Choice on the simplex domain*

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Abstract. One unit of a good has to be divided among a group of agents who each are entitled to a minimal share, and these shares sum up to less than one. The associated set of choice problems consists of the unit simplex and all its full-dimensional subsimplices with the same orientation. We characterize all choice rules that are independent of irrelevant alternatives, continuous, and monotonic. We also consider the issue of rationalizability and show that in general, excluding cycles of any fixed length does not imply the strong axiom of revealed preference, that is, the exclusion of cycles of any length. For continuous three-agent choice rules, however, excluding cycles of length three implies the strong axiom of revealed preference. *Journal of Economic Literature* Classification Nos.: D11, D71.

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1 Introduction

In this paper we consider the following division problem: one unit of a perfectly divisible good is to be divided among n agents who each are entitled to a minimal share of this good, where these shares sum up to less than one; in other words, there is a surplus. Thus, the set of division problems consists of the unit simplex and all its full-dimensional subsimplices with the same orientation. Our approach to this division problem is choice-theoretic: we interpret a chosen division to be superior to those that are feasible but not chosen, and therefore assume that it should again be chosen if it is still available when the minimal entitlements increase. Expressed more formally, we consider choice rules that are independent of irrelevant alternatives in the sense of (assumption no. 7 in) Nash (1950). As is the case in all choice environments, this independence property is a necessary (but, in general, not sufficient) condition for rationalizability. We think of it as a fundamental requirement because it represents a coherence condition that is compelling in many situations. Note that, again in agreement with other models that involve the division of objects among a group of agents, the decision to impose this independence axiom reflects that we do not focus on strategic aspects of the problem under consideration. This observation illustrates a parallel with Nash's (1950) axiomatic approach to the bargaining problem; other than that, however, the two types of problem have different specifications, and results obtained in one setting cannot be directly imported into the other.

The other basic property we impose on choice rules is continuity with respect to the minimal shares. When combined with independence of irrelevant alternatives, the condition has an additional powerful consequence: if an agent receives more than the minimal share in a given division problem and this minimal share decreases, then the chosen division does not change (Lemma 1). We use the terms division problem and choice problem interchangeably in this paper.

For the general n-agent choice problem we add a third condition, called monotonicity. Suppose that, in a given division problem, an agent receives its minimal share at a chosen division. If this minimal share increases, then none of the other agents should benefit. This is a plausible fairness condition and facilitates the description of possible choice rules.

Our first main result (Theorems 1 and 2 in Section 3) is the characterization of all choice rules satisfying independence of irrelevant alternatives, continuity, and monotonicity. We refer to the resulting choice rules as path rules. A path collection is a set of monotonic and continuous curves for the set of agents N and for all subsets of N, in the simplex that represents the choice problem when all minimal shares are zero. These curves additionally satisfy a consistency condition. The associated rule assigns to the subsimplex corresponding to a choice problem the point where one of these curves enters the subsimplex; consistency ensures that if two different curves enter the subsimplex, they enter at the same point. For the case $n = 3$ we obtain a larger class of path rules by weakening the monotonicity condition (Theorem 3). Thus, the characterization result provides a precise answer to the question of what exactly the axioms entail.

In the final part of the paper (Section 5) we investigate the issue of rationalizability. In our setting, independence of irrelevant alternatives is equivalent to the weak axiom of revealed preference, which requires that in the preference relation induced by a choice rule there are no cycles of length two. The strong axiom of revealed preference demands that there are no cycles of any length. We first show that excluding cycles of length K or less does not necessarily exclude cycles of length $K + 1$ (Theorem 4)—thus, in particular, independence of irrelevant alternatives does not guarantee rationalizability, that is, the strong axiom of revealed preference. Second, we show that for the case of three agents adding continuity and non-existence of cycles of length three implies the strong axiom of revealed preference (Theorem 5). An extension of the latter result for $n > 3$ remains an open question.

The choice problem examined in this paper can be seen as the counterpart of the familiar bankruptcy problem, where the shares are claims which sum up to more than one and thus cannot all be satisfied. See Thomson (2015) for an overview of the literature on bankruptcy problems. Stovall (2014) considers the bankruptcy problem from a choice-theoretic perspective similar to ours.

Our paper contributes to the literature on revealed preference and rationalizability, starting with the seminal work of Samuelson (1938) and later Arrow (1959), Uzawa (1960), and Richter (1966, 1971). We refer to Bossert and Suzumura (2010) for a recent overview of this literature. The present paper is perhaps most closely related to Bossert and Peters (2009), where choice sets are compact and convex subsets of $\mathbb{R}^n$. Here, the domain of choice sets—the standard simplex in $\mathbb{R}^n$ and all its subsimplices with the same orientation—is much smaller. One of the common denominators in the literature (see, for instance, Rose, 1958, Peters and Wakker, 1991; 1994, Bossert, 1994, Blackorby, Bossert, and Donaldson, 1995, and Bossert and Peters, 2009) is that while in two dimensions (which corresponds to the case $n = 3$ in the present paper) cycles of any length are excluded if cycles of length two or three are excluded, this observation does not carry over to more than two dimensions. Thus, as mentioned above, the case of more than three agents remains an unsolved problem.

2 Basic definitions

We consider n-dimensional choice problems, where $n \in \mathbb{N} \setminus \{1\}$. The set of agents is $N = \{1, \ldots, n\}$. For $x, y \in \mathbb{R}^N$, we write $x \geq y$ if $x_i \geq y_i$ for all $i \in N$ and $x > y$ if $x_i > y_i$ for all $i \in N$. For $S \subseteq N$ and $x \in \mathbb{R}^N$ we adopt the notational conventions $x_S = (x_i)_{i \in S} \in \mathbb{R}^S$ and $x(S) = \sum_{j \in S} x_j$, with $x(\emptyset) = 0$. Further, we let $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \mid x \geq 0\}$ and $\Delta = \{x \in \mathbb{R}^N_+ \mid x(N) = 1\}$. We use $e^i \in \mathbb{R}^N$ to denote the i^{th} unit vector.

The elements of the set

$$A = \left\{\alpha \in \mathbb{R}^n_+ \,|\, \alpha(N) < 1\right\}$$

are used to parametrize the choice problems considered in this paper. For any $\alpha \in A$, the feasible set corresponding to α is given by

$$\Delta(\alpha) = \{x \in \mathbb{R}^n \,|\, x \geq \alpha \text{ and } x(N) = 1\}.$$

Observe that each $\Delta(\alpha)$ is a full-dimensional subsimplex of the unit simplex Δ in $\mathbb{R}^N$ with the same orientation. It represents the division problem where each agent i is entitled to a minimal share α_i. For $\alpha \in A$ and $i \in N$, the vector that is obtained by replacing α_i with α'_i is denoted by (α_{-i}, α'_i).

A *choice rule* is a map $C\colon A \to \Delta$ such that $C(\alpha) \geq \alpha$ for all $\alpha \in A$. We interpret $C(\alpha)$ as the choice from the simplex $\Delta(\alpha)$. Note that we restrict attention to single-valued choice rules.

The following properties of choice rules are considered in this paper.

Independence of irrelevant alternatives. For all $\alpha, \beta \in A$,

$$[C(\alpha) \geq \beta \geq \alpha] \;\Rightarrow\; C(\beta) = C(\alpha).$$

Continuity. For all sequences $\langle \alpha^q \rangle_{q \in \mathbb{N}}$ such that $\alpha^q \in A$ for all $q \in \mathbb{N}$ and all $\alpha \in A$,

$$\lim_{q \to \infty} \alpha^q = \alpha \;\Rightarrow\; \lim_{q \to \infty} C(\alpha^q) = C(\alpha).$$

Monotonicity. For all $\alpha \in A$, all $i \in N$ and $\alpha'_i \in (\alpha_i, 1)$ such that $(\alpha_{-i}, \alpha'_i) \in A$,

$$C(\alpha)_i = \alpha_i \;\Rightarrow\; C(\alpha_{-i}, \alpha'_i)_j \leq C(\alpha)_j \text{ for all } j \in N \setminus \{i\}.$$

The inequality $\beta \geq \alpha$ in our definition of the independence property is equivalent to the set inclusion $\Delta(\beta) \subseteq \Delta(\alpha)$.

Independence of irrelevant alternatives and continuity are well-established properties in choice theory. Monotonicity says that if the i^{th} constraint at a point $\alpha \in A$ is active at the choice from $\Delta(\alpha)$ and this constraint is increased, then the choice in the new situation cannot increase at any other constraint. In other words, if agent i obtains its minimal share and in the new choice (division) problem this minimal share increases, so that agent i will receive more, then no other agent will be better off in the new problem. This is a plausible property for a choice rule, and it will facilitate the description of the class of path choice rules in the next section.

The following lemma says that under independence of irrelevant alternatives and continuity, if an agent receives more than its minimal share, then the chosen outcome will not change if this minimal share decreases. This is a quite intuitive observation: under the assumption that the constraint corresponding to the individual in question is not active at the originally chosen point, any deviation as a consequence of relaxing the constraint has to happen in a continuous manner, which immediately generates a conflict with the independence property.

Lemma 1. *Let $n \in \mathbb{N} \setminus \{1\}$, let C be a choice rule satisfying independence of irrelevant alternatives and continuity, and let $\alpha \in A$, $i \in N$, and $t_i \in [0, \alpha_i)$. If $\alpha_i < C(\alpha)_i$, then $C(\alpha) = C(\alpha_{-i}, t_i)$.*

Proof. By way of contradiction, suppose that $C(\alpha) \neq C(\alpha_{-i}, t_i)$. By continuity, there exists $t'_i \in [t_i, \alpha_i)$ such that

$$C(\alpha_{-i}, t'_i) \geq \alpha \;\text{ and }\; C(\alpha_{-i}, t'_i) \neq C(\alpha).$$

Since $C(\alpha_i, t'_i) \geq \alpha \geq (\alpha_{-i}, t'_i)$, independence of irrelevant alternatives implies that

$$C(\alpha_{-i}, t'_i) = C(\alpha)$$

which is a contradiction. ■

Lemma 1, which will be extremely useful throughout the paper, is illustrated for $n = 3$ in Figure 1. Let $\alpha = (0, \frac{1}{2}, \frac{1}{8})$ and $y = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4}) = C(0, \frac{1}{2}, \frac{1}{8})$. We have $\alpha_3 = \frac{1}{8} < \frac{1}{4} = y_3$. Let $t_3 = 0$ and suppose, by way of contradiction, that $C(0, \frac{1}{2}, 0) = z = (\frac{7}{16}, \frac{1}{2}, \frac{1}{16}) \neq y$. Continuity implies that there exists $t'_3 \in [0, \frac{1}{8})$ such that

$$C\left(0, \frac{1}{2}, t'_3\right) \geq \alpha \text{ and } C\left(0, \frac{1}{2}, t'_3\right) = z' \neq y.$$

But we have $(0, \frac{1}{2}, \frac{1}{8}) \leq (0, \frac{1}{2}, t'_3)$ and independence of irrelevant alternatives implies that $C(0, \frac{1}{2}, \frac{1}{8}) = C(0, \frac{1}{2}, t'_3)$. This contradicts $C(0, \frac{1}{2}, \frac{1}{8}) = y$.

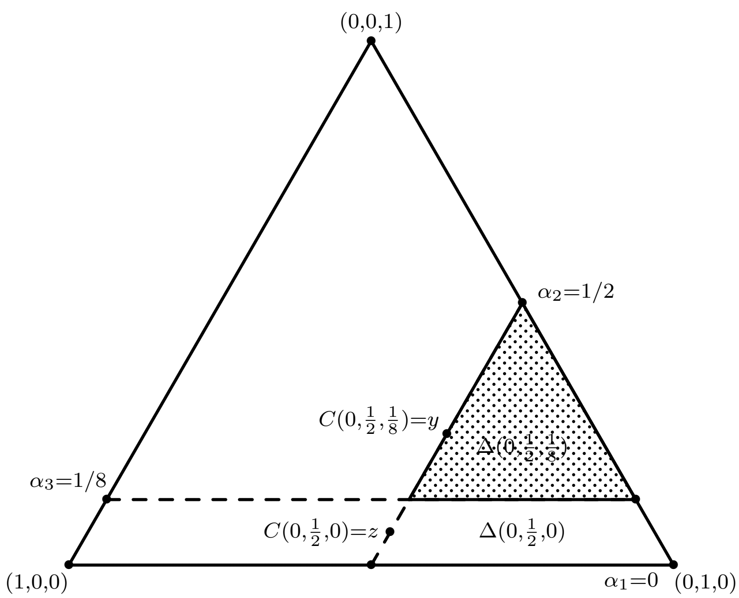

Figure 1: Proof of Lemma 1.

3 Path rules

In this section we define path rules and show that these are exactly all choice rules satisfying independence of irrelevant alternatives, continuity, and monotonicity. We first define S-paths (Definition 1), then path collections (Definition 2), and finally path rules (Definition 3).

Definition 1.
(a) Let $S \subseteq N$ with $|S| \geq 2$. An S-*path* is a $(1 + |S|)$-tuple $(p, (f^i)_{i \in S})$ where

(i) $p \in \Delta$ is such that $p(S) > 0$;

(ii) for each $i \in S$, $f^i\colon [p_i, p(S)] \to \{x \in \Delta \mid x_{N\setminus S} = p_{N\setminus S}\}$ is a continuous function such that $f^i(t_i)_i = t_i$ for each $t_i \in [p_i, p(S)]$.

(b) An S-path $(p, (f^i)_{i\in S})$ is *monotonic* if, for all $i \in S$ and all $t_i, t_i' \in [p_i, p(S)]$, $t_i' > t_i$ implies $f^i(t_i')_{S\setminus\{i\}} \leq f^i(t_i)_{S\setminus\{i\}}$.

Thus, an S-path is obtained by picking a point in the unit simplex Δ, fixing the coordinates of agents outside S, and starting from this point taking for each $i \in S$ a continuous increasing curve to the i-vertex of the subsimplex for S; monotonicity of such an S-path means that these curves do not increase in any of the other coordinates.

Observe that if $|S| = 2$ in Definition 1, say $S = \{i, j\}$, then an S-path $(p, (f^i, f^j))$ is uniquely determined by its starting point p. In this case, $f^i(t_i)_i = t_i$ and $f^i(t_i)_j = p_i + p_j - t_i$ for every $t_i \in [p_i, p_i + p_j]$; and $f^j(t_j)_j = t_j$ and $f^j(t_j)_i = p_i + p_j - t_j$ for every $t_j \in [p_j, p_i + p_j]$.

Definition 2.
(a) A *path collection Π* assigns to every $S \subseteq N$ with $|S| \geq 2$ a collection of S-paths, described recursively as follows:

⟨0⟩ $(p, (f^i)_{i\in N})$ is an N-path.

⟨1⟩ For all $i \in N$ and all $t_i \in [p_i, 1]$,

$$\big(f^i(t_i), (f^{i,t_i;j})_{j\in N\setminus\{i\}}\big)$$

is an $N \setminus \{i\}$-path.

(Hence, writing $q = f^i(t_i) \in \Delta$, for every $j \in N \setminus \{i\}$, $f^{i,t_i;j}\colon [q_j, q(N \setminus \{i\})] \to \{x \in \Delta \mid x_i = q_i\}$ is a continuous function such that $f^{i,t_i;j}(t_j)_j = t_j$ for every $t_j \in [q_j, q(N \setminus \{i\})]$. Thus, the superscripts of this function indicate that agent i is fixed at t_i and agent j is being considered. Observe that $q(N \setminus \{i\}) = f^i(t_i)(N \setminus \{i\}) = 1 - f^i(t_i)_i = 1 - t_i = 1 - q_i$.)

⟨k⟩ When $n \geq 4$, for $2 \leq k \leq n-2$, all distinct $i_1, \ldots, i_k \in N$, and all $t_{i_k} \in \left[f^{i_1,t_{i_1};\ldots;i_{k-2},t_{i_{k-2}};i_{k-1}}(t_{i_{k-1}})_{i_k}, 1 - \sum_{\ell=1}^{k-1} t_{i_\ell}\right]$,

$$\big(f^{i_1,t_{i_1};\ldots;i_{k-1},t_{i_{k-1}};i_k}(t_{i_k}), (f^{i_1,t_{i_1};\ldots;i_k,t_{i_k};j})_{j\in N\setminus\{i_1,\ldots,i_k\}}\big)$$

is an $N \setminus \{i_1, \ldots, i_k\}$-path, where $f^{i_1,t_{i_1};\ldots;i_k,t_{i_k};j}$ is defined in analogy to ⟨1⟩ (which is the case $k = 1$).

(b) Let $n \geq 4$. The path collection Π is *consistent* if for all $2 \leq k \leq n-2$, all distinct $i_1, \ldots, i_k$, and all permutations π of $i_1, \ldots, i_k$, if

$$t_{\pi(i_\ell)} \in \left[f^{\pi(i_1),t_{\pi(i_1)};\ldots;\pi(i_{\ell-2}),t_{\pi(i_{\ell-2})};\pi(i_{\ell-1})}(t_{\pi(i_{\ell-1})})_{\pi(i_\ell)}, 1 - \sum_{j=1}^{\ell-1} t_{\pi(i_j)}\right] \tag{1}$$

for every $\ell = 2, \ldots, k$, then

$$f^{i_1,t_{i_1};\ldots;i_{k-1},t_{i_{k-1}};i_k}(t_{i_k}) = f^{\pi(i_1),t_{\pi(i_1)};\ldots;\pi(i_{k-1}),t_{\pi(i_{k-1})};\pi(i_k)}(t_{\pi(i_k)})$$

for all $t_{i_1}, \ldots, t_{i_k}$ in the respective domains.

(c) The path collection Π is *continuous* if for the functions $f^{i_1,t_{i_1};\ldots;i_k,t_{i_k};j}$ in Step $\langle 1\rangle$ or Step $\langle k\rangle$ in (a) we have that, for all sequences $\langle (t^q_{i_1}, \ldots, t^q_{i_k})\rangle_{q\in\mathbb{N}}$ and all $(t'_{i_1}, \ldots, t'_{i_k})$,

$$\lim_{q\to\infty}(t^q_{i_1}, \ldots, t^q_{i_k}) = (t'_{i_1}, \ldots, t'_{i_k}) \;\Rightarrow\; \lim_{q\to\infty} f^{i_1,t^q_{i_1};\ldots;i_k,t^q_{i_k};j}(t_j) = f^{i_1,t'_{i_1};\ldots;i_k,t'_{i_k};j}(t_j)$$

for every t_j in the domain.

(d) The path collection Π is *monotonic* if each S-path belonging to it is monotonic, as defined in Definition 1(b).

Thus, a path collection starts with an N-path, consisting of n continuous curves from a point p to each of the vertices. Next, for each curve f^i and each point $f^i(t_i)$ on that curve, we have an $N \setminus \{i\}$ path in the subsimplex obtained by fixing the i^{th} coordinate at t_i. This process is repeated until we arrive at (uniquely determined) S-paths for $|S| = 2$.

Consistency of a path collection means that if we proceed a number of steps along the paths in the collection and reach a certain point, then we reach the same point if we permute the steps in this procedure; condition (1) restricts this requirement to well-defined movements.

Continuity requires that, in addition to the implicit property that all paths themselves are continuous, all these paths behave continuously with respect to changes in starting points.

Monotonicity simply means that all paths in the path collection are monotonic.

We will now define a choice rule based on a consistent, continuous, and monotonic path collection.

Definition 3.

Let Π be a path collection as in Definition 2(a), and let Π be consistent, continuous, and monotonic, as defined in Definition 2(b,c,d). The choice rule C^Π is determined via the following algorithm. Let $\alpha \in A$.

$\langle 0\rangle$ If $p^0 := p \geq \alpha$ then $C^\Pi(\alpha) = p^0$.
Otherwise, go to Step 1.

$\langle 1\rangle$ Let $N^1 = \{i \in N \mid \alpha_i > p^0_i\}$ and let $i_1, \ldots, i_{\ell_1}$ be an ordering of N^1.
If $p^1 := f^{i_1,\alpha_{i_1};\ldots;i_{\ell_1-1},\alpha_{i_{\ell_1-1}};i_{\ell_1}}(\alpha_{i_{\ell_1}}) \geq \alpha$, then $C^\Pi(\alpha) = p^1$.
Otherwise, go to Step 2.

$\langle 2\rangle$ Let $N^2 = \{i \in N \mid \alpha_i > p^1_i\}$ and let $i_{\ell_1+1}, \ldots, i_{\ell_2}$ be an ordering of N^2.
If $p^2 := f^{i_1,\alpha_{i_1};\ldots;i_{\ell_2-1},\alpha_{i_{\ell_2-1}};i_{\ell_2}}(\alpha_{i_{\ell_2}}) \geq \alpha$, then $C^\Pi(\alpha) = p^2$.
Otherwise, go to Step 3.

$\langle k \rangle$ Let $N^k = \{i \in N \mid \alpha_i > p_i^{k-1}\}$ and let $i_{\ell_{k-1}+1}, \ldots, i_{\ell_k}$ be an ordering of N^k.
If $p^k := f^{i_1,\alpha_{i_1};\ldots;i_{\ell_k-1},\alpha_{i_{\ell_k-1}};i_{\ell_k}}(\alpha_{i_{\ell_k}}) \geq \alpha$, then $C^{\Pi}(\alpha) = p^k$.
Otherwise, go to Step $k+1$.

In words, in order to determine the outcome in a particular division problem $\Delta(\alpha)$, we start from the solution p of the problem Δ and successively increase, along the curves in the N-path, the shares of all agents for which such an increase is possible. If we reach the subsimplex $\Delta(\alpha)$, the algorithm terminates and the choice is determined. Otherwise, we continue with the next set of agents along curves in the associated paths and continue until we reach $\Delta(\alpha)$. The consistency requirement guarantees that, in the involved consecutive steps, the order in which we increase the coordinates does not matter. The proof of the following theorem formalizes this. For the case $n = 3$, Section 4 provides some diagrammatic illustrations of the process involved in this construction.

Theorem 1. *Let Π be a consistent, continuous, and monotonic path collection. Then C^{Π} is well-defined, continuous, monotonic, and independent of irrelevant alternatives.*

Proof. Clearly, since the algorithm defining C^{Π} follows a sequence of functions in paths, it terminates at or before Step $n-1$ for every $\alpha \in A$. If the algorithm terminates at Step 1, then by consistency of Π the ordering of the elements of N^1 does not matter; moreover, monotonicity of each of the curves used in Step 1 ensures that Step 1 is well-defined for any ordering. Similarly, if the algorithm terminates at Step 2, then the ordering of the elements of N^2 does not matter, etc. Hence, C^{Π} is well-defined.

Continuity of C^{Π} follows from continuity of Π and continuity of the functions in every S-path in Π.

To prove monotonicity of C^{Π}, let $\alpha \in A$, $i \in N$, and $\alpha'_i \in (0,1)$ be such that $\alpha'_i > \alpha_i$ and $(\alpha_{-i}, \alpha'_i) \in A$, and let $j \in N$. Let $C^{\Pi}(\alpha)_i = \alpha_i$. We have to show that $C^{\Pi}(\alpha_{-i}, \alpha'_i)_j \leq C^{\Pi}(\alpha)_j$. Let N_α and $N_{\alpha'}$ denote the unions of the sets $N^1_\alpha, N^2_\alpha, \ldots$ and $N^1_{\alpha'}, N^2_{\alpha'}, \ldots$ in the algorithm for $C^{\Pi}(\alpha)$ and $C^{\Pi}(\alpha')$, respectively. If $i \notin N_{\alpha'}$, then $i \notin N_\alpha$ which, in turn, implies $C^{\Pi}(\alpha) = C^{\Pi}(\alpha')$ so that in particular $C^{\Pi}(\alpha_{-i}, \alpha'_i)_j \leq C^{\Pi}(\alpha)_j$. If $i \in N_{\alpha'}$, then if (say) $C^{\Pi}(\alpha') = f^{i_1,\alpha'_{i_1};\ldots;i_{\ell_k-1},\alpha'_{i_{\ell_k-1}};i_{\ell_k}}(\alpha'_{i_{\ell_k}})$, we may assume that $i = i_{\ell_k}$. Then either $C^{\Pi}(\alpha) = f^{i_1,\alpha_{i_1};\ldots;i_{\ell_k-1}}(\alpha_{i_{\ell_k-1}})$ (if the i^{th} coordinate of this point is equal to α_i) or $C^{\Pi}(\alpha) = f^{i_1,\alpha_{i_1};\ldots;i_{\ell_k-1},\alpha_{i_{\ell_k-1}};i_{\ell_k}}(\alpha_{i_{\ell_k}})$ (if the i^{th} coordinate of the previous point is strictly smaller than α_i). In either case, $C^{\Pi}(\alpha_{-i}, \alpha'_i)_j \leq C^{\Pi}(\alpha)_j$ follows from the monotonicity of the function $f^{i_1,\alpha_{i_1};\ldots;i_{\ell_k-1},\alpha_{i_{\ell_k-1}};i_{\ell_k}}$.

In order to show that C^{Π} is independent of irrelevant alternatives, let $\alpha, \beta \in A$ with $C^{\Pi}(\alpha) \geq \beta \geq \alpha$. We wish to show that $C^{\Pi}(\alpha) = C^{\Pi}(\beta)$. Clearly, if $p \geq \alpha$, then $p = C^{\Pi}(\alpha) \geq \beta$, hence $C^{\Pi}(\beta) = p = C^{\Pi}(\alpha)$. Now denote the sets $N^1, N^2, \ldots$ and the points $p^1, p^2, \ldots$ in the algorithm applied to α and β by $N^1_\alpha, N^2_\alpha, \ldots$; $N^1_\beta, N^2_\beta, \ldots$, and $p^1_\alpha, p^2_\alpha, \ldots$; $p^1_\beta, p^2_\beta, \ldots$, respectively. Suppose $C^{\Pi}(\alpha)$ is determined in Step $k \geq 1$. Then, for each $i \in N^1_\alpha \cup \ldots \cup N^k_\alpha$, we have $\alpha_i = C^{\Pi}(\alpha)_i \geq \beta_i \geq \alpha_i$ so that $\alpha_i = \beta_i$. This implies $N^1_\alpha \subseteq N^1_\beta$. Suppose that $j \notin N^1_\alpha \cup \ldots \cup N^k_\alpha$ but $j \in N^1_\beta$. Then $\beta_j > p_j \geq \alpha_j$ since, in particular, $j \notin N^1_\alpha$. Moreover, the monotonicity of Π implies that $p_j \geq C^{\Pi}(\alpha)_j$. Hence $p_j \geq \beta_j$, a contradiction. Therefore, $N^1_\beta \subseteq N^1_\alpha \cup \ldots \cup N^k_\alpha$, but then we have $N^1_\beta = N^1_\alpha$

since $\alpha_i = \beta_i$ for all $i \in N_\alpha^1 \cup \ldots \cup N_\alpha^k$. Next, suppose again that $j \notin N_\alpha^1 \cup \ldots \cup N_\alpha^k$, and assume that $N_\beta^r = N_\alpha^r$ for all $r = 1, \ldots, \ell - 1$, where $\ell \in \{2, \ldots, k\}$. Suppose that $j \in N_\beta^\ell$. Then $\beta_j > (p_\beta^{\ell-1})_j = (p_\alpha^{\ell-1})_j \geq \alpha_j$, and by monotonicity $(p_\alpha^{\ell-1})_j \geq C^\Pi(\alpha)_j$. Hence $(p_\beta^{\ell-1})_j = (p_\alpha^{\ell-1})_j \geq C^\Pi(\alpha)_j \geq \beta_j$, a contradiction. This implies $N_\beta^\ell \subseteq N_\alpha^1 \cup \ldots \cup N_\alpha^k$, and thus again $N_\beta^\ell = N_\alpha^\ell$. Since $C^\Pi(\alpha) \geq \beta$, we conclude that $C^\Pi(\beta) = C^\Pi(\alpha)$. ■

The next result establishes the converse of Theorem 1.

Theorem 2. *Let C be a choice rule satisfying independence of irrelevant alternatives, continuity, and monotonicity. Then there exists a consistent, continuous, and monotonic path collection Π such that $C = C^\Pi$.*

Proof. We iteratively construct a path collection Π as follows.

⟨0⟩ Let $p = C(0)$ and for every $i \in N$ let f^i be defined by $f^i(t_i) = C(t_i e^i)$ for every $t_i \in [p_i, 1]$. Then $f^i(t_i) = t_i$ by Lemma 1. Moreover, the functions f^i are continuous and monotonic by the continuity and monotonicity of C. Hence, $(p, (f^i)_{i \in N})$ is an N-path which is monotonic.

⟨1⟩ For all distinct $i, j \in N$ and $t_i \in [p_i, 1]$, let $f^{i,t_i;j}$ be defined by $f^{i,t_i;j}(t_j) = C(t_i e^i + t_j e^j)$ for every $t_j \in [f^i(t_i)_j, 1 - t_i]$. Then $f^{i,t_i;j}(t_j) = t_j$ by Lemma 1. Moreover, the functions $f^{i,t_i;j}$ are continuous and monotonic by the continuity and monotonicity of C. Hence, the pair $(f^i(t_i), (f^{i,t_i;j})_{j \in N \setminus \{i\}})$ is an $N \setminus \{i\}$-path which is monotonic.

⟨k⟩ For $2 \leq k \leq n - 2$, all distinct $i_1, \ldots, i_k \in N$, and $t_{i_1}, \ldots, t_{i_{k-1}}$ with $\sum_{j=1}^{k-1} t_{i_j} < 1$, let, for every $t_{i_k} \in [f^{i_1,t_{i_1};\ldots;i_{k-1}}(t_{i_{k-1}})_{i_k}), 1 - \sum_{j=1}^{k-1} t_{i_j}]$, $f^{i_1,t_{i_1};\ldots;i_{k-1},t_{i_{k-1}};i_k}(t_{i_k}) = C(\sum_{j=1}^{k} t_{i_j} e^{i_j})$. Then $f^{i_1,t_{i_1};\ldots;i_{k-1},t_{i_{k-1}};i_k}(t_{i_k}) = t_{i_k}$ by Lemma 1. Moreover, the functions $f^{i_1,t_{i_1};\ldots;i_{k-1},t_{i_{k-1}};i_k}$ are continuous and monotonic by the continuity and monotonicity of C. Hence, $(f^{i_1,t_{i_1};\ldots;i_{k-1}}(t_{i_{k-1}}), f^{i_1,t_{i_1};\ldots;i_{k-1},t_{i_{k-1}};j})_{j \in N \setminus \{i_1,\ldots,i_{k-1}\}})$ is an $N \setminus \{i_1, \ldots, i_{k-1}\}$-path which is monotonic.

The consistency of Π follows by construction, and the continuity of Π follows by the continuity of C. Finally, we have $C(\alpha) = C^\Pi(\alpha)$ for all $\alpha \in A$ by the construction of Π. ■

4 The case $n = 3$

In the case $n = 3$, a path collection is completely described by its N-path in Step 0 of Definition 2(a), since—as already observed—for distinct $i, j \in N$ the $\{i, j\}$-paths in Step 1 are uniquely determined by (ii) in Definition 1(a). This makes it is also easy to see that path collections are continuous and consistent by definition. Thus, for $N = \{1, 2, 3\}$ a path collection can be identified with an N-path $\Pi = (p, (f^i)_{i \in N})$. We weaken monotonicity of an N-path to the following condition: $\Pi = (p, (f^i)_{i \in N})$ is *weakly monotonic* if for all $i, j \in N$ and all $t_i \in [p_i, 1]$ and $t_j \in [p_j, 1]$ such that $t_i + t_j < 1$,

$$\left[f^i(t_i)_j \geq t_j \text{ and } f^j(t_j)_i \geq t_i\right] \Rightarrow f^i(t_i) = f^j(t_j). \tag{2}$$

For a monotonic N-path the premise in (2) is only satisfied for $t_i = p_i$ and $t_j = p_j$, in which case $f^i(t_i) = f^j(t_j) = p$, hence a monotonic N-path satisfies (2).

Now suppose that the choice rule C is independent of irrelevant alternatives and continuous. As in the proof of Theorem 2, let $p = C(0)$ and for each $i \in N$ and $t_i \in [p_i, 1]$ let $f^i(t_i) = C(t_i e^i)$. Then, by independence of irrelevant alternatives and continuity (and, in particular, Lemma 1), $\mathit{\Pi} = (p, (f^i)_{i\in N})$ is an N-path such that $C(\alpha) = C^{\mathit{\Pi}}(\alpha)$ for every $\alpha \in A$. Moreover, suppose that the premise of (2) holds. Then $f^i(t_i), f^j(t_j) \in \Delta(t_i e^i + t_j e^j)$; since $\Delta(t_i e^i + t_j e^j) \subseteq \Delta(t_i e^i) \cap \Delta(t_j e^j)$, independence of irrelevant alternatives of C implies $f^i(t_i) = f^j(t_j)$. Hence (2) holds.

Conversely, for an N-path $\mathit{\Pi} = (p, (f^i)_{i\in N})$ satisfying weak monotonicity we construct a choice rule $C^{\mathit{\Pi}}$ analogously to Definition 3—this construction now greatly simplifies as follows. Let $\alpha \in A$ and define $N_{\alpha>p} = \{i \in N \mid \alpha_i > p_i\}$. Let $d(\cdot,\cdot)$ denote the Euclidean distance. Then

(a) if $N_{\alpha>p} = \emptyset$, then $C^{\mathit{\Pi}}(\alpha) = p$;

(b) if $N_{\alpha>p} \neq \emptyset$, then $C^{\mathit{\Pi}}(\alpha) = \arg\min_{x\in\Delta(\alpha)}\{d(x, f^i(\alpha_i)) \mid i \in N_{\alpha>p}\}$.

It is not hard to show that, by (2), $C^{\mathit{\Pi}}(\alpha)$ in (b) is well-defined; and that $C^{\mathit{\Pi}}$ is independent of irrelevant alternatives and continuous. Summarizing, we have the following theorem.

Theorem 3. *Let $n = 3$. A choice rule C satisfies independence of irrelevant alternatives and continuity if and only if there is a weakly monotonic N-path $\mathit{\Pi} = (p, (f^i)_{i\in N})$ such that $C = C^{\mathit{\Pi}}$.*

For $n = 3$, an N-path $\mathit{\Pi} = (p, f) = (p, (f^i)_{i\in N})$ and the associated choice rule $C^{\mathit{\Pi}}$ is presented in the following example. Figures 2 to 5 provide illustrations for different choices of $(f^i)_{i\in N}$ and different values of the parameter $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Example 1. Throughout the example, the point p is given $p = (\frac{3}{8}, \frac{1}{8}, \frac{1}{2})$ and in each of Figures 2 to 4, the set $\Delta(\alpha)$ is indicated by the shaded area. The horizontal arrow pointing towards the vertex $(0, 1, 0)$ is used to indicate that the path associated with f^2 extends to that vertex along the bottom edge of the triangle once this edge is reached.

(a) In Figure 2, we let $\alpha = (0, 0, 0)$. The three curves leading from p to each of the three vertices of $\Delta(0, 0, 0)$ are associated with the functions f^1, f^2, and f^3. We obtain $C^{\mathit{\Pi}}(\alpha) = p$ by definition.

(b) In Figure 3, we illustrate the choice for $\alpha = (0, 0, \frac{1}{4})$. Because $N_{\alpha>p} = \emptyset$, part (a) of the definition of a choice rule associated with a path collection implies that $C^{\mathit{\Pi}}(\alpha) = p = (\frac{3}{8}, \frac{1}{8}, \frac{1}{2})$.

(c) Figure 4 depicts the choice for $\alpha = (\frac{1}{4}, \frac{3}{8}, \frac{1}{8})$. We have $p_1 = \frac{3}{8} \geq \frac{1}{4} = \alpha_1$, $p_2 = \frac{1}{8} < \frac{3}{8} = \alpha_2$ and $p_3 = \frac{1}{2} \geq \frac{1}{8} = \alpha_3$ and, thus, $N_{\alpha>p} = \{2\}$. Furthermore, $f^2(\alpha_2) = (\frac{1}{4}, \frac{3}{8}, \frac{3}{8}) = y \geq (\frac{1}{4}, \frac{3}{8}, \frac{1}{8}) = \alpha$. According to part (b) of our definition, we obtain $C^{\mathit{\Pi}}(\alpha) = (\frac{1}{4}, \frac{3}{8}, \frac{3}{8})$.

(d) Finally, Figure 5 illustrates the case $\alpha = (\frac{1}{2}, \frac{3}{8}, 0)$. According to our definition, we obtain $C^{\mathit{\Pi}}(\alpha) = (\frac{1}{2}, \frac{3}{8}, \frac{1}{8}) = y$. Note that we have $N_{\alpha>p} = \{1, 2\}$, $C_2(\alpha) = y_2 = \frac{3}{8} = \alpha_2$, and $C_1(\alpha) = y_1 = \frac{1}{2} = \alpha_1$.

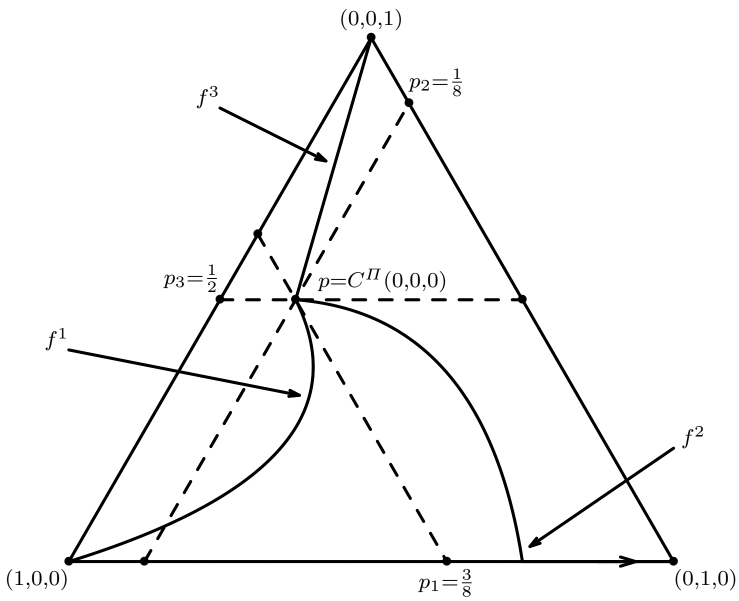

Figure 2: Example 1, part (a).

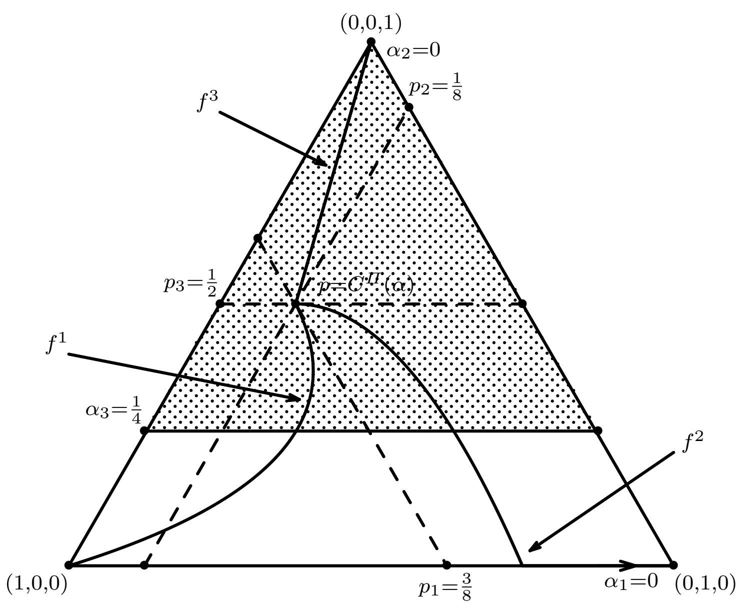

Figure 3: Example 1, part (b).

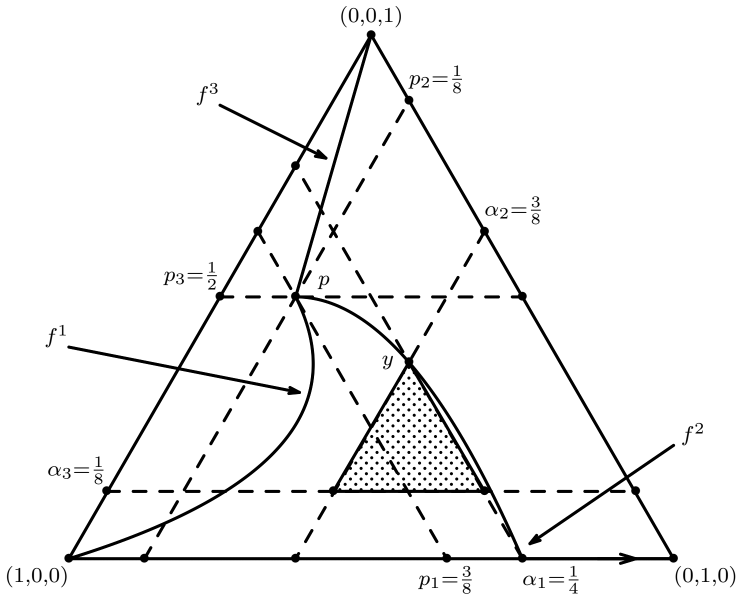

Figure 4: Example 1, part (c).

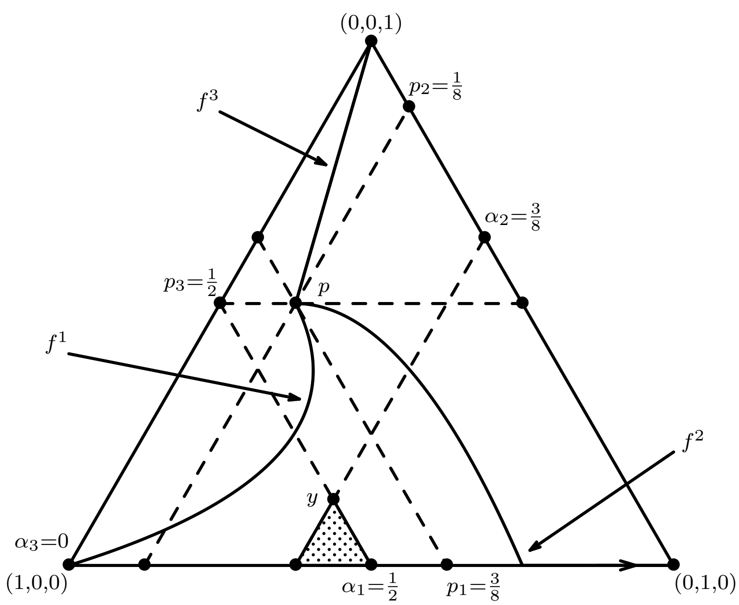

Figure 5: Example 1, part (d).

5 Rationalizability

The notion of rationalizability is a fundamental issue in the theory of individual and collective choice. In our context, a choice rule C is rationalizable if there exists an ordering R on Δ such that, for all $\alpha \in A$ and for all $x \in \Delta(\alpha)$,

$$C(\alpha) = x \;\Leftrightarrow\; xRy \text{ for all } y \in \Delta(\alpha).$$

The direct revealed preference relation R^C corresponding to the choice rule C is defined by letting, for all $x, y \in \Delta$, xR^Cy if there exists $\alpha \in A$ such that $C(\alpha) = x$ and $y \geq \alpha$. A revealed-preference cycle in C of length $K \in \mathbb{N} \setminus \{1\}$ is a set of distinct alternatives $\{x^1, \ldots, x^K\}$ such that $x^k R^C x^{k+1}$ for all $k \in \{1, \ldots, K\}$, where $x^{K+1} = x^1$. The following axioms are well-established in the theory of rational choice.

Weak axiom of revealed preference. There is no revealed-preference cycle of length two in C.

Strong axiom of revealed preference. For all $K \in \mathbb{N} \setminus \{1\}$, there is no revealed-preference cycle of length K in C.

It is well-known that, because our choice rule is single-valued, rationalizability is equivalent to the strong axiom of revealed preference. The strong axiom of revealed preference implies the weak axiom which, in turn, implies independence of irrelevant alternatives. This is true for a choice rule with an arbitrary domain. Because the domain of our choice rule is closed under intersection, the weak axiom of revealed preference is equivalent to the independence property. In some situations, the weak axiom of revealed preference is not sufficient to imply the strong axiom of revealed preference (and thus rationalizability) but ruling out cycles of length three does imply rationalizability. This is the case, in particular, if we restrict attention to the weakly monotone path rules for $n = 3$; see Theorem 5 below. This raises the question of whether for choice rules that are not necessarily weakly monotone path rules, ruling out cycles of a given length excludes longer cycles as well.

On our domain, independence of irrelevant alternatives (or, equivalently, the weak axiom of revealed preference) is not sufficient for rationalizability. In answer to the question raised at the end of the previous paragraph, we show the stronger result that for $n \geq 3$ and for any $K \in \mathbb{N}$, ruling out the existence of revealed-preference cycles of any length less than $K+1$ is not sufficient to rule out cycles of length $K+1$. This is established in the following theorem. For the case $n = 3$, we denote the left, right and top vertices of a simplex $\Delta(\alpha)$ by $\ell(\alpha)$, $r(\alpha)$, and $t(\alpha)$, hence $\ell(\alpha) = (1 - \alpha_2 - \alpha_3, \alpha_2, \alpha_3)$, $r(\alpha) = (\alpha_1, 1 - \alpha_1 - \alpha_3, \alpha_3)$, and $t(\alpha) = (\alpha_1, \alpha_2, 1 - \alpha_1 - \alpha_2)$.

Theorem 4. *For each $K \in \mathbb{N}$, there exists a choice rule C^{K+1} such that there is a revealed-preference cycle of length $K+1$ in C^{K+1} and no revealed-preference cycle of length less than $K+1$ in C^{K+1}.*

Proof. We first prove the theorem for the case $n = 3$. Next, we extend the constructed choice rules to the case $n > 3$. Until further notice we let $n = 3$.

For the case $K = 1$, consider the choice rule C^2 defined by letting, for all $\alpha \in A$,

$$C^2(\alpha) = \begin{cases} \ell(\alpha) & \text{if } \alpha = (0,0,0), \\ r(\alpha) & \text{if } \alpha \neq (0,0,0). \end{cases}$$

Then

$$\begin{aligned} C^2(0,0,0) = (1,0,0) \quad &\text{and} \quad \left(\frac{1}{2},\frac{1}{2},0\right) \geq (0,0,0); \\ C^2\left(\frac{1}{2},0,0\right) = \left(\frac{1}{2},\frac{1}{2},0\right) \quad &\text{and} \quad (1,0,0) \geq \left(\frac{1}{2},0,0\right) \end{aligned}$$

so that the points $(1,0,0)$ and $(\frac{1}{2},0,0)$ form a cycle of length two (and, trivially, there is no cycle of length less than two).

For the case $K = 2$, consider the choice rule C^3 defined by letting, for all $\alpha \in A$,

$$C^3(\alpha) = \begin{cases} \ell(\alpha) & \text{if } \alpha_3 \geq \frac{1}{2}\ , \\ r(\alpha) & \text{if } \alpha_1 \geq \frac{1}{2}\ , \\ t(\alpha) & \text{if } \alpha_2 \geq \frac{1}{2}\ , \\ \arg\min_{x \geq \alpha} d(x, (\frac{1}{3},\frac{1}{3},\frac{1}{3})) & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} C^3\left(0,\frac{1}{2},0\right) = \left(0,\frac{1}{2},\frac{1}{2}\right) \quad &\text{and} \quad \left(\frac{1}{2},\frac{1}{2},0\right) \geq \left(0,\frac{1}{2},0\right); \\ C^3\left(\frac{1}{2},0,0\right) = \left(\frac{1}{2},\frac{1}{2},0\right) \quad &\text{and} \quad \left(\frac{1}{2},0,\frac{1}{2}\right) \geq \left(\frac{1}{2},0,0\right); \\ C^3\left(0,0,\frac{1}{2}\right) = \left(\frac{1}{2},0,\frac{1}{2}\right) \quad &\text{and} \quad \left(0,\frac{1}{2},\frac{1}{2}\right) \geq \left(0,0,\frac{1}{2}\right) \end{aligned}$$

so that we have a cycle of length three formed by the points $(0,\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2},0)$ and $(\frac{1}{2},0,\frac{1}{2})$ but there is no cycle of length two.

For the case $K = 3$, consider the choice rule C^4 defined by letting, for all $\alpha \in A$,

$$C^4(\alpha) = \begin{cases} C^4_*(\alpha) & \text{if } \alpha_3 \geq \frac{1}{2}\ , \\ C^3(\alpha) & \text{if } \alpha_3 < \frac{1}{2}\ , \end{cases}$$

where $C^4_*\colon \{\alpha \in A \mid \alpha_3 \geq \frac{1}{2}\} \to \Delta(0,0,0)$ is defined by

$$C^4_*(\alpha) = \begin{cases} \ell(\alpha) & \text{if } \alpha_1 \geq \frac{1}{4} \text{ or } \alpha_2 \geq \frac{1}{4}\ , \\ \arg\min_{x \geq \alpha} d(x, (\frac{1}{8},\frac{1}{8},\frac{3}{4})) & \text{otherwise.} \end{cases}$$

Now we obtain

$$\begin{aligned}
C^4\left(0,\frac{1}{2},0\right) &= \left(0,\frac{1}{2},\frac{1}{2}\right) \quad \text{and} \quad \left(\frac{1}{2},\frac{1}{2},0\right) \geq \left(0,\frac{1}{2},0\right);\\
C^4\left(\frac{1}{2},0,0\right) &= \left(\frac{1}{2},\frac{1}{2},0\right) \quad \text{and} \quad \left(\frac{1}{2},0,\frac{1}{2}\right) \geq \left(\frac{1}{2},0,0\right);\\
C^4\left(\frac{1}{4},0,\frac{1}{2}\right) &= \left(\frac{1}{2},0,\frac{1}{2}\right) \quad \text{and} \quad \left(\frac{1}{4},\frac{1}{4},\frac{1}{2}\right) \geq \left(\frac{1}{4},0,\frac{1}{2}\right);\\
C^4\left(0,\frac{1}{4},\frac{1}{2}\right) &= \left(\frac{1}{4},\frac{1}{4},\frac{1}{2}\right) \quad \text{and} \quad \left(0,\frac{1}{2},\frac{1}{2}\right) \geq \left(0,\frac{1}{4},\frac{1}{2}\right)
\end{aligned}$$

so that C^4 has a cycle of length four formed by the points $(0,\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2},0)$, $(\frac{1}{2},0,\frac{1}{2})$, $(\frac{1}{4},\frac{1}{4},\frac{1}{2})$ but no cycles of smaller length.

This last choice rule can be generalized to all $K \geq 3$ as follows. Consider the choice rule C^{K+1} defined by letting, for all $\alpha \in A$,

$$C^{K+1}(\alpha) = \begin{cases} C_*^{K+1}(\alpha) & \text{if } \alpha_3 \geq \frac{1}{2} \text{ and } \alpha_2 \geq \frac{2^{K-3}-1}{2^{K-2}},\\ C^K(\alpha) & \text{otherwise,}\end{cases}$$

where $C_*^{K+1}\colon \{\alpha \in A \mid \alpha_3 \geq \frac{1}{2} \text{ and } \alpha_2 \geq \frac{2^{K-3}-1}{2^{K-2}}\} \to \Delta(0,0,0)$ is defined by

$$C_*^{K+1}(\alpha) = \begin{cases} \ell(\alpha) & \text{if } \alpha_1 \geq \frac{1}{2^{K-1}} \text{ or } \alpha_2 \geq \frac{2^{K-2}-1}{2^{K-1}},\\ \arg\min_{x\geq\alpha} d(x,(\frac{1}{2^K},\frac{2^K-2^{K-1}-3}{2^K},\frac{2^{K-1}+2}{2^K})) & \text{otherwise.}\end{cases}$$

We have

$$\begin{aligned}
C^{K+1}\left(0,\frac{1}{2},0\right) &= \left(0,\frac{1}{2},\frac{1}{2}\right) \quad \text{and} \quad \left(\frac{1}{2},\frac{1}{2},0\right) \geq \left(0,\frac{1}{2},0\right);\\
C^{K+1}\left(\frac{1}{2},0,0\right) &= \left(\frac{1}{2},\frac{1}{2},0\right) \quad \text{and} \quad \left(\frac{1}{2},0,\frac{1}{2}\right) \geq \left(\frac{1}{2},0,0\right);\\
C^{K+1}\left(\frac{1}{4},0,\frac{1}{2}\right) &= \left(\frac{1}{2},0,\frac{1}{2}\right) \quad \text{and} \quad \ldots \geq \left(\frac{1}{4},0,\frac{1}{2}\right);\\
&\vdots\\
C^{K+1}\left(\frac{1}{2^{K-1}},\frac{2^{K-3}-1}{2^{K-2}},\frac{1}{2}\right) &= \left(\frac{1}{2^{K-2}},\frac{2^{K-3}-1}{2^{K-2}},\frac{1}{2}\right) \quad \text{and} \quad \left(\frac{1}{2^{K-1}},\frac{2^{K-2}-1}{2^{K-1}},\frac{1}{2}\right)\\
&\qquad \geq \left(\frac{1}{2^{K-1}},\frac{2^{K-3}-1}{2^{K-2}},\frac{1}{2}\right);\\
C^{K+1}\left(0,\frac{2^{K-2}-1}{2^{K-1}},\frac{1}{2}\right) &= \left(\frac{1}{2^{K-1}},\frac{2^{K-2}-1}{2^{K-1}},\frac{1}{2}\right) \quad \text{and} \quad \left(0,\frac{1}{2},\frac{1}{2}\right) \geq \left(0,\frac{2^{K-2}-1}{2^{K-1}},\frac{1}{2}\right)
\end{aligned}$$

so that C^{K+1} has a cycle of length $K+1$ formed by the points $(0,\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2},0)$, $(\frac{1}{2},0,\frac{1}{2})$, $\ldots$, $(\frac{1}{2^{K-2}},\frac{2^{K-3}-1}{2^{K-2}},\frac{1}{2})$, $(\frac{1}{2^{K-1}},\frac{2^{K-2}-1}{2^{K-1}},\frac{1}{2})$, but no cycles of smaller length.

We now extend the choice rules constructed above to the case $n > 3$. Thus, let C be a choice rule for $N = \{1, 2, 3\}$ such that there is a revealed preference cycle of length $K+1$ (where $K \in \mathbb{N}$), but no revealed preference cycle of length less than $K+1$. Let $N = \{1, \ldots, n\}$ with $n \geq 4$. We define the choice rule C^* as follows. For each $\alpha \in A$,

$$C^*(\alpha) = \begin{cases} C(\alpha_1, \alpha_2, \alpha_3, 0, \ldots, 0) & \text{if } \alpha_4 = \ldots = \alpha_n = 0 \\ (\alpha_1, \ldots, \alpha_{n-1}, 1 - \alpha(N \setminus \{n\})) & \text{otherwise.} \end{cases}$$

In words, if agents $4, \ldots, n$ have minimal shares zero then we apply C to $\{1, 2, 3\}$ and agents $4, \ldots, n$ receive zero; otherwise, agent n receives its maximal possible share.

Then C^* is a choice rule for N such that there is a revealed preference cycle of length $K+1$, but no revealed preference cycle of length less than $K+1$. The proof of this is straightforward and left to the reader. ■

For $n = 3$, the cases $K = 1$, $K = 2$, $K = 3$, and $K = 4$ are illustrated in Figures 6 to 9.

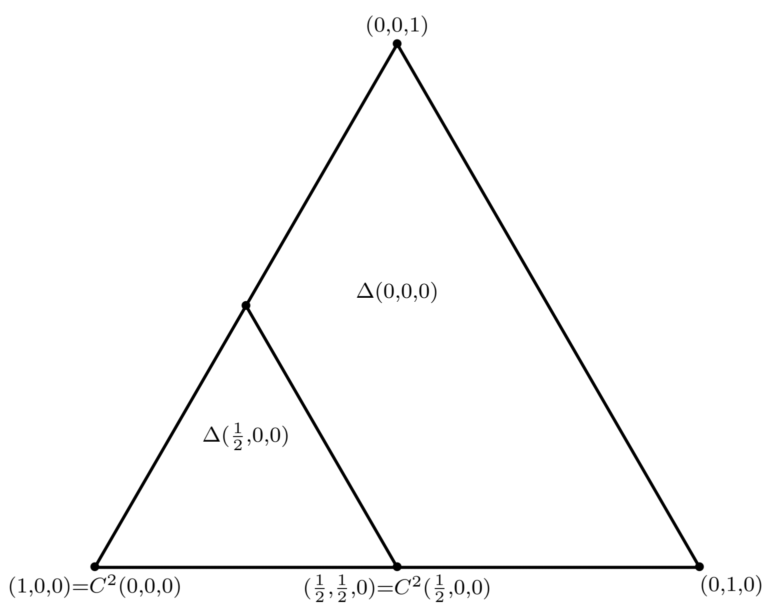

Figure 6: Theorem 4, the case $K=1$.

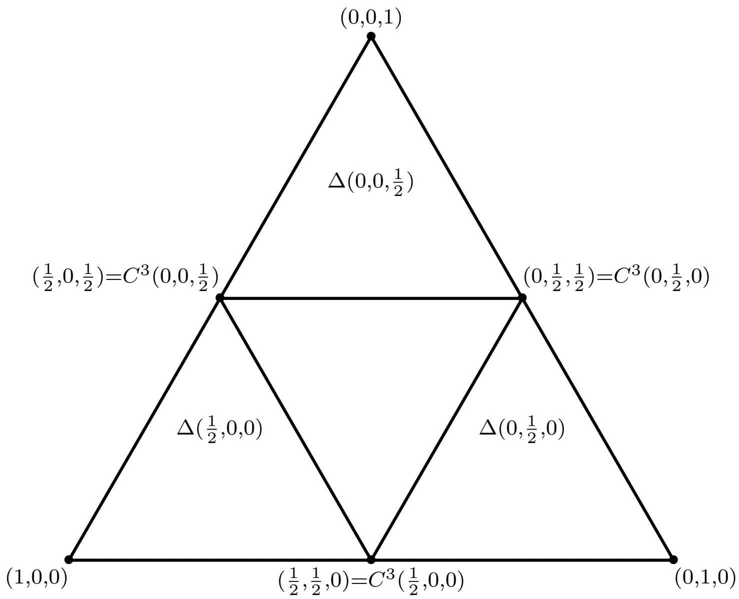

Figure 7: Theorem 4, the case K=2.

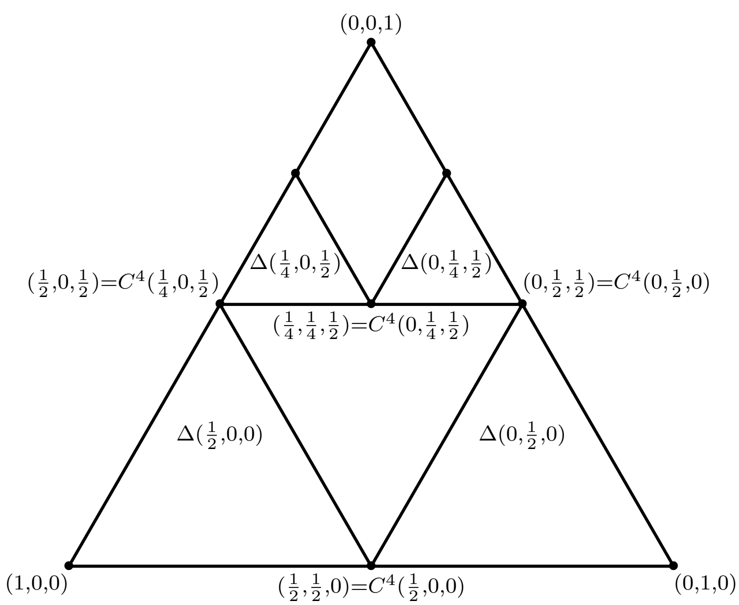

Figure 8: Theorem 4, the case K=3.

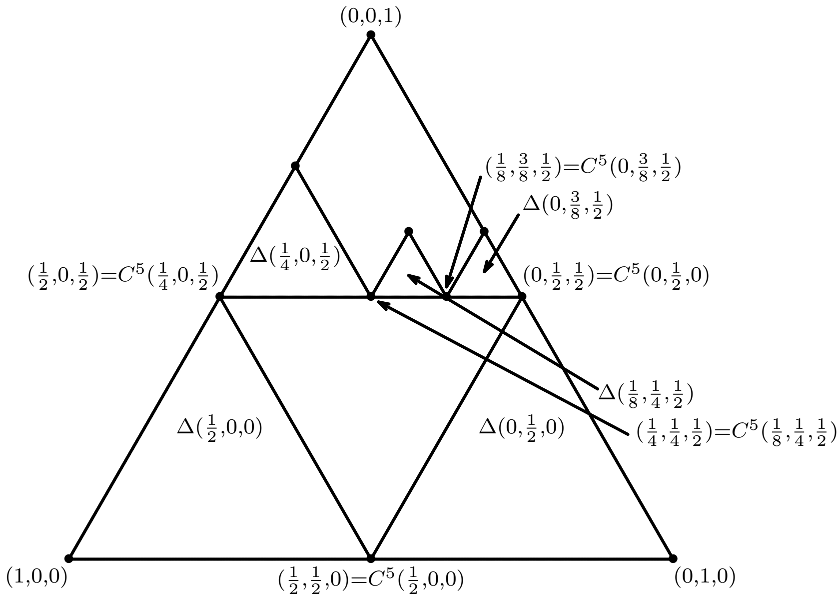

Figure 9: Theorem 4, the case K=4.

Even if continuity and monotonicity are added to independence of irrelevant alternatives, a choice rule on the simplex domain need not be rationalizable. Let $n = 3$ and consider, for example, the choice rule C^{Π} defined by letting $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$,

$$f^1(t_1) = \begin{cases} (t_1, t_1, 1-2t_1) & \text{if } t_1 \in [\frac{1}{3}, \frac{1}{2}], \\ (t_1, 1-t_1, 0) & \text{if } t_1 \in (\frac{1}{2}, 1], \end{cases}$$

$$f^2(t_2) = \begin{cases} (1-2t_2, t_2, t_2) & \text{if } t_2 \in [\frac{1}{3}, \frac{1}{2}], \\ (0, t_2, 1-t_2) & \text{if } t_2 \in (\frac{1}{2}, 1], \end{cases}$$

$$f^3(t_3) = \begin{cases} (t_3, 1-2t_3, t_3) & \text{if } t_1 \in [\frac{1}{3}, \frac{1}{2}], \\ (1-t_3, 0, t_3) & \text{if } t_3 \in (\frac{1}{2}, 1]. \end{cases}$$

Because C^{Π} is associated with an N-path, it satisfies independence of irrelevant alternatives, continuity, and monotonicity. But this choice rule is not rationalizable because there exists a cycle of length three formed by the points $(\frac{1}{2}, 0, \frac{1}{2}) = C^{\Pi}(0, 0, \frac{1}{2})$, $(0, \frac{1}{2}, \frac{1}{2}) =$

$C^{\Pi}(0, \frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2}, 0) = C^{\Pi}(\frac{1}{2}, 0, 0)$. See Figure 10 for an illustration.

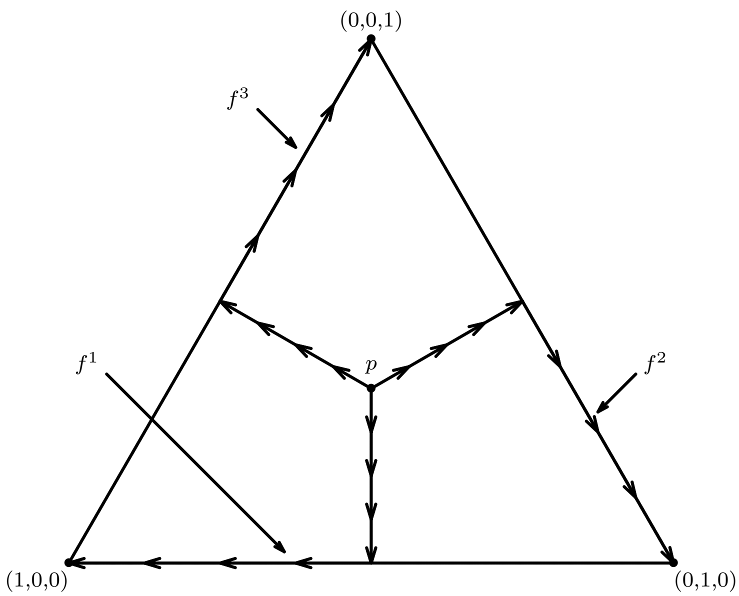

Figure 10: A non-rationalizable path rule C^{Π}.

It is straightforward to verify that, for $n = 3$, some choice rules associated with N-paths are rationalizable. A simple example is given by the choice rule C^{Π} with $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$,

$$f^1(t_1) = (t_1, \tfrac{1-t_1}{2}, \tfrac{1-t_1}{2}) \text{ for all } t_1 \in [\tfrac{1}{3}, 1],$$
$$f^1(t_1) = (t_1, \tfrac{1-t_1}{2}, \tfrac{1-t_1}{2}) \text{ for all } t_1 \in [\tfrac{1}{3}, 1],$$
$$f^1(t_1) = (t_1, \tfrac{1-t_1}{2}, \tfrac{1-t_1}{2}) \text{ for all } t_1 \in [\tfrac{1}{3}, 1].$$

This choice rule is rationalized by the Euclidean distance from p multiplied by -1, that is, we have, for all $\alpha \in A$ and for all $x \in \Delta(\alpha)$,

$$C^{\Pi}(\alpha) = x \iff d(p, x) \leq d(p, y) \text{ for all } y \in \Delta(\alpha).$$

See Figure 11.

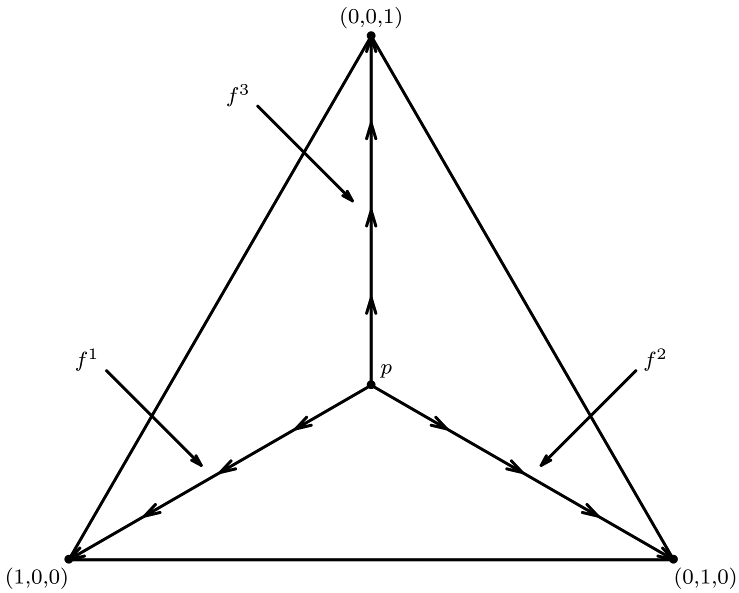

Figure 11: A rationalizable path rule c^{Π}.

For $n = 3$, it turns out that in the presence of independence of irrelevant alternatives and continuity of the choice rule, it is necessary and sufficient to exclude cycles of length three in order to obtain rationalizability. In view of Theorem 3, this statement is equivalent to the following result.

Theorem 5. *Let $n = 3$ and let $C = C^{\Pi}$, where Π is a weakly monotonic N-path. Then C satisfies the strong axiom of revealed preference if and only if there is no revealed-preference cycle of length three in C.*

Proof. If C satisfies the strong axiom of revealed preference, then clearly there is no revealed-preference cycle of length three.

Now suppose that there is no revealed-preference cycle of length three in C. Because the simplex domain is closed under intersection and choice rules corresponding to weakly monotonic N-paths satisfy independence of irrelevant alternatives, they also satisfy the weak axiom of revealed preference and, thus, there is no revealed-preference cycle of length two in C. Let $K > 3$ and let $x^1, \ldots, x^K$ be distinct alternatives such that $x^k R^C x^{k+1}$ for all $k \in \{1, \ldots, K\}$, where $x^{K+1} = x^1$. The proof is complete once we establish that there is a revealed-preference cycle of length less than K in C.

(a) Let $\alpha \in A$ such that $\Delta(\alpha)$ is the (unique) smallest choice set containing the points $x^1, \ldots, x^K$, and write $D = \Delta(\alpha)$. Then $\{x^1, \ldots, x^K\} \cap \{\ell(\alpha), r(\alpha), t(\alpha)\} = \emptyset$ since otherwise by Lemma 1 there would be a cycle of length two. Also, note that each edge of D contains an element of $\{x^1, \ldots, x^K\}$ in its interior. Suppose, without loss of generality, that x^1 is in the relative interior of the left edge of D and that $x_3^2 \geq x_3^1$. Then by Lemma 1 we have $x^1 = C(\{x \in D \mid x_3 \geq x_3^1\})$ and $x^2 \in \{x \in D \mid x_3 \geq x_3^1\}$. See Figure 12. If $\{x \in D \mid x_3 \geq x_3^1\} \cap \{x^3, \ldots, x^K\} \neq \emptyset$ then we can construct a shorter cycle and are done. So we now assume $\{x \in D \mid x_3 \geq x_3^1\} \cap \{x^3, \ldots, x^K\} = \emptyset$, that is, $x_3^3, \ldots, x_3^K < x_3^1$.

Next let

$$E = \{x \in D \mid x_2 \geq x_2^2\}.$$

If $x^k \in E$ for some $k \in \{4, \ldots, K\}$, then we have $x^2 R^C x^k$ and therefore a cycle of length less than K in C. Now we assume that $x^4, \ldots, x^K \notin E$. Consider the parallelograms

$$F = \{x \in D \mid x_2 \leq x_2^2 \text{ and } x_3 \leq x_3^1\}$$

and

$$G = \{x \in D \mid x_1 \leq x_1^2 \text{ and } x_3 \leq x_3^1\}$$

(the shaded regions in Figure 12). Suppose x^2 is not on the right edge of D, that is, $G \neq \emptyset$. Since D is minimal there is a $k \neq 1, 2$ with $x^k \in G$. If $x^{k+1} \in F$, then we also have $x^k R^C x^2$ because $x^k R^C x^{k+1}$ and thus there is a shorter cycle. Now assume $x^{k+1} \in G$ and, by repeating this argument, $x^K \in G$. Since $x^K R^C x^1$, we obtain again $x^K R^C x^2$, and thus a shorter cycle.

For the remainder of the proof we assume that x^2 is on the right edge of D, that is, $G = \emptyset$.

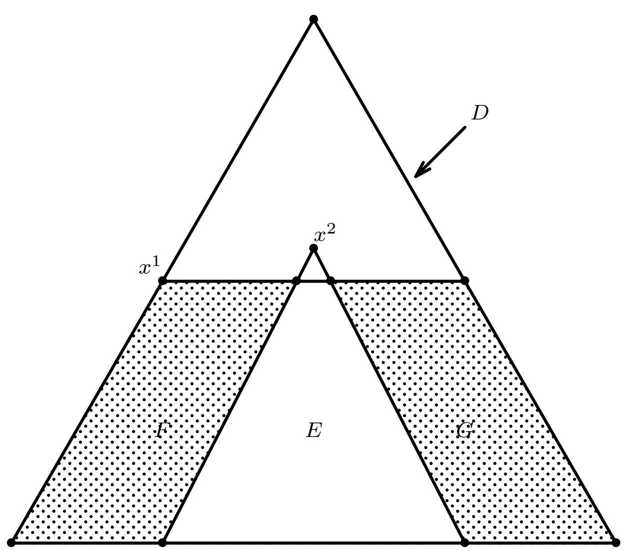

Figure 12: Theorem 5, part (a).

(b) Let

$$H = \{x \in D \mid x_1 \geq x_1^1\}$$

(see Figure 13). Since $x^2 R^C x^3$ we have $x^3 \in E$, since otherwise by Lemma 1 we would have $x^2 R^C x^1$, a contradiction. Consider a choice set (triangle) D' so that $x^3 R^C x^4$. If x^3 is on the left edge including the left and top vertices of D', we have $x^4 \in E$ and therefore, by Lemma 1, $x^2 R^C x^4$ and thus a shorter cycle. So we now assume that x^3 is on the right or bottom edge of D' including the right vertex. If x^3 is on the bottom edge of D' excluding the vertices, Lemma 1 implies $x^3 R^C x^1$ and thus we have a cycle of length three, a contradiction. Hence, x^3 is on the right edge or the right vertex of D'. This implies, moreover, that $x_1^3 > x_1^1$; otherwise, again by Lemma 1, we have $x^3 R^C x^1$ and thus a cycle of length three, a contradiction. In particular, $x^3 \in E \cap H$ and x^3 is not on the right edge or top or right vertices of $E \cap H$.

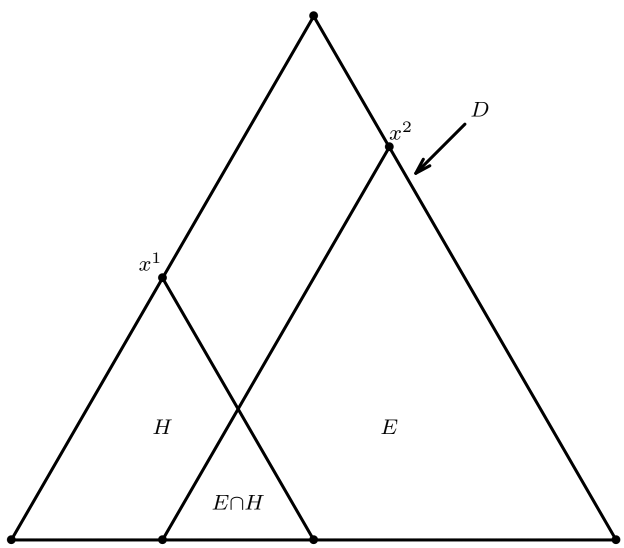

Figure 13: Theorem 5, part (b).

(c) To conclude the proof, we repeat the arguments in (a) and (b) but now for $x^2, \ldots, x^5$ instead of $x^1, \ldots, x^4$, respectively, where $x^5 = x^1$ in case $K = 4$. Define, analogously to E and H, the sets E' and H' by

$$E' = \{x \in D \mid x_1 \geq x_1^3\} \text{ and } H' = \{x \in D \mid x_3 \geq x_3^2\}.$$

Note that the arguments in (a) and (b) for $x^1, \ldots, x^4$ either resulted in shorter cycles or, in the end, in $x^3 \in E \cap H$. Since $E' \cap H' = \emptyset$, analogous arguments for $x^2, \ldots, x^5$ result in shorter cycles. ■

6 Concluding remarks

The main open problem in connection to this paper is the rationalizability question if $n > 3$. On the one hand, the existing literature (see Section 1) would suggest that even under additional conditions exclusion of cycles of a particular length does not imply the strong axiom of revealed preference. On the other hand, the domain of choice problems in the present paper is rather restricted, so that this general insight may not carry over.

Finally, we note that the axioms of independence of irrelevant alternatives and continuity are incompatible with the assumption that requires a choice rule to assume all possible values in its range Δ. If $p_1 = 1$, the point e^i for any $i \in \{2, \ldots, n\}$ will never be chosen, as can be seen by applying Lemma 1; similarly, if $p_1 \neq 1$, e^1 is never chosen.

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