$	$\mathcal{S}_F$ partition, $R_{\mathcal{S}_F}$ complete and acyclic

Table 1: Summary of the main results

5 Concluding remarks

The main results of the paper are summarized in Table 1. The main feature distinguishing these results from the existing literature is that preferences over sets rather than singletons are considered. No additional structure on the set of alternatives is presumed. Of course, more specific results may be attainable if this set has more structure.

An open question is how to obtain a set-theoretic characterization of WIIA choice correspondences. In an earlier version of this paper[4] we introduced the concept of a weak sets, and some of the results obtained for IIA correspondences and strong sets have analogues for WIIA correspondences and weak sets, but these weak sets do not fully determine the correspondence; in other words, we do not obtain an analogue of Corollary 3.

A similar question can be asked with respect to RIIA choice correspondences. Other open questions concern the implications of different IIA-extensions, such as the extensions discussed in Section 1.3.1.

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