Self-implementation of social choice correspondences in Nash equilibrium*

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Abstract

A social choice correspondence is Nash self-implementable if it can be implemented in Nash equilibrium by a social choice function that selects from it as the game form. We provide a fairly complete characterization of all unanimous and anonymous social choice correspondences that can be Nash implemented by an anonymous selection. In particular, such social choice correspondences include the top correspondence and are included in the Pareto correspondence.

Keywords Social choice correspondence, Nash implementation, self-implementation
JEL-classification D71, D72

1 Introduction

A social choice correspondence assigns a set of alternatives to a profile of agents' preferences. A specific social choice correspondence – for instance, a particular political election system – may have been adopted since it has some properties that are regarded as attractive, but these properties may be lost if agents behave strategically – do not vote according to their true preferences. Thus, agents should be provided with incentives in order to reach a truthful alternative, that is, an alternative assigned by the social choice correspondence to the true preference profile. This leads to the classical question of implementation, which dates back to Hurwicz (1972) and Maskin (1999): given a social choice correspondence, can we find a game form such that for every preference profile the equilibrium outcomes of the game form coincide with the outcomes of the social choice correspondence? Here, 'equilibrium' may refer to Nash equilibrium or to other equilibrium concepts, for instance strong equilibrium.

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Game forms in the literature used for implementing social choice correspondences tend to be quite complicated. Typically, they may ask individuals to report complete strategy profiles, and they use integers or similar devices as part of strategies in order to endow individuals or coalitions – depending on the equilibrium concept – with sufficient possibilities to deviate, thus avoiding equilibria with outcomes that are not contained in the set assigned by the social choice correspondence. Indeed, the complexity of the message space or space of strategies is an issue that has gathered quite some attention in the literature (see already Mount and Reiter, 1974). Recently, Koray and Yildiz (2018) consider implementation via rights structures, based on an idea by Sertel (2001), which also avoids the use of integer devices. For (relatively) recent surveys on implementation see Jackson (2001) and Börgers (2015).

In the present paper we restrict our attention to direct game forms which are single-valued selections (social choice functions) from the social choice correspondence. This means, simply, that the strategy space of each agent consists of the set of all preferences – agents are not required to report complete profiles, integers, or any similar or other information – and the outcome function selects (at every preference profile) from the social choice correspondence to be implemented. Our main motivation for studying this case is twofold. First, considering that a social choice correspondence in vigor in a group or society is usually regarded as appropriate or desirable by its members, choosing according to a selection from it seems to be natural, both for simplicity and in order to convince agents to accept the game form (the rules of the game). Several papers have discussed the advantages of direct game forms. For example, Hammond (1996) shows that such game forms, in which individual strategies are direct reports of their preferences, are implicit in Sen's model of rights (Sen, 1970). Second, but related, it seems to be an obvious and natural question to establish which social choice correspondences can be implemented this way, i.e., by a direct game form selecting from the correspondence. One could call this kind of implementation 'natural', but this word already has different meanings in this context (e.g., Saijo et al, 1996). Therefore, we use the term 'self-implementation', an expression which has already been used informally, but with the same meaning, for social choice functions (see Remark 4.2.7 in Peleg, 2002).

We will focus attention on self-implementation in Nash equilibrium.[1] Moreover, we consider only social choice correspondences which are unanimous and anonymous, and only anonymous selections (social choice functions). Under these restrictions our results are as follows. Let the 'top correspondence' TC assign to each preference profile all alternatives that occur at top for at least one agent, and let the 'Pareto correspondence' PC assign all Pareto optimal alternatives. Then every anonymous and unanimous Nash self-implementable

[1]For self-implementation in strong equilibrium (Aumann, 1959) see Peleg and Peters (2018).

social choice correspondence contains TC and is contained in PC.[2] [3] If there are exactly two alternatives, then TC and PC of course coincide; and we show that TC is Nash self-implementable if and only if the number of agents is not equal to 2, 3, 4, or 6; moreover, we describe all implementing selections (Corollary 4.8). If there are exactly two agents, then no anonymous and unanimous social choice correspondence is Nash self-implementable, a result which also follows from Hurwicz and Schmeidler (1978). If there are three agents and three alternatives then only TC is Nash self-implementable. If there are at least three agents and and at least four alternatives then TC and PC are Nash self-implementable but also some social choice correspondences in between these two; the characterization of all such correspondences is still open. The remaining case is the case with three alternatives and at least four agents: then TC is Nash self-implementable but it is an open question whether TC is uniquely Nash self-implementable.

To the best of our knowledge, this concept of Nash self-implementation has not been explicitly studied in the literature, although some of our results can be derived from existing results, for instance the two-agent case mentioned above. Nevertheless, the proofs below are self-contained. Also, the social choice correspondences prominent in this paper have occurred frequently in the literature, for instance the Pareto Correspondence only recently in Mukherjee et al (2019).

The outline of the paper is as follows. Section 2 presents the main definitions and a few basic results. Sections 3 and 4 deal with the two agents and two alternatives cases, respectively. The case with at least four alternatives is presented in Section 5, while Section 6 treats the three alternatives case. Section 7 concludes.

2 Definitions and preliminary results

Let A be the set of m *alternatives*, $m \geq 2$, and let $N = \{1, \ldots, n\}$, $n \geq 2$, be the set of *agents*. Let L be the set of *preferences*, i.e., reflexive, complete, antisymmetric and transitive binary relations, on A. Then L^N is the set of (preference) *profiles*. A *social choice correspondence* (SCC) is a function $H : L^N \to 2^A \setminus \{\emptyset\}$. A *social choice function* (SCF) is a function $F : L^N \to A$. For a social choice function F and a profile R^N we denote by (F, R^N) the (ordinal) game with the agents as players who each have strategy set L, and for a (strategy) profile $Q^N \in L^N$ the outcome (alternative) $F(Q^N)$ is evaluated by R^i for each $i \in N$. Profile Q^N is a *Nash equilibrium* of the game (F, R^N) if $F(Q^N)R^iF(Q^{N\setminus\{i\}}, \widetilde{Q}^i)$ for every $i \in N$ and $\widetilde{Q}^i \in L$, where $(Q^{N\setminus\{i\}}, \widetilde{Q}^i)$ is obtained from Q^N by replacing Q^i by $\widetilde{Q}^i$.

A social choice function F is a *selection* from a social choice correspondence H if $F(R^N) \in H(R^N)$ for every $R^N \in L^N$. We say that F *implements* H *in*

[2] In particular, this means that Nash self-implementability implies that the implementing game form (social choice function) is acceptable – see Hurwicz and Schmeidler (1978) and Dutta (1984).

[3] The latter is not a new observation. E.g., Peleg et al (2005) show that Maskin monotonicity (which is implied by Nash implementability) and unanimity imply Pareto optimality.

Nash equilibrium if for every $R^N \in L^N$ we have

$$H(R^N) = \{x \in A \mid x = F(Q^N) \text{ for some Nash equilibrium } Q^N \text{ of } (F, R^N)\}. \tag{1}$$

We call social choice correspondence H *Nash self-implementable* if (1) holds for some selection F from H.

For a preference $R \in L$ the *top alternative* $t(R) \in A$ is the alternative such that $t(R)Rx$ for all $x \in A$. The social choice correspondence TC, defined by

$$TC(R^N) = \{t(R^i) \mid i \in N\}$$

for all $R^N \in L^N$ is called the *top correspondence*. An alternative x is *Pareto optimal* in R^N if there is no $y \in A \setminus \{x\}$ such that yR^ix for all $i \in N$. The correspondence PC, defined by

$$PC(R^N) = \{x \in A \mid x \text{ is Pareto optimal in } R^N\}$$

for all $R^N \in L^N$ is called the *Pareto correspondence*. Clearly, $TC(R^N) \subseteq PC(R^N)$ for all $R^N \in L^N$.

A social choice correspondence H is *unanimous* if for all $R^N \in L^N$ and $x \in A$ such that $t(R^i) = x$ for all $i \in N$ we have $H(R^N) = \{x\}$. It is *anonymous* if for every permutation π of N and every $R^N \in L^N$, we have $H(R^N) = H(Q^N)$ where $Q^i = R^{\pi(i)}$ for every $i \in N$. Throughout, we will focus on unanimous and anonymous social choice correspondences. Clearly, TC and PC are unanimous and anonymous.

If a social choice correspondence H is unanimous and anonymous and F is a selection from it, then clearly also F is unanimous, but F is not necessarily anonymous. In what follows we will always consider implementation by anonymous social choice functions.

The following lemmas collect some basic results which will be useful throughout. In these lemmas H is a unanimous and anonymous social choice correspondence which is Nash implemented by an anonymous selection F. The first lemma implies that for every preference profile and every alternative there is an agent who can make sure that this alternative will be chosen. Furthermore, H contains all top alternatives and is contained in the set of Pareto optimal alternatives.

Lemma 2.1. *Let H be a unanimous and anonymous social choice correspondence and let the anonymous selection F implement H in Nash equilibrium. Let $R^N \in L^N$ and $x \in A$.*

(a) *There is an $i \in N$ and $Q^i \in L$ such that $F(R^{N\setminus\{i\}}, Q^i) = x$.*

(b) *If $R^i = R^j$ for all $i, j \in N$, then there is a $Q \in L$ such that $F(R^{N\setminus\{k\}}, Q) = x$ for all $k \in N$.*

(c) $TC(R^N) \subseteq H(R^N) \subseteq PC(R^N)$.

Proof. (a) Let $y = F(R^N)$. If $y = x$ we are done. Otherwise, consider a profile $Z^N \in L^N$ with $t(Z^i) = x$ for all $i \in N$ and yZ^iz for all $z \in A \setminus \{x\}$ (i.e., x is ranked first and y second by every agent). Then $H(Z^N) = \{x\}$ by unanimity, and therefore R^N is not a Nash equilibrium of (F, Z^N). Hence, there is an $i \in N$ and a $Q^i \in L$ such that $F(R^{N\setminus\{i\}}, Q^i)Z^iy$, which implies that $F(R^{N\setminus\{i\}}, Q^i) = x$.

(b) This follows directly from (a) and anonymity of F.

(c) Suppose that $z \in A \setminus PC(R^N)$. Then there is a $\widetilde{z} \neq z$ such that $\widetilde{z}R^iz$ for all $i \in N$. If $Z^N \in L^N$ such that $F(Z^N) = z$, then by (a) there is an $i \in N$ and $Q^i \in L$ such that $F(Z^{N\setminus\{i\}}, Q^i) = \widetilde{z}$, so that Z^N is not a Nash equilibrium in (F, R^N). Hence, $z \notin H(R^N)$, and thus $H(R^N) \subseteq PC(R^N)$.[4]

Finally, suppose that $v \in TC(R^N)$, without loss of generality $t(R^1) = v$. Consider now a profile $Z^N \in L^N$ such that $Z^i = Z^j$ and $\widetilde{v}Z^iv$ for all $i, j \in N$ and $\widetilde{v} \in A$. By part (b) there is a $\widetilde{Z}^1 \in L$ such that $F(Z^{N\setminus\{1\}}, \widetilde{Z}^1) = v$, and since $F(Z^{N\setminus\{1\}}, \widetilde{Z}^1) \in H(Z^{N\setminus\{1\}}, \widetilde{Z}^1) \subseteq PC(Z^{N\setminus\{1\}}, \widetilde{Z}^1)$ by the first part of (c) we have that $v\widetilde{Z}^1\widetilde{v}$ for all $\widetilde{v} \in A$. Let Q^N be a Nash equilibrium in $(F, (Z^{N\setminus\{1\}}, \widetilde{Z}^1))$ such that $F(Q^N) = v$. Then we have $F(Q^{N\setminus\{i\}}, V^i) = v$ for all $i \in N \setminus \{1\}$ and $V^i \in L$ since $\widetilde{v}Z^iv$ for all $i \in N \setminus \{1\}$ and $\widetilde{v} \in A$. Since, moreover, $v = t(R^1)$, it follows that Q^N is a Nash equilibrium in (F, R^N). Therefore, $v = F(Q^N) \in H(R^N)$, and thus $TC(R^N) \subseteq H(R^N)$. □

The second lemma says that for every alternative x there is a collection of n preferences with one of them different from the $n-1$ other preferences, such that x is assigned by F to every profile containing that particular preference and at least $n-2$ of the other preferences.

Lemma 2.2. *Let H be a unanimous and anonymous social choice correspondence and let the anonymous selection F implement H in Nash equilibrium. Let $x \in A$. Then there are preferences $P_{x,1}, \ldots, P_{x,n-1}, P_{x,n} \in L$ with $P_{x,n} \neq P_{x,j}$ for all $j \in N \setminus \{n\}$ such that $F(R^N) = x$ for all $R^N \in L^N$ for which there is a permutation π of N and a $k \in N \setminus \{n\}$ with $R^{\pi(n)} = P_{x,n}$ and $R^{\pi(j)} = P_{x,j}$ for all $j \in \{1, \ldots, n-1\} \setminus \{k\}$.*

Proof. Let $Z^N \in L^N$ be a profile with $Z^i = Z^j$ and yZ^ix for all $i, j \in N$ and $y \in A$. By Lemma 2.1(b),(c) we have $F(Z^{N\setminus\{n\}}, V^n) = x$ for some $V^n \in L$ with $t(V^n) = x$. Let Q^N be a Nash equilibrium in $(F, (Z^{N\setminus\{n\}}, V^n))$. Then, clearly, $F(Q^{N\setminus\{i\}}, V^i) = x$ for every $i \in N \setminus \{n\}$ and $V^i \in L$. Also, $Q^n \neq Q^i$ for all $i \in N \setminus \{i\}$ since otherwise by anonymity $F(Q^{N\setminus\{i\}}, V^i) = x$ for every $i \in N$ and $V^i \in L$, contradicting Lemma 2.1(a). The lemma follows by taking $P_{x,i} = Q^i$ for every $i \in N$ and anonymity. □

[4]This part of the proof also follows from the facts that Nash-implementability of a social choice correspondence implies Maskin monotonicity, and this condition together with unanimity implies Pareto optimality. See e.g. Peleg et al (2005), Lemma 3.1.

3 Two agents

For the case of two agents we have the following impossibility result. We provide a proof, but the result also follows from a more general result in Hurwicz and Schmeidler (1978).

Theorem 3.1. *Let $n = 2$ and let H be a unanimous and anonymous social choice correspondence. Then H is not Nash implementable by an anonymous selection from it.*

Proof. Let F be an anonymous selection from H, and let $(R^1, R^2) \in L^N$ such that $t(R^1) \neq t(R^2)$. Let $x = F(R^1, R^2) \in H(R^1, R^2)$, and suppose that $(Q^1, Q^2) \in L^N$ is a Nash equilibrium in $(F, (R^1, R^2))$ such that $F(Q^1, Q^2) = x$. We derive a contradiction, which completes the proof. By applying Lemma 2.1(b) to the profile (Q^1, Q^1) it follows that $F(Q^1, R) = t(R^2)$ for some $R \in L$. By applying Lemma 2.1(b) to the profile (Q^2, Q^2) it follows that $F(R', Q^2) = t(R^1)$ for some $R' \in L$. Since (Q^1, Q^2) is a Nash equilibrium in $(F, (R^1, R^2))$ it follows that $t(R^1) = x = t(R^2)$, which is the desired contradiction. □

4 Two alternatives

Throughout this section $A = \{a, b\}$, and in view of Theorem 3.1 we assume $n \geq 3$. We say that a profile R^N is of the form $a^k b^{n-k}$ if a occurs exactly k times on top at R^N, where $0 \leq k \leq n$. We also use this expression to denote a(ny) profile of that form. This notation is convenient since we only consider anonymous social choice functions and correspondences. We use the expressions ab and ba for the preferences with a and b on top, respectively.

In Lemmas 4.1–4.6 we assume that H is a unanimous and anonymous social choice correspondence and that the anonymous selection F from H implements H in Nash equilibrium.

Lemma 4.1. *$H = TC = PC$.*

Proof. Follows directly from Lemma 2.1(c). □

Lemma 4.2. *$F(a^1 b^{n-1}) = a$ and $F(a^{n-1} b^1) = b$.*

Proof. Since $F(b^n) = b$ by unanimity of H, Lemma 2.1(b) implies $F(a^1 b^{n-1}) = a$. Since $F(a^n) = a$ by unanimity of H, Lemma 2.1(b) implies $F(a^{n-1} b^1) = b$. □

Lemma 4.3. *$F(a^2 b^{n-2}) = a$ and $F(a^{n-2} b^2) = b$.*

Proof. We only prove the first claim, the proof of the other claim is similar. By Lemma 4.2 we have $F(R^N) = a$, where $R^N = a^1 b^{n-1}$. Let Q^N be a Nash equilibrium in $(F, a^1 b^{n-1})$ with $F(Q^N) = a$, say $Q^N = a^k b^{n-k}$ for some $k \in \{1, \ldots, n-2, n\}$. By Lemma 2.1(a) and anonymity of F we must have

$$F(a^{k-1} b^{n-k+1}) = b \tag{2}$$

or

$$F(a^{k+1}b^{n-k-1}) = b. \tag{3}$$

Without loss of generality in our profile $R^N = a^1b^{n-1}$ let $R^1 = ab$. If both (2) and (3) are true then there is $j \in N \setminus \{1\}$ such that $F(Q^{N\setminus\{j\}}, \widetilde{Q}^j) = b$ for $\widetilde{Q}^j$ with $t(\widetilde{Q^j}) \neq t(Q^j)$. This contradicts that Q^N is a Nash equilibrium in (F, a^1b^{n-1}). If only (3) is true then, since Q^N is a Nash equilibrium in (F, a^1b^{n-1}), $Q^j = ab$ for all $j \in N \setminus \{1\}$. Then by Lemma 4.2, also $Q^1 = ab$, but $Q^N = a^n$ is not a Nash equilibrium in (F, a^1b^{n-1}) again by Lemma 4.2. So we have that only (2) is true, which implies, again by the fact that Q^N is a Nash equilibrium in (F, a^1b^{n-1}), that $Q^j = ba$ for all $j \in N \setminus \{1\}$. Since $F(Q^N) = a$, this implies that $Q^1 = ab$, hence $Q^N = a^1b^{n-1}$, and since (3) does not hold, $F(a^2, b^{n-2}) = a$. □

Lemma 4.4. *$n \neq 3$ and $n \neq 4$.*

Proof. If $n = 3$, then $F(a^2b) = b$ by Lemma 4.2, and $F(a^2b) = a$ by Lemma 4.3, contradiction. If $n = 4$, then both $F(a^2b^2) = a$ and $F(a^2b^2) = b$ by Lemma 4.3, contradiction. □

Lemma 4.5. *There is no $k \in \{2, \ldots, n-2\}$ such that $F(a^{k-1}, b^{n-k+1}) = F(a^kb^{n-k}) = F(a^{k+1}, b^{n-k-1})$.*

Proof. This follows from Lemma 2.1(a) by considering a profile a^kb^{n-k}. □

Lemma 4.6. *$n \neq 6$.*

Proof. For $n = 6$, by Lemmas 4.2 and 4.3 we have $F(a^1b^5) = F(a^2b^4) = a$, hence by Lemma 4.5, $F(a^3b^3) \neq a$. Again by Lemmas 4.2 and 4.3, $F(a^4b^2) = F(a^5b^1) = b$, hence by Lemma 4.5, $F(a^3b^3) \neq b$. This contradiction completes the proof. □

Lemmas 4.1–4.6 provide necessary conditions for a unanimous and anonymous social choice correspondence to be Nash implementable by an anonymous selection. We now drop our assumptions on H and F and proceed with the converse.

Lemma 4.7. *Let social choice function F be an anonymous selection from PC satisfying*

(i) *$F(a^n) = F(a^1b^{n-1}) = F(a^2b^{n-2}) = a$,*

(ii) *$F(b^n) = F(a^{n-1}b^1) = F(a^{n-2}b^2) = b$,*

(iii) *there is no $k \in \{2, \ldots, n-2\}$ such that $F(a^{k-1}, b^{n-k+1}) = F(a^kb^{n-k}) = F(a^{k+1}, b^{n-k-1})$.*

Then F Nash implements PC.

Proof. Define the correspondence $H : L^N \to 2^{\{a,b\}}$ by

$$H(R^N) = \{x \in \{a,b\} \mid x = F(Q^N) \text{ for some Nash equilibrium } Q^N \text{ in } (F, R^N)\}$$

for every $R^N \in L^N$. It is sufficient to prove that $H = PC$.

If $R^N = a^n$ then $F(R^N) = a$ by (i), and R^N is a Nash equilibrium in (F, R^N), so that $a \in H(R^N)$. Suppose that $Q^N \in L^N$ such that $F(Q^N) = b$. By (i), (ii), and (iii) it follows that there is some $i \in N$ and $\widetilde{Q}^i \in L$ such that $F(Q^{N\setminus\{i\}}, \widetilde{Q}^i) = a$. Hence Q^N is not a Nash equilibrium in (F, R^N). Therefore, $H(a^n) = \{a\} = PC(a^n)$.

Similarly one proves $H(b^n) = \{b\} = PC(b^n)$.

Now let $1 \leq k \leq n-1$ and consider a profile $R^N = a^k b^{n-k}$. Let, without loss of generality, $R^i = ab$ for $i = 1, \ldots, k$. Let $Q^N = a^{n-1}b^1$ with $Q^n = ba$. Then $F(Q^n) = b$ by (ii). To show that Q^N is a Nash equilibrium in (F, R^N) it is sufficient to observe that $F(ba, Q^{N\setminus\{i\}}) = b$ for every $i \in \{1, \ldots, k\}$ by (ii). Analogously, $V^N = a^1 b^{n-1}$ with $t(V^1) = a$ is a Nash equilibrium in (F, R^N) with $F(V^N) = a$. Hence, $H(a^k b^{n-k}) = \{a, b\} = PC(a^k b^{n-k})$ for $1 \leq k \leq n-1$, which completes the proof of the lemma. □

We summarize the results of this section as follows.

Corollary 4.8. *Let $A = \{a, b\}$.*

(a) *If $n \in \{2, 3, 4, 6\}$ then there exists no unanimous and anonymous social choice correspondence that is Nash implementable by an anonymous selection.*

(b) *If $n \notin \{2, 3, 4, 6\}$ then the only unanimous and anonymous social choice correspondence that is Nash implementable by an anonymous selection F is the Pareto correspondence PC. Moreover, F Nash implements PC if and only if it satisfies conditions* (i)–(iii) *in Lemma 4.7.*

Observe that the Pareto and top correspondences coincide in this case. Also, the implementing selection F in Corollary 4.8 is not necessarily unique. The conditions (i)–(iii) in Lemma 4.7 uniquely determine $F(a^k b^{n-k})$ for $k = n - 3, \ldots, n$ and for $k = 0, \ldots, 3$, but for other values of k there is freedom of choice subject to condition (iii). For $n = 5$ and $n = 7$ we have uniqueness of F, but for $n = 8$ and $n = 9$ there are each time two possibilities. For $n = 8$ we may have $F(a^4 b^4) = a$ or $F(a^4 b^4) = b$. For $n = 9$ we may have $F(a^5 b^4) = a$ and $F(a^6 b^3) = b$ or $F(a^5 b^4) = b$ and $F(a^6 b^3) = a$. The number of possible selections expands rapidly as n increases.

5 At least three agents and at least four alternatives

In game forms (mechanisms) that are used to implement social choice correspondences often strategies (messages) include mentioning an integer number, or sending a comparable message, in order to provide the agents with additional strategic power. Since in the case of self-implementation agents can exclusively report a preference, in Section 5.1 we will label these in a convenient way in order to replace the standard integer messaging. If there are at least four alternatives then there are many different preferences and therefore many different strategies or messages. This case is treated in Sections 5.2, 5.3, and 5.4. The case $m = 3$ is more subtle and will be studied in Section 6.

Throughout this section we write $A = \{x_1, \ldots, x_m\}$ and $M = \{1, \ldots, m\}$. Also, for a preference $R \in L$ we use notations such as $R = xyz\ldots$; $R = x_{i_1} \ldots x_{i_m}$; etc., meaning that $xRyRzR\ldots$; $x_{i_1}R \ldots Rx_{i_m}$; etc.

5.1 Labelling preferences

In this section we describe a labelling of the preferences. In fact, this will just be alphabetic labelling, with $x_1, \ldots, x_m$ as the 'alphabet'. Just for completeness, we include the formula.

For $\emptyset \neq L \subseteq M$ and $k \in L$ let $\pi(k, L)$ be the number of elements of L smaller than or equal to k, formally: $\pi(k, L) = |\{\ell \in L \mid \ell \leq k\}|$. Observe that $\pi(k, M) = k$ for every $k \in M$. We define

$$\nu(x_{i_1} \ldots x_{i_m}) = \sum_{\ell=1}^{m} (\pi(i_\ell, \{i_\ell, \ldots, i_m\}) - 1)\,(m - \ell)! + \pi\,(i_{m-1}, \{i_{m-1}, i_m\})$$

as the *label* of preference $R = x_{i_1} \ldots x_{i_m}$.

Example 5.1.
(a) Suppose $m = 3$ and let $A = \{a, b, c\}$ with $a = x_1$, $b = x_2$, $c = x_3$. The following table gives the six different preferences as columns with their top alternatives in the first row and their labels in the last row.

	a	a	b	b	c	c
R	b	c	a	c	a	b
	c	b	c	a	b	a
$\nu(R)$	1	2	3	4	5	6

(b) Let $A = \{a, b, c, d, e\}$ with $a = x_1$, etc. Then preference $R = cbaed$ has the label

$$\nu(R) = (3 - 1)(5 - 1)! + (2 - 1)(5 - 2)! + (1 - 1)(5 - 3)! + 2 = 56.$$

Preferences $c\ldots$ have labels 49–72, preferences $cb\ldots$ have labels 55–60, and preferences $cba\ldots$ have the labels 55 and 56.

As illustrated in Example 5.1, this way of labelling has the property that all preferences that share the same top alternative(s), are labelled consecutively with no gaps. This is important later on.

5.2 The top correspondence

In this subsection we show that the top correspondence can be Nash self-implemented for the case with at least three agents and at least four alternatives.

For each $j \in M$ we define the preferences $\widetilde{P}_j$ and P_j by

$$\widetilde{P}_j = \begin{cases} x_j x_1 \ldots x_{j-1} x_{j+1} \ldots x_m & \text{if } j > 1 \\ x_1 \ldots x_m & \text{if } j = 1 \end{cases}$$

and

$$P_j = \begin{cases} x_j x_m \ldots x_{j+1} x_{j-1} \ldots x_1 & \text{if } j < m \\ x_m \ldots x_1 & \text{if } j = m \end{cases}$$

In words, $\widetilde{P}_j$ is the preference with x_j on top and all the other alternatives in the ordering $x_1 \ldots x_m$, while P_j has x_j on top and the remaining alternatives in reverse ordering.

We define the selection F_1 from the top correspondence TC as follows. Let $R^N \in L^N$, and denote $\nu(R^N) = \sum_{i=1}^n \nu(R^i)$.

(1.1) If there is $j \in M$ and distinct $\ell, \ell' \in N$ such that $R^\ell = \widetilde{P}_j$ and $R^i = P_j$ for all $i \in N \setminus \{\ell, \ell'\}$, then $F_1(R^N) = x_j$.

(1.2) Otherwise, if $TC(R^N) = \{x_{j_1}, \ldots, x_{j_t}\}$ with $j_1 < \ldots < j_t$, then $F_1(R^N) = x_{j_s}$ where $s = 1 + \big(\nu(R^N) \mod t\big)$.

In words, F_1 picks x_j if one agent has preference $\widetilde{P}_j$ and all other agents except at most one have preference P_j, for some $j \in M$. In all other cases, F_1 chooses from the top alternatives by using modulo counting based on the labels of the preferences. Clearly, F_1 is an anonymous selection from TC. We have the following result.

Theorem 5.2. *F_1 implements TC in Nash equilibrium.*

Proof. (i) Let $R^N \in L^N$ and $x_j \in TC(R^N)$. Take $\ell \in N$ with $t(R^\ell) = x_j$. Then $Q^N \in L^N$ with $Q^\ell = \widetilde{P}_j$ and $Q^i = P_j$ for all $i \in N \setminus \{\ell\}$ is a Nash equilibrium in (F_1, R^N) such that $F_1(Q^N) = x_j$.

(ii) Let $R^N \in L^N$ and suppose that $Q^N \in L^N$ is a Nash equilibrium in (F_1, R^N) with, say, $F_1(Q^N) = x_j$. We have to show that $x_j \in TC(R^N)$. Suppose not.

If Q^N is of the format as in (1.1) with $Q^\ell = \widetilde{P}_j$ for some agent ℓ with, say, $t(R^\ell) = x_k$ for some $k \neq j$, then by (1.2) there is $\hat{Q}^\ell \in L$ with $t(\hat{Q}^\ell) = x_k$ such that $F_1(Q^{N \setminus \{\ell\}}, \hat{Q}^\ell) = x_k$: observe that $TC(Q^{N \setminus \{\ell\}}, \hat{Q}^\ell)$ contains at most three alternatives and that there are $(m-1)! \geq 6$ consecutive labels associated with preferences that have x_k on top. This contradicts that Q^N is a Nash equilibrium in (F_1, R^N).

If Q^N is of the format as in (1.2) then $TC(Q^N)$ contains at most m elements and each agent i can obtain $t(R^i)$ be deviating from Q^N by a preference $\hat{Q}^i \in L$ with $t(\hat{Q}^i) = t(R^i)$: there are $(m-1)!$ such preferences labelled consecutively, and $(m-1)! > m$ for $m \geq 4$. Hence, $F_1(Q^{N \setminus \{i\}}, \hat{Q}^i) = t(R^i)$ for at least one such $\hat{Q}^i$, contradicting again that Q^N is a Nash equilibrium in (F_1, R^N). $\square$

5.3 The Pareto correspondence

Here, we show that for at least three agents and at least four alternatives also the Pareto correspondence is Nash self-implementable, by a selection F_2 that can be seen as a refinement of F_1. Before we can define F_2 we need a few notations and a definition. First, if a subset B of A is used in the notation of a preference R, e.g., $R = \ldots B \ldots$, this means that each alternative of $A \setminus B$ is ranked either above or below the alternatives of B, and that the alternatives of B are ranked in the order $x_1 \ldots x_m$: that is, if $x_j, x_k \in B$ and $j < k$ then $x_j R x_k$. The notation $R = \ldots \overline{B} \ldots$ has a similar meaning except that now the alternatives of B are ranked in reverse order: if $x_j, x_k \in B$ and $j < k$ then $x_k R x_j$.

Let $R^N \in L^N$. We call R^N an $(x, (B_i)_{i \in K})$-*profile* if $x \in A$, $\emptyset \neq K \subseteq N$, and $(B_i)_{i \in K}$ is a partition of $A \setminus \{x\}$ such that:

- $R^i = x\,(A \setminus (B_i \cup \{x\}))\,B_i$ for every $i \in K$,
- $R^i = x\,\overline{A \setminus \{x\}}$ for every $i \in N \setminus K$.

In words, for an agent $i \in K$, in R^i the top alternative is x, next the alternatives that are not in B_i are ranked in the order $x_1 \ldots x_m$, and finally the alternatives in B_i are ranked in the same order. For an agent not in K also x is the top alternative, but all remaining alternatives are ranked in reverse order. Observe that a profile with agent $\ell \in N$ having a preference $\widetilde{P}_j$ for some $j \in M$ and all other agents having preference P_j, as defined in Section 5.2, is a special case: it is an $(x_j, (B_i)_{i \in K})$-profile with $K = \{\ell\}$ and $B_\ell = A \setminus \{x_j\}$.

We first show that in an $(x_j, (B_i)_{i \in K})$-profile the set K and the sets B_i are uniquely determined.

Lemma 5.3. *Let R^N be an $(x, (B_i)_{i \in K})$-profile and also an $(x, (C_i)_{i \in K'})$-profile. Then $K = K'$ and $B_i = C_i$ for each $i \in K$.*

Proof. First observe that, since $m \geq 4$, by definition of the preferences we have $N \setminus K = N \setminus K'$, and thus also $K = K'$. Next, the only possible preference of an agent $i \in K$ where B_i and C_i could be different would be a preference $R^i = x(A \setminus \{x\})$, but then there can be only one such preference in R^K (since such preferences share the same bottom alternative), so that $B_i = (A \setminus \{x\}) \setminus \cup_{\ell \in K \setminus \{i\}} B_\ell = (A \setminus \{x\}) \setminus \cup_{\ell \in K \setminus \{i\}} C_\ell = C_i$. □

In view of Lemma 5.3, if R^N is an $(x, (B_i)_{i \in K})$-profile, it is *the* $(x, (B_i)_{i \in K})$-profile.

We next define the social choice function F_2, which is a selection from the Pareto correspondence (actually, from the top correspondence), by the following procedure. Let $R^N \in L^N$.

(2.1) If there is an $\ell \in N$ and $x_j \in A$ such that $R^\ell \neq P_j$ and $(R^{N \setminus \{\ell\}}, P_j)$ is the $(x_j, (B_i)_{i \in K})$-profile, then $F_2(R^N) = x_j$. (Recall that $P_j = x_j\,\overline{A \setminus \{x_j\}}$ and observe that, in this case, $\ell \notin K$.)

(2.2) If there is an $\ell \in N$, $x_j \in A$, and a $Q \in L$ such that $(R^{N\setminus\{\ell\}}, Q)$ is the $(x_j, (B_i)_{i\in K})$-profile with $\ell \in K$, then

$$F_2(R^N) = \begin{cases} t(R^\ell) & \text{if } t(R^\ell) \in B_\ell \\ x_j & \text{otherwise.} \end{cases}$$

(2.3) Otherwise, if $TC(R^N) = \{x_{j_1}, \ldots, x_{j_t}\}$ with $j_1 < \ldots < j_t$, then $F_2(R^N) = x_{j_s}$ where $s = 1 + \big(\nu(R^N) \mod t\big)$.

In words, if R^N is the $(x_j, (B_i)_{i\in K})$-profile except for the preference of exactly one agent outside K, then F_2 assigns x_j. If R^N is the $(x_j, (B_i)_{i\in K})$-profile except for the preference of at most one agent ℓ inside K, then F_2 assigns x_j unless agent $\ell \in K$ reports a preference with an alternative from B_ℓ on top: this is case (2.2), which is well-defined in view of Lemma 5.3. Also observe that cases (2.1) and (2.2) are exclusive. In all remaining cases F_2, like F_1 above, chooses from the top alternatives by using modulo counting based on the labels of the preferences: this is (2.3). Observe that if R^N is the $(x_j, (B_i)_{i\in K})$-profile then $F(R^N) = x_j$: this follows from (2.2).

Clearly, F_2 is an anonymous selection from the top correspondence, hence also from PC. We will prove that F_2 Nash implements the Pareto correspondence and to this end first prove that it satisfies the necessary condition stated in Lemma 2.1(a).

Lemma 5.4. *Let $R^N \in L^N$ and $y \in A$. Then there is an $i \in N$ and $Q^i \in L$ such that $F_2(R^{N\setminus\{i\}}, Q^i) = y$.*

Proof. We are done if $F_2(R^N) = y$. Now suppose $F_2(R^N) = x_j \neq y$. We distinguish three cases.

(a) If $(R^{N\setminus\{\ell\}}, P_j)$ is the $(x_j, (B_i)_{i\in K})$-profile for some agent ℓ with $R^\ell \neq P_j$, then by (2.3) every agent $i \in N \setminus \{\ell\}$ has a preference $Q^i \in L$ with $t(Q^i) = y$ such that $F_2(R^{N\setminus\{i\}}, Q^i) = y$ (namely, such a profile has at most three top alternatives and there are at least six preferences with top alternative y that have consecutive labels).

(b) If R^N is as in (2.2), with notations as there, then either $y \in B_\ell$ and then $F_2(R^{N\setminus\{\ell\}}, Q^\ell)$ for any $Q^\ell \in L$ with $t(Q^\ell) = y$, or $y \notin B_\ell$. In the latter case, if $R^\ell = Q$, then by (2.2) agent $i \in K \setminus \{\ell\}$ with $y \in B_i$ has a preference $Q^i \in L$ with $t(Q^i) = y$ such that $F_2(R^{N\setminus\{i\}}, Q^i) = y$; and if $R^\ell \neq Q$ then by (2.3) each agent $i \in N \setminus \{\ell\}$ has a preference $Q^i \in L$ with $t(Q^i) = y$ such that $F_2(R^{N\setminus\{i\}}, Q^i) = y$ (by an argument similar as in case (a)).

(c) If (2.3) applies to R^N then there is an agent $i \in N$ such that (2.3) still applies to every $(R^{N\setminus\{i\}}, Q^i)$ with $t(Q^i) = y$. By a similar argument as in the last paragraph of the proof of Theorem 5.2, there is a Q^i with $t(Q^i) = y$ and $F_2(R^{N\setminus\{i\}}, Q^i) = y$. □

Theorem 5.5. *F_2 implements PC in Nash equilibrium.*

Proof. (a) First, let $R^N \in L^N$ and $x \in PC(R^N)$. For each agent $i \in N$ let $L(x, R^i)$ denote the strict lower contour set of x at R^i, that is, $L(x, R^i) = \{y \in$

$A \setminus \{x\} \mid xR^iy\}$. Define $B_1 = L(x, R^1)$, $B_2 = L(x, R^2) \setminus B_1$, ..., and in general $B_i = L(x, R^i) \setminus (B_1 \cup \ldots \cup B_{i-1})$ for every $i \in N$. Let $K = \{i \in N \mid B_i \neq \emptyset\}$. Since $x \in PC(R^N)$, we have $\cup_{i \in N} L(x, R^i) = A \setminus \{x\}$, and in turn this implies that $(B_i)_{i \in K}$ is a partition of $A \setminus \{x\}$. Now let Q^N be the $(x, (B_i)_{i \in K})$-profile. Then $F_2(Q^N) = x$ by (2.2). By (2.1), $F_2(Q^{N \setminus \{i\}}, V^i) = x$ for every $i \notin K$ and $V^i \in L$. By (2.2), $F_2(Q^{N \setminus \{i\}}, V^i) \in \{x\} \cup B_i \subseteq \{x\} \cup L(x, R^i)$ for every $i \in K$ and $V^i \in L$. Hence, Q^N is a Nash equilibrium in (F_2, R^N).

(b) For the converse, let $x \in A$ and $Q^N \in L^N$ such that $F_2(Q^N) = x$. Let $R^N \in L^N$ such that $x \notin PC(R^N)$. It is sufficient to prove that Q^N is not a Nash equilibrium in (F, R^N). Since $x \notin PC(R^N)$ there is a $y \in A \setminus \{x\}$ such that yR^ix for every $i \in N$. By Lemma 5.4 there is an $\ell \in N$ and $V^\ell \in L$ such that $F_2(Q^{N \setminus \{\ell\}}, V^\ell) = y$. This implies that Q^N is not a Nash equilibrium in (F, R^N). □

5.4 Between top and Pareto

In the case of four or more alternatives and more than two agents there are also other correspondences, apart from TC and PC, that are Nash self-implementable, as we will show now.

For $B \subseteq A$ define the social choice correspondence H_B by $H_B(R^N) = TC(R^N) \cup (PC(R^N) \cap B)$ for every $R^N \in L^N$. Then H_B is unanimous and anonymous, and $TC(R^N) = H_\emptyset(R^N) \subseteq H_B(R^N) \subseteq H_A(R^N) = PC(R^N)$ for every $R^N \in L^N$ for every $B \subseteq A$. In words, H_B picks all top alternatives, and from the Pareto optimal alternatives only those that are in B.

Fix $B \subseteq A$. We define the anonymous selection F_B from H_B as follows (notations as earlier).

(B.1) If there is $j \in M$ and distinct $\ell, \ell' \in N$ such that $R^\ell = \widetilde{P}_j$ and $R^i = P_j$ for all $i \in N \setminus \{\ell, \ell'\}$, then $F_B(R^N) = x_j$.

(B.2) If there is an $\ell \in N$ and $x_j \in B$ such that $R^\ell \neq P_j$ and $(R^{N \setminus \{\ell\}}, P_j)$ is the $(x_j, (B_i)_{i \in K})$-profile, then $F_B(R^N) = x_j$.

(B.3) If there is an $\ell \in N$, $x_j \in B$, and a $Q \in L$ such that $(R^{N \setminus \{\ell\}}, Q)$ is the $(x_j, (B_i)_{i \in K})$-profile with $\ell \in K$, then

$$F_B(R^N) = \begin{cases} t(R^\ell) & \text{if } t(R^\ell) \in B_\ell \\ x_j & \text{otherwise.} \end{cases}$$

(B.4) Otherwise, if $TC(R^N) = \{x_{j_1}, \ldots, x_{j_t}\}$ with $j_1 < \ldots < j_t$, then $F_B(R^N) = x_{j_s}$ where $s = 1 + \big(\nu(R^N) \mod t\big)$.

Observe that (B.1) and (B.4) correspond to (1.1) and (1.2), defining the selection F_1 implementing the top correspondence, while (B.2) and (B.3) correspond to (2.1) and (2.2) in the definition of F_2, implementing the Pareto correspondence, but restricted to the set B.

Theorem 5.6. *Let $B \subseteq A$. Then F_B implements H_B in Nash equilibrium.*

This theorem can be proved by similar arguments as in the proofs of Theorems 5.2 and 5.5: we omit the details.

It is an open question whether there are other Nash self-implementable anonymous and unanimous social choice correspondences in between the top and the Pareto correspondence.

6 Three alternatives

The case of three alternatives is more subtle, due to the fact that there are less different preferences and, thus, less different labels available for messaging.

As in the previous section, $A = \{x_1, \ldots, x_m\}$, with now $m = 3$. Throughout, however, we write $a = x_1$, $b = x_2$, and $c = x_3$. Instead of $\widetilde{P}_1, P_1, \widetilde{P}_2, P_2, \widetilde{P}_3$, P_3 (introduced in Section 5.2) we write $\widetilde{P}_a, P_a, \widetilde{P}_b, P_b, \widetilde{P}_c, P_c$, respectively, and repeat the table in Example 5.1:

R	$\widetilde{P}_a$	P_a	$\widetilde{P}_b$	P_b	$\widetilde{P}_c$	P_c
	a	a	b	b	c	c
	b	c	a	c	a	b
	c	b	c	a	b	a
$\nu(R)$	1	2	3	4	5	6

We first consider the top correspondence and next the Pareto correspondence.

6.1 The top correspondence

Theorem 5.2 still holds for $m = 3$ but needs a different proof.

Theorem 6.1. *Let $m = 3$ and $n \geq 3$. Then F_1 implements TC in Nash equilibrium.*

Proof. (i) Let $R^N \in L^N$ and let $x \in TC(R^N)$. Take an agent $i \in N$ such that $t(R^i) = x$, and let Q^N be the profile with $Q^i = \widetilde{P}_x$ and $Q^j = P_x$ for all $j \in N \setminus \{i\}$. As in the proof of Theorem 5.2 it follows that Q^N is a Nash equilibrium in (F_1, R^N) with $F_1(Q^N) = x$.

(ii) For the converse, suppose that Q^N is a Nash equilibrium in (F_1, R^N) for some $Q^N, R^N \in L^N$. Without loss of generality let $F_1(Q^N) = a$. We wish to show that $a \in TC(R^N)$. Suppose not. Then without loss of generality a is ranked second everywhere at R^N, hence for every agent i, $R^i = \widetilde{P}_b$ or $R^i = \widetilde{P}_c$. We distinguish cases according to Q^N and show that in each case we have a contradiction. In these cases we assign specific roles to agents, which is without loss of generality in view of Anonymity. As before, we denote $\nu(Q^N) = \sum_{i \in N} \nu(Q^i)$, i.e., the sum of the labels of the preferences in Q^N.

(1) $Q^1 = \widetilde{P}_a$, $Q^2 = \ldots = Q^{n-1} = P_a$. We consider the following subcases.

(1a) $Q^n = \widetilde{P}_a$. Then for each agent $i \in \{2, \ldots, n-1\}$ we have $F_1(Q^{N\setminus\{i\}}, V^i) = b$ for $V^i = P_b$ or $V^i = \widetilde{P}_b$, and $F_1(Q^{N\setminus\{i\}}, V^i) = c$ for $V^i = P_c$ or $V^i = \widetilde{P}_c$.

(1b) $Q^n = P_a$. Similar to case (1a) but now for agent 1.

(1c) $Q^n \in \{P_b, \widetilde{P}_b\}$. Then for each agent $i \in \{1, \ldots, n-1\}$ we have $F_1(Q^{N\setminus\{i\}}, V^i) = b$ for $V^i = P_b$ or $V^i = \widetilde{P}_b$. Now suppose that $R^1 = \ldots = R^{n-1} = \widetilde{P}_c$ and $R^n \in \{\widetilde{P}_b, \widetilde{P}_c\}$, and suppose that $F_1(P_c, Q^{N\setminus\{1\}}) \neq c$ and $F_1(\widetilde{P}_c, Q^{N\setminus\{1\}}) \neq c$. This implies that $(\nu(Q^N) - 1 + 5) \mod 3 = 0$ and $(\nu(Q^N) - 1 + 6) \mod 3 = 1$. In this case, let agent 2 deviate to $\widetilde{P}_c$. Then $(\nu(Q^N) - 2 + 5) \mod 3 = 2$, implying that $F_1(Q^{N\setminus\{2\}}, \widetilde{P}_c) = c$.

(1d) $Q^n \in \{P_c, \widetilde{P}_c\}$. Then for each agent $i \in \{1, \ldots, n-1\}$ we have $F_1(Q^{N\setminus\{i\}}, V^i) = c$ for $V^i = P_c$ or $V^i = \widetilde{P}_c$. Now suppose that $R^1 = \ldots = R^{n-1} = \widetilde{P}_b$ and $R^n \in \{\widetilde{P}_b, \widetilde{P}_c\}$, and suppose that $F_1(P_b, Q^{N\setminus\{1\}}) \neq b$ and $F_1(\widetilde{P}_b, Q^{N\setminus\{1\}}) \neq b$. This implies that $(\nu(Q^N) - 1 + 3) \mod 3 = 2$ and $(\nu(Q^N) - 1 + 4) \mod 3 = 0$. In this case, let agent 2 deviate to $\widetilde{P}_b$. Then $(\nu(Q^N) - 2 + 3) \mod 3 = 1$, implying that $F_1(Q^{N\setminus\{2\}}, \widetilde{P}_b) = b$.

(2) Q^N is not of the form as in (1). We consider the following subcases.

(2a) $TC(Q^N) = \{a\}$. Then, if $Q^i = P_a$ for some $i \in N$ then $F_1(Q^{N\setminus\{i\}}, V^i) = b$ for $V^i = P_b$ or $V^i = \widetilde{P}_b$, and $F_1(Q^{N\setminus\{i\}}, V^i) = c$ for $V^i = P_c$ or $V^i = \widetilde{P}_c$. Otherwise, the same holds for every $i \in N$.

(2b) $TC(Q^N) = \{a, b\}$. Then for every $i \in N$ there is $V^i \in L$ such that $F_1(Q^{N\setminus\{i\}}, V^i) = b$. Now suppose that $R^i = \widetilde{P}_c$ for every $i \in N$. If, at Q^N, there is a unique agent i with $Q^i \in \{P_b, \widetilde{P}_b\}$, then that agent has $V^i \in L$ such that $F_1(Q^{N\setminus\{i\}}, V^i) = c$. Suppose there are at least two agents at Q^N who have b on top. Suppose $Q^i = \widetilde{P}_b$ for some $i \in N$. Then $\nu(Q^{N\setminus\{i\}}, \widetilde{P}_c) \mod 3 = (\nu(Q^N) - 3 + 5) \mod 3 = 2$, so that $F_1(Q^{N\setminus\{i\}}, \widetilde{P}_c) = c$. Otherwise there is an agent $i \in N$ with $Q^i = P_b$. In that case, $\nu(Q^{N\setminus\{i\}}, P_c) \mod 3 = (\nu(Q^N) - 4 + 6) \mod 3 = 2$, so that $F_1(Q^{N\setminus\{i\}}, P_c) = c$.

(2c) $TC(Q^N) = \{a, c\}$. This case is almost identical to case (2b) by switching the roles of b and c.

(2d) $TC(Q^N) = \{a, b, c\}$. Then each for agent $i \in N$ with $t(Q^i) = a$ there are V^i, W^i such that $F_1(Q^{N\setminus\{i\}}), V^i) = b$ and $F_1(Q^{N\setminus\{i\}}), W^i) = c$. □

6.2 The Pareto correspondence

We show that the Pareto correspondence cannot be Nash self-implemented if, besides $m = 3$, also $n = 3$.

Theorem 6.2. *Let $n = 3$ and let the unanimous and anonymous social choice correspondence H be implementable in Nash equilibrium by the anonymous selection F. Then H is the top correspondence.*

Proof. Without loss of generality let $R^N = (P_a, P_a, P_b)$ and let $Q^N \in L^N$ such that $F(Q^N) = c$. In view of Lemma 2.1 it is sufficient to show that Q^N is not a Nash equilibrium in (F, R^N). Let $\{P_{x,i} \mid x \in \{a,b\},\ i \in N\}$ be a collection of preferences as in Lemma 2.2. If $Q^1 \in \{P_{a,i} \mid i \in N\}$ then $F(Q^1, V^2, Q^3) = a$ for some $V^2 \in \{P_{a,i} \mid i \in N\}$, so that Q^N is not a Nash equilibrium in (F, R^N). A similar argument holds if $Q^1 \in \{P_{b,i} \mid i \in N\}$ and if $Q^2 \in \{P_{x,i} \mid i \in N,\ x \in \{a,b\}\}$. If $Q^1 = Q^2$ then by Lemma 2.1(b), $F(Q^1, Q^2, V^3) = b$ for some $V^3 \in L$, so again Q^N is not a Nash equilibrium in (F, R^N). Hence, $Q^1 \neq Q^2$. If $Q^1 = P_{c,3}$ or $Q^2 = P_{c,3}$ then $F(Q^1, Q^2, V^3) = c$ for every $V^3 \in L$. By Lemma 2.1(a) this implies that there is a $V^1 \in L$ with $F(V^1, Q^2, Q^3) = a$ or a $V^2 \in L$ with $F(Q^1, V^2, Q^3) = a$, and in both cases Q^N is not a Nash equilibrium in (F, R^N). If both $Q^1 \neq P_{c,3}$ and $Q^2 \neq P_{c,3}$ then $F(V^1, Q^2, P_{c,3}) = F(Q^1, V^2, P_{c,3}) = c$ for all $V^1, V^2 \in L$. By Lemma 2.1(a) this implies that there is a $V^3 \in L$ such that $F(Q^1, Q^2, V^3) = b$, so that also in this case Q^N is not a Nash equilibrium in (F, R^N). □

The case of more than three agents is still open. In other words, for $m = 3$ and $n \geq 4$, does there exist a unanimous and anonymous social choice correspondence H with $H \neq TC$ and $TC(R^N) \subseteq H(R^N) \subseteq PC(R^N)$ for all $R^N \in L^N$, which is implementable in Nash equilibrium by an anonymous selection?

7 Concluding remarks

The following table summarizes the results of the paper. A $\ominus$ sign means nonexistence of a unanimous and anonymous social choice correspondence Nash implementable by an anonymous selection. The question marks indicate the open problems mentioned earlier.

		n		
		2	3	≥ 4
	2	$\ominus$	$\ominus$	$\ominus$ for $n = 4, 6$; $PC = TC$ otherwise
m	3	$\ominus$	only TC	TC, more?
	≥ 4	$\ominus$	TC, PC, and more	TC, PC, F_B ($B \subseteq A$), more?

We conclude with a further consideration of the requirement that the implementing social choice function be a selection from the social choice correspondence that is to be implemented, in other words, the self-implementation assumption. We do this by providing two examples. The first example is an implementation of TC by a social choice function that not always selects from TC, and thus shows the (not surprising) fact that a self-implementable social choice correspondence may be implementable by a direct mechanism that is not a selection. The second example shows that a subcorrespondence of TC may be implementable by a direct mechanism that is not a selection.

Example 7.1. Let $n \geq 3$, $m \geq 4$, and consider the social choice function $\hat{F}_1$ defined as follows.

(i) $\hat{F}_1(R^N) = x_2$ if R^N is a profile such that there is one agent with preference $x_1x_2x_3x_4 \ldots x_m$ and all other agents have preference $x_1x_3x_2x_4 \ldots x_m$.

(ii) $\hat{F}_1(R^N) = F_1(R^N)$ otherwise, where F_1 is defined by (1.1) and (1.2).

Then $\hat{F}_1$ Nash-implements TC, which can be seen by adding to the proof of Theorem 5.2 the consideration that a profile R^N as in (i) cannot be a Nash equilibrium in the game (F, Q^N) for any $Q^N \in L^N$. However, $\hat{F}_1$ is not a selection from TC.

Example 7.2. Let $n \geq 3$, $m \geq 4$, and let the social choice correspondence H be defined by

$$H(R^N) = \begin{cases} \{x_1\} & \text{if } t(R^i) = x_1 \text{ for all } i \in N \\ TC(R^N) \setminus \{x_1\} & \text{otherwise} \end{cases}$$

for all $R^N \in L^N$. We define the social choice function $\widetilde{F}_1$ as follows. Let $R^N \in L^N$, and denote $\nu(R^N) = \sum_{i=1}^{n} \nu(R^i)$.

(1a) If there is $\ell \in N$ such that $R^\ell = \widetilde{P}_1$ and $R^i = P_1$ for all $i \in N \setminus \{\ell\}$, then $\widetilde{F}_1(R^N) = x_1$.

(1b) If there is $j \in M \setminus \{1\}$ and distinct $\ell, \ell' \in N$ such that $R^\ell = \widetilde{P}_j$ and $R^i = P_j$ for all $i \in N \setminus \{\ell, \ell'\}$, then $\widetilde{F}_1(R^N) = x_j$.

(2) Otherwise, if $H(R^N) = \{x_{j_1}, \ldots, x_{j_t}\}$ with $j_1 < \ldots < j_t$, then $\widetilde{F}_1(R^N) = x_{j_s}$ where $s = 1 + \big(\nu(R^N) \mod t\big)$.

By adding to the proof of Theorem 5.2 the consideration that a profile as in case (1a) can only be a Nash equilibrium if every agent has x_1 as top alternative in the true profile, we see that $\widetilde{F}$ implements H. However, (2) implies that $\widetilde{F}$ is not a selection from H.

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