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Interfaces with Other Disciplines

Fraction auctions: The tradeoff between efficiency and running time

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ABSTRACT

This paper studies the sales of a single indivisible object where bidders have continuous valuations. In Grigorieva et al. [14] it was shown that, in this setting, query auctions necessarily allocate inefficiently in equilibrium. In this paper we propose a new sequential auction, called the *c*-fraction auction. We show the existence of an ex-post equilibrium, called bluff equilibrium, in which bidders behave truthfully except for particular constellations of observed bids at which it is optimal to pretend a slightly higher valuation. We show *c*-fraction auctions guarantee approximate efficiency at any desired level of accuracy, independent of the number of bidders, when bidders choose to play the bluff equilibrium. We discuss the running time and the efficiency in the bluff equilibrium. We show that by changing the parameter *c* of the auction we can trade off efficiency against running time.

1. Introduction

The English auction remains the predominant auction format used in practice, though we know since Vickrey [25] that it is strategically equivalent to the second-price, sealed bid auction. Rothkopf et al. [21] argued that third parties being able to "capture fractions of the economic rent revealed by the second price procedure" is one of the reasons why we hardly observe any Vickrey auctions in practice. Engelbrecht-Wiggans and Kahn [11] support this argument in their analysis. They study a model of a procurement auction where the winner of the auction might have to undergo a negotiation with a third party after the auction. In this negotiation, information of the third party about the winner's cost revealed in the auction can have a negative influence on the winner's surplus and the auctioneer's revenue. Also for combinatorial auctions—settings with multiple, heterogenous goods and bidder valuations for bundles of items—iterative auctions have been the most popular approach: first in form of simultaneous ascending auctions (Cramton [7]), later in form of ascending or descending combinatorial auctions (de Vries et al. [10] and Mishra and Veeramani [18]) and clock and clock-proxy auctions (Ausubel et al. [1]).

Most ascending auctions studied in the theoretical literature should be implemented by a continuous, increasing price clock, where bidders drop out whenever the price exceeds what they are willing to pay. However, in practice we see almost exclusively implementations of the following two variants: (1) a discrete clock is increased by increments chosen by the auctioneer; (2) bidders submit increasing bids which exceed the current high bid plus some minimum increment, usually in terms of a percentage of the standing high bid. Cramton [7] reports that "bid increments in the 5 to 10 percent range are required to get the auction to conclude in a manageable number of rounds". For example, if the auction is used to decide on the sales of radio spectrum, such increments are in the order of hundreds of millions of dollars, and can cause bidders to drop out because the next bid would exceed their valuation.

When the auctioneer sets bid increments he has to deal with a tradeoff between efficiency and running time of the auction. As bid increments define a decomposition of the continuum of valuations into intervals, a large bid increment might lead to an inefficient allocation of the item because it prohibits distinguishing between bidders whose valuations are in the same interval. Small increments decrease the chance of allocative inefficiency, but increase the running time of the auction, and thereby participation costs, substantially [9]. Despite its practical relevance, the tradeoff between these two goals has found very little attention in the academic literature. The furthest reaching theoretical evaluation of discrete bid levels has been given in David et al. [9], following up on [8]. They provide a recipe on how to set a finite number of discrete bid levels in order to maximize expected revenue of the

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auctioneer. Thereby they extend the analysis of Rothkopf and Harstad [22], which was limited to either 2 bidders or 2 bid levels. However, the recipe of David et al. can be solved analytically in a few cases only. In other cases, the auctioneer has to rely on numerical calculations. The key qualitative insight of these articles is that *decreasing* increments are preferable to constant or increasing bid increments—the common practice.

In this article we introduce a discrete query auction, called c-fraction auction, for the sale of a single item. The auction gives the auction designer full control of the tradeoff between inefficiency and running time. Based on a prior of bidders' valuations, the auctioneer can choose a single parameter, called c throughout, to regulate both running time and expected inefficiency. The practical implementation requires the computation of values of the quasi-inverse of cumulative distribution functions as the only numerical tool. The auction is detail-free from the bidders' perspectives by offering them an ex-post Nash equilibrium, called *bluff equilibrium*, that differs only slightly from truth-telling (see Section 3 for details). In other words, the equilibrium analysis does not have to make assumptions about bidders' beliefs of other bidders' valuations.

Our main contribution is to provide a detailed analysis of the performance of the c-fraction auction under the so-called bluff equilibrium. First, we investigate the running time of the auction according to two measures, the expected number of rounds and the expected number of queries performed in the auction.[5] For both measures, we first derive an exact recursive formula and then give an upper bound for the function defined by this formula. We prove that, for a fixed c, the expected number of rounds is bounded by a function that is logarithmic in the number of players, while the expected number of queries is bounded by a function that is linear in the number of players. We also give an impression of the number of rounds and number of queries by using computer simulations.

Second, we analyze the level of inefficiency of the auction. As measures of inefficiency, we employ the probability of inefficient allocation and the expected loss of welfare. For the probability of inefficient allocation, we prove that it is no more than c for any number of players, again after deriving an exact recursive formula. This is remarkable because for an ascending auction with constant or increasing bid increments, and, say, uniform i.i.d. valuations, the probability of inefficient allocation is increasing in the number of bidders and converging to 1. Indeed, the number n_{max} of bidders with a value in the largest interval is increasing, and as such an auction cannot distinguish between them, the probability of inefficient allocation is $\frac{n_{max}-1}{n_{max}}$. With respect to the expected loss of welfare, we derive for each distribution a constant $\gamma(c)$ such that the expected loss of welfare is bounded from above by $\gamma(c)\, c$ for any number of players. When valuations are uniformly drawn from $[0,1)$ it holds that $\gamma(c) = c$ and for the exponential distribution with parameter λ we have that $\gamma(c) = -(\ln(1-c))/\lambda$. Our results imply that by choosing the appropriate c, the minimum level of efficiency can be determined by the auctioneer *before* it is known how many players will participate in the auction. We also give an impression of the expected loss of welfare by using computer simulations. Furthermore, we show that there is a tradeoff between efficiency and running time: to increase the efficiency of the auction we have to pay by increasing the number of rounds and increasing the number of queries.

In Grigorieva et al. [14] we study the limitations of query auctions with respect to the objective of economic efficiency maximization in a setting with valuations distributed according to a continuous density function. We show that any ex-post equilibrium in an ex-post individual rational query auction that ends with positive probability after a finite number of queries cannot be fully efficient. This result implies that in the setting of continuous valuations, full efficiency can only be achieved at the expense of an infinite running time of a query auction for almost all realizations of valuations. Our results on the c-fraction auction prove a counterpart of this negative result: for any $c > 0$, the probability of inefficient allocation can be limited by c using an individually rational query auction with an ex-post equilibrium that ends after a finite number of rounds for *all* realizations of valuations. Furthermore, the number of rounds is logarithmic in the number of bidders, and independent of the range of valuations.

Our paper is closely related to David et al. [9]. However, they focus on revenue maximization and touch upon efficiency losses only on the side by showing that revenue-optimal discrete bid levels achieve better economic efficiency than equidistant levels. Though the paper is more ambitious since it tries to solve the optimization problem, a practical implementation requires complex numerical calculations and at least a prior on the number of bidders. David et al. [9] provide a Bayesian machine learning model, based on closing prices of previous auctions, to compute revenue maximizing discrete bid levels when no ex-ante information on the number of bidders and their distribution of valuations is available.

The earliest paper on discrete bid levels in continuous settings that we could trace is by Chwe [4]. Chwe studies the impact on revenue when discrete bids rather than continuous bids are used in sealed-bid, first price auctions. He analyzes equilibrium bidding strategies and shows that revenue with discrete bids is always lower than with continuous bids. Yu [26] extends his equilibrium analysis to English, Vickrey, and Dutch auctions.

Another stream of literature studies iterative auctions from the viewpoint of preference elicitation. Determining one's valuation with a precision up to the last digit can be computationally demanding, see for example Larson and Sandholm [16], Parkes [20], and Sandholm [23]. In combinatorial auctions, the full revelation of agents' preferences may require a prohibitive amount of communication, see for example Nisan and Segal [19]. Such considerations lead to an interest in auctions where agents need not reveal their information entirely but only partially. One approach is to limit communication in a sealed bid auction to a finite number of possible bits, see Blumrosen et al. [2]. Another approach is incremental elicitation of valuations in multiple rounds. It has been recognized that multi-round mechanisms can reduce the amount of preferences that need to be revealed and reduce the amount of computation and communication, compared to single-round mechanisms advocated by the revelation principle, see Blumrosen et al. [3] and Conitzer and Sandholm [6]. Incremental elicitation of bidder valuations has been modeled by query auctions (see e.g., Conen and Sandholm [5]). In a query auction the auctioneer sequentially queries the agents about specific aspects of their preferences. As an answer to the query, an agent can choose one of a finite set of actions. Through incremental querying, the auctioneer gradually collects the information on agents' valuations. By using a query strategy in which previously revealed information guides the selection of subsequent queries, elicitation is targeted towards pertinent information. Incremental querying has been applied in different settings (see e.g. Conen and Sandholm [5] and Grigorieva et al. [13]) and it has been shown that only a small fraction of agents' valuation information needs to be revealed before the (approximately) optimal allocation can be determined (Grigorieva et al. [12], Hudson and Sandholm [15]).

When evaluating the effectiveness of elicitation we may generally care about the running time expressed by the number of queries that are required to determine an optimal allocation (Sandholm and Boutilier [24]). Since information about agents' valuations becomes more refined with each query, a higher

[5] As a query we consider each separate question of the auctioneer to an active player. As a round we consider a sequence of queries in which each active player is asked to act exactly once.

number of queries leads to a better allocation. Our proposed c-fraction auction provides a framework that lets the auctioneer explicitly trade off the two goals.

The paper is organized as follows. Section 2 introduces the rules of the c-fraction auction and Section 3 derives what we call the bluff equilibrium. Section 4 addresses the probabilities by which bidders make *yes* and *no* responses. In Section 5, the running time of the auction is analyzed. Section 6 is devoted to the analysis of the efficiency of the auction and remarks about the tradeoff between running time and efficiency. Section 7 provides some concluding remarks.

2. The c-fraction auction

Suppose a single indivisible object is auctioned to a set $N = \{1,\ldots,n\}$ of players. The players have quasi-linear utilities. We assume independent private valuations drawn from a common continuous probability distribution with density f and cumulative density F. The support of f belongs to $\mathbb{R}_+$. The minimum of this support is denoted by α and the supremum by β, where we allow β to be infinite and assume that β strictly exceeds every possible valuation.[6] Since F may not be strictly increasing, it may not have an inverse. By $F^{-1}:[0,1)\to\mathbb{R}$ we denote the function such that $F^{-1}(y) = \max\{x\in\mathbb{R}_+ | F(x) = y\}$.

Before the start of the auction there is a lottery that determines an ordering of the players. Without loss of generality we assume that this ordering is $1 \prec 2 \prec \cdots \prec n-1 \prec n$. A player with a lower ranking is called a predecessor.

The auction runs for a number of rounds. A round r is characterized by a payment p_r, a query price q_r, an upper bound u_r, rand a set of active players A_r. The payment specifies the price to be paid if an active player wins in this round. The query price is used by the auctioneer to ask the active players whether their valuation is larger than or equal to the query price. Active players are queried publicly in increasing order—player i before player $i+1$—so that an active player can observe the bids of his predecessors. In each round the query price q_r is chosen from the open interval (p_r, u_r), where u_r is allowed to be infinite.

The initial set of active players is $A_1 = N$. The auction starts with $p_1 = \alpha$, $u_1 = \beta$, and some q_1 in (p_1, u_1). Given the current set A_r, the payment p_r, the query price q_r, the upper bound u_r, rand the bids of players in round r, the characteristics of the next round $r+1$ are defined as follows. If all active players submit a *no* bid, they all remain active, i.e. $A_{r+1} = A_r$, the payment remains the same, and the upper bound is set to the previous query price, i.e. $p_{r+1} = p_r$ and $u_{r+1} = q_r$. If at least two active players submit a *yes* bid, all players that said *yes* remain active, the upper bound remains the same, and the payment is set equal to the previous query price, i.e. $u_{r+1} = u_r$ and $p_{r+1} = q_r$. In both cases, as a function of the bounds, a new query price q_{r+1} in (p_{r+1}, u_{r+1}) is determined in the way specified below. If only one active player submits a *yes* bid, the auction stops, this player wins the auction, and pays p_r. If such a moment does not occur, i.e. at least two players remain active forever, the winner is determined according to the order of players: among those players who remain active the player with the highest ranking wins. The price the winner pays is equal to the limit of the sequence of payments $(p_r)_{r\in\mathbb{N}}$ that occur in the subsequent rounds in the auction. Since the sequence of payments is increasing, this limit is equal to the supremum of the payments, and is denoted by p_∞.

For the c-fraction auction, where $c \in (0,1)$, the query price q_r is chosen as the maximal q for which

$$\frac{F(q) - F(p_r)}{F(u_r) - F(p_r)} = c,$$

i.e. c is equal to the probability that a valuation belongs to the interval $(p_r, q_r]$ conditional on this valuation being in the interval $(p_r, u_r]$. For the uniform distribution for example it holds that $q_r = p_r + c(u_r - p_r)$. Equivalently, we can define

$$q_r = F^{-1}((1-c)F(p_r) + cF(u_r)).$$

Consider the following strategy for player i having valuation v_i. This player says *yes* in round r if and only if

1. $v_i \geqslant q_r$, or
2. $p_r \leqslant v_i < q_r$ and no active predecessor of i said *yes* in round r.

In the second part of the definition, we mean by an active predecessor of i in round r a player in A_r that is a predecessor of i. Because this part of the definition involves a certain amount of bluff, we call this strategy the bluff strategy. Formally, the auction is given by an extensive form game, the bluff strategy of player i is a function from the set of decision nodes of player i to $\{yes, no\}$, and is denoted by b_i. The following example illustrates how the auction proceeds when all players follow the bluff strategy.

Example 1. Suppose five players, with valuations uniformly distributed on $[0,1)$, participate in the c-fraction auction, where c is equal to 0.5. Players have the following private valuations: 0.43, 0.71, 0.38, 0.79, and 0.86. The auction proceeds as follows.

Round r	Payment p_r	Query price q_r	Set of active players A_r	Player 1	Player 2	Player 3	Player 4	Player 5
1	0	0.5	{1,2,3,4,5}	yes	yes	no	yes	yes
2	0.5	0.75	{1,2,4,5}	no	yes	–	yes	yes
3	0.75	0.875	{2,4,5}	–	no	–	yes	no

In the first round, player 1, having no predecessor and having a valuation larger than p_1, says *yes*. Every other player, having now an active predecessor who said *yes*, says *yes* if and only if his valuation is greater than or equal to $q_1 = 0.5$. All players except player 3 say *yes* and therefore remain active. The payment and the query price increase to 0.5 and 0.75, respectively. Since $v_1 < p_2$ player 1 says *no* in the second round. Now player 2 has no active predecessor who said *yes* and since $v_2 \geqslant p_2$ he says *yes*. Players 4 and 5 say *yes* since their valuations are larger than $q_2 = 0.75$. Again the payment and the query price increase. In the third round player 2 says *no*, player 4, having now no active predecessors who said *yes*, says *yes*, rand player 5 says *no*. In this round there is only one *yes* decision, meaning that the auction ends. Player 4 wins the auction and pays 0.75.

[6] Requiring all valuations to be strictly smaller than β is a mild, though non-standard, technical assumption that is helpful for some of our theorems. See also footnote 8.

Notice that the outcome in the example is not efficient-the winner is not the player with the highest valuation. Later in the paper we investigate how inefficient this auction is by analyzing the probability of an inefficient allocation and the expected loss of welfare.

3. Bluff equilibria

In this section we show that the profile of bluff strategies constitutes an ex-post individually rational ex-post equilibrium called the *bluff equilibrium*. An ex-post equilibrium is a strategy profile such that, given any realization of valuations, the plan of action prescribed to a bidder in the auction by his strategy is a best response to the plans of action prescribed by the strategies of the other bidders given their valuations. A strategy is ex-post individually rational if for every realization of valuations and for any profile of actions of the player's opponents, the strategy leads to non-negative utility.

Proposition 3.1. *The bluff strategy is ex-post individually rational.*

Proof. Consider an arbitrary strategy profile where player i follows his bluff strategy. If i is not the winner, then he has utility 0. Thus assume that i is the winner. Note that i must be an active player in all rounds of the auction, as only active players can become winners. We show that $p_r \leqslant v_i$ in each round r, which implies that i has positive utility. By construction, $p_r \leqslant v_i$ for $r = 1$. Suppose the inequality holds up to round r. We show that it also holds in round $r + 1$, or the auction ends in round r. There are two cases.

Case 1. A predecessor of i says *yes* in round r. To become the winner, i must say yes as well. As i plays the bluff strategy, it holds that $v_i \geqslant q_r$. Furthermore, we have that $p_{r+1} = q_r$.

Case 2. No predecessor of i says *yes* in round r.When i says *no*, all successors of i in A_r must also say *no* for i to be the winner, so $p_{r+1} = p_r \leqslant v_i$. When i says *yes* and all successors say *no*, the auction ends. When i says *yes* and one successor says *yes*, either $p_{r+1} = q_r \leqslant v_i$ or $p_{r+1} = q_r > v_i$. We show that the latter case contradicts the assumption that i is the winner. Note that i will say *no* in each future round r' where he is active, as $v_i < p_{r+1} \leqslant q_{r'}$. When any other active player says *yes* in any future round r', i will not be the winner. When all other active players—and there is at least one in round $r + 1$—keep saying *no*, the set of active players stays the same in all future rounds. In this case an active player with higher rank than i wins. □

Theorem 3.2. *The bluff strategy profile is an ex-post Nash equilibrium.*

Proof. Let v be a realization of valuations and f_i be a strategy for player i, i.e. a function from the set of decision nodes for player i to $\{yes, no\}$. We show that f_i is not a profitable deviation from b_i against b_{-i}. Let h be the first decision node at which player i following f_i deviates from the bluff strategy, and let r denote the corresponding round. (Obviously, if f_i coincides with b_i at all decision nodes, f_i is not a profitable deviation.) Notice that since we treat v as given and apart from v there is no imperfect information, the node h is well-defined. We consider two cases.

Case 1. Let h be such that at least one predecessor of i in A_r has said *yes*. If $f_i(h) = no$ and $b_i(h) = yes$, the payoff of playing f_i is 0, while according to Proposition 3.1 the payoff of playing the bluff strategy is at least 0. Consider the case where $f_i(h) = yes$ and $b_i(h) = no$. When player i says *yes* in h, there are at least two players who say *yes* in round r. The winning payment will be at least q_r. Further, $v_i < q_r$ because $b_i(h) = no$. Hence, the payoff of playing f_i is non-positive while the payoff of playing b_i is 0.

Case 2. Let h be such that none of the predecessors of i in A_r has said *yes*. If $f_i(h) = yes$ and $b_i(h) = no$, we know that $v_i < p_r$. Since the payment of the winner is at least p_r, playing f_i leads to a non-positive payoff, while playing according to b_i guarantees a non-negative payoff. Consider the case where $f_i(h) = no$ and $b_i(h) = yes$, so $v_i \geqslant p_r$. Suppose that v is such that all successors of i say *no* if i says *yes*. Then, following b_i, player i wins at a price p_r while following f_i he might win at a price at least p_r. Now suppose that v is such that there is a successor j of i that says *yes* if player i says *yes*. Since player j uses the bluff strategy, he will also say *yes* when player i switches to *no*. But then the payoff of playing f_i would be 0 while the payoff of playing b_i is non-negative. □

Theorem 3.3. *The allocation under the bluff equilibrium is not ex-post efficient.*

Proof. See the example in Section 2. □

We argue next that c-fraction auctions have a finite running time under the bluff equilibrium for *any* realization of valuations. Notice that this statement is stronger than just saying that we have a finite running time almost surely. We also argue that when every bidder plays according to his bluff strategy, in any round of the auction there is at least one player who says *yes*.

Theorem 3.4. *The bluff equilibrium has a finite running time for every realization of valuations. Moreover, the query price increases from round to round up to the moment where the winner is found.*

Proof. We first claim that the winner of the auction, say player j, says *yes* in every round of the auction. Suppose not, then let r be the first round in which he says *no*. Now all players in A_r say *no* in round r, since otherwise j cannot be the winner of the auction. Either $r = 1$ or in round $r - 1$ player j said *yes* and so did all players in A_r. If $r = 1$, then $v_j \geqslant p_1 = \alpha$, so player j says *yes*, leading to a contradiction. If in round $r - 1$ all players in A_r said *yes*, then so did a player i with a predecessor in A_r, implying that $v_i \geqslant q_{r-1} = p_r$. Observing a *no* from all his predecessors in round r, player i should say *yes* in round r when following the bluff strategy, a contradiction. We have shown that the winner of the auction says *yes* in every round of the auction. It then follows that the query price increases from round to round up to the moment where the winner is found.

Suppose the valuation v is such that the running time of the auction is infinite. Since by the previous paragraph the query price increases from round to round, we have $p_\infty = \beta$. Since the winner of the auction, say player j, says *yes* in every round r, we have for all $r \in \mathbb{N}$, $v_j \geqslant p_r$, so $v_j \geqslant \beta$, ra contradiction to our assumption that β strictly exceeds every possible valuation.[7] □

As a corollary to Theorem 3.4 we find an easy characterization of the query price in round r.

[7] Without the assumption that β strictly exceeds every possible valuation, the running time would be infinite exactly when two or more bidders have valuation β.

Corollary 3.5. *In the bluff equilibrium it holds that* $q_r = F^{-1}(1-(1-c)^r)$, $r \in \mathbb{N}$.

4. Probability distribution of player actions

In this section we evaluate the probability of saying *yes* and *no* by an active player under the bluff strategy. Knowing these probabilities enables us to derive recursive formulas for the expected number of rounds and the expected number of queries performed in the auction. Throughout this section, as well as in remaining sections, it is assumed that every player uses the bluff strategy.

Recall that in any round r of the c-fraction auction, the query price q_r is determined such that, conditional on v_i belonging to $[p_r, u_r)$, the probability that v_i is in $[p_r, q_r)$ is equal to c. We define $i_r = \min\{i | i \in A_r\}$—among the active players in round r the one with the lowest ranking—and $j_r = \min\{i | i \in A_r, i \neq i_r\}$ —among the active players in round r the one with the second lowest ranking.

Proposition 4.1. *In the bluff equilibrium it holds that a player* $i \in A_r$ *says yes in round r with probability* $1-c$*, except when* $i = i_1$ *or [*$i = j_r$ *and* i_r *says no], in which case player i says yes with probability 1.*

Proof. First let us observe that when player i_r says *no* for the first time, player j_r says *yes* with certainty. Indeed, in all previous rounds player i_r said *yes* and since j_r is active in round r also he said *yes* in those rounds. Both the payment and the query price have increased so that $p_r = q_{r-1}$. Since player j_r follows the bluff strategy, his previous *yes* decision implies that $v_{j_r} \geqslant q_{r-1} = p_r$. If in round r player i_r says *no*, player j_r is in the situation where he does not have an active predecessor with a *yes* decision and therefore says *yes* whenever his valuation is not smaller than p_r. It follows that after a round r where player i_r says *no* for the first time, he drops out, j_r says *yes* and $i_{r+1} = j_r$. Otherwise, $i_{r+1} = i_r$.

Next we determine the probability that player i_r says *yes* in round r. Having no active predecessor, player i_r says *yes* if and only if $v_{i_r} \geqslant p_r$. Since $p_1 = \alpha$ player i_1 says *yes* with certainty in round 1. For $r > 1$, we have that either $i_r = i_{r-1}$ or $i_r = j_{r-1}$. In both cases the decision of player i_r in round $r-1$ was *yes* which happens if and only if $v_{i_r} \geqslant p_{r-1}$. Thus, the probability that i_r says yes is equal to $\mathbb{P}(v_{i_r} \geqslant p_r | v_{i_r} \geqslant p_{r-1}) = \mathbb{P}(v_{i_r} \geqslant q_{r-1} | v_{i_r} \geqslant p_{r-1}) = 1-c$. The last equality holds because q_{r-1} satisfies

$$\frac{F(q_{r-1}) - F(p_{r-1})}{F(u_{r-1}) - F(p_{r-1})} = c,$$

and Theorem 3.4 implies that $u_{r-1} = \beta$.

Next we determine the probability of saying *yes* in round r for any player $i \in A_r \backslash \{i_r\}$. We distinguish two cases.

First, consider the case where player i_r says *yes* in round r. From the fact that $i \in A_r$, it follows that player i said *yes* in round $r-1$ which is the case if and only if $v_i \geqslant q_{r-1}$. In round r he says *yes* if and only if $v_i \geqslant q_r$. Thus, $\mathbb{P}(v_i \geqslant q_r | v_i \geqslant q_{r-1}) = \mathbb{P}(v_i \geqslant q_r | v_i \geqslant p_r) = 1-c$. The last equality holds for the same reason as above.

Secondly, consider the case where player i_r says *no* in round r. As we have already shown, player j_r says *yes* with certainty. For any player $i \in A_r \backslash \{i_r, j_r\}$, the situation is the same as in the previous case and thus such a player i says *yes* with probability $1-c$. □

Due to the query price setting rule of the c-fraction auction, Proposition 4.1 holds regardless of the distribution from which valuations are drawn. Moreover, the probability of saying *yes* or *no* by an active player does not depend on the round with the exception of round 1. This enables us to derive recursive formulas for the expected number of rounds and the expected number of queries performed in the auction.

5. Running time of the auction

In this section we investigate the expected running time of the c-fraction auction if the bluff strategies are played. We analyze two measures, namely the expected number of rounds and the expected number of queries performed in the auction before the winner is found. As a query we consider each separate question of the auctioneer to an active player. For both measures, we first derive a recursive formula and then give an upper bound for the function defined by this formula.

5.1. The expected number of rounds

Let $e_c(k)$ be the expected number of rounds of the auction with k active players, given that the decision of the active player with the lowest ranking is *yes* in the current round, and let $e_c^*(k)$ be the expected number of rounds given that this decision is *no*.

Consider round r with n active players and suppose that the decision of player i_r in the current round is *yes*. The current round contributes 1 to $e_c(n)$. Now let us compute the expected number of remaining rounds. If all active players apart from player i_r say *no*, the auction stops after this round. If k, where $1 \leqslant k \leqslant n-1$, r active players apart from player i_r say *yes*, then the auction continues with $k+1$ active players. Using Proposition 4.1, the probability of this situation given the *yes* decision of player i_r is $\binom{n-1}{k}(1-c)^k c^{n-1-k}$. In the next round player $i_{r+1} = i_r$ says *yes* or *no* with probability $1-c$ and c respectively. Thus if k active players apart from player i_r say *yes* in round r, the expected number of remaining rounds is equal to $(1-c)e_c(k+1) + ce_c^*(k+1)$. Hence, for any $n \geqslant 2$,

$$e_c(n) = 1 + \sum_{k=1}^{n-1} \binom{n-1}{k} (1-c)^k c^{n-1-k} \left[(1-c)e_c(k+1) + ce_c^*(k+1)\right]. \tag{1}$$

Proposition 4.1 states that if player i_{r+1} says *no*, player j_{r+1} says *yes* with certainty, which causes player i_{r+1} to drop out of the auction. For $k > 1$ this yields $e_c^*(k+1) = e_c(k)$, as the expected number of rounds with player i_{r+1} saying no is equal to the expected number of rounds without this player and the player with lowest rank among the remaining k players saying no. Observe that $e_c^*(2) = 1$ as exactly one of the two players says *yes*.

We denote $P_k^n = \binom{n}{k}(1-c)^k c^{n-k}$ and rewrite Eq. (1) as follows:

$$\begin{aligned}
e_c(n) &= 1 + \sum_{k=1}^{n-1} P_k^{n-1}\left[(1-c)e_c(k+1) + ce_c^*(k+1)\right] \\
&= 1 + (1-c)\sum_{k=1}^{n-2} P_k^{n-1} e_c(k+1) + (1-c)P_{n-1}^{n-1} e_c(n) \\
&\quad + c\sum_{k=2}^{n-1} P_k^{n-1} e_c^*(k+1) + cP_1^{n-1} e_c^*(2) \\
&= 1 + (1-c)\sum_{k=1}^{n-2} P_k^{n-1} e_c(k+1) + (1-c)^n e_c(n) \\
&\quad + c\sum_{k=2}^{n-1} P_k^{n-1} e_c(k) + (n-1)(1-c)c^{n-1} \\
&= 1 + (1-c)^n e_c(n) + (n-1)(1-c)c^{n-1} \\
&\quad + (1-c)\sum_{k=2}^{n-1} P_{k-1}^{n-1} e_c(k) + c\sum_{k=2}^{n-1} P_k^{n-1} e_c(k) \\
&= 1 + (1-c)^n e_c(n) + (n-1)(1-c)c^{n-1} \\
&\quad + \sum_{k=2}^{n-1}\left[(1-c)P_{k-1}^{n-1} + cP_k^{n-1}\right] e_c(k) \\
&= 1 + (1-c)^n e_c(n) + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1} P_k^n e_c(k).
\end{aligned}$$

Table 1
The expected number of rounds $e_c(n)$ in the c-fraction auction.

n	c										
	1/10	1/8	1/6	1/4	1/3	1/2	2/3	3/4	5/6	7/8	9/10
2	5.737	4.733	3.727	2.714	2.200	1.667	1.375	1.267	1.171	1.127	1.101
3	8.901	7.230	5.555	3.873	3.021	2.143	1.663	1.483	1.319	1.240	1.193
4	11.273	9.102	6.927	4.742	3.638	2.505	1.891	1.660	1.446	1.341	1.277
5	13.172	10.600	8.024	5.437	4.131	2.794	2.076	1.807	1.557	1.431	1.353
6	14.753	11.848	8.938	6.016	4.542	3.035	2.230	1.931	1.654	1.512	1.423
7	16.109	12.918	9.721	6.513	4.895	3.241	2.361	2.037	1.738	1.584	1.486
8	17.296	13.854	10.407	6.947	5.203	3.421	2.475	2.129	1.813	1.650	1.545
9	18.350	14.686	11.016	7.334	5.477	3.581	2.576	2.211	1.879	1.709	1.598
10	19.299	15.435	11.565	7.681	5.724	3.726	2.667	2.283	1.939	1.762	1.647
20	25.647	20.443	15.233	10.006	7.373	4.690	3.275	2.760	2.312	2.102	1.971
30	29.417	23.418	17.412	11.387	8.353	5.264	3.637	3.048	2.524	2.281	2.140
40	32.109	25.541	18.967	12.372	9.052	5.673	3.894	3.255	2.681	2.406	2.249
50	34.203	27.194	20.177	13.140	9.596	5.991	4.095	3.414	2.808	2.508	2.333
60	35.918	28.547	21.168	13.768	10.042	6.252	4.260	3.543	2.913	2.595	2.405
70	37.370	29.693	22.007	14.299	10.419	6.472	4.399	3.652	3.002	2.672	2.469
80	38.628	30.686	22.735	14.760	10.746	6.664	4.520	3.747	3.077	2.740	2.527
90	39.740	31.563	23.377	15.167	11.035	6.833	4.627	3.831	3.143	2.801	2.580
100	40.735	32.348	23.952	15.532	11.294	6.984	4.722	3.907	3.201	2.855	2.629

This can be rewritten to

$$[1-(1-c)^n]e_c(n) = 1+(n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} e_c(k). \quad (2)$$

This formula is valid for any $n \geqslant 2$.

Now notice that since in the first round player i_1 says *yes* with certainty, the expected number of rounds of the auction with n players is equal to $e_c(n)$. Thus, using Formula 2, we can compute the expected number of rounds in the auction of n players. Plugging in $n = 2$ yields $e_c(2) = \frac{1+c(1-c)}{c(2-c)}$. All other values can be determined recursively. Table 1 presents the computational results for different values of c in the auction with up to 100 players. Notice that the bisection character of the c-fraction auction guarantees a remarkably low number of rounds. For instance, with 100 bidders and $c = 1/2$, the auction terminates in less than 7 rounds on average.

Fig. 1(a) shows how for a fixed value of c the expected number of rounds increases in the number of players who participate in the auction. Furthermore, Fig. 1(b) demonstrates how for a fixed number of players the expected number of rounds decreases as c increases.

We show next that the expected number of rounds of the auction is bounded from above by a function that is logarithmic in the number of players. We determine this bound for $c \leqslant 1/2$, since the bound for $\bar{c} = 1/2$ is also valid for $c > 1/2$. First, we introduce some notation and several lemmas.

For any $n \geqslant 2$, define $D_n = \prod_{k=1}^{n} \frac{1}{1-(1-c)^k}$. Also, define $E_2 = \frac{1+c(1-c)}{c(2-c)}$ and, for any $n > 2$,

$$E_n = 1+(n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} E_k.$$

Lemma 5.1. *For any $n \geqslant 2$, $e_c(n) < E_n \cdot D_n$.*

Proof. The proof is by induction on n. The basis of the induction is trivial since $e_c(2) = E_2$ and $D_2 > 1$. Suppose that $e_c(k) < E_k \cdot D_k$ is true for any $2 \leqslant k \leqslant n-1$. Notice that $D_n > D_{n-1} > \cdots > D_2 > 1$. Thus, using the recursive formula for $e_c(n)$ and the induction hypothesis,

$$\begin{aligned}
[1-(1-c)^n]e_c(n) &= 1+(n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} e_c(k) \\
&< 1+(n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} E_k D_k \\
&< 1+(n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} E_k D_{n-1} \\
&< D_{n-1}\left[1+(n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} E_k\right] \\
&= E_n \cdot D_{n-1},
\end{aligned}$$

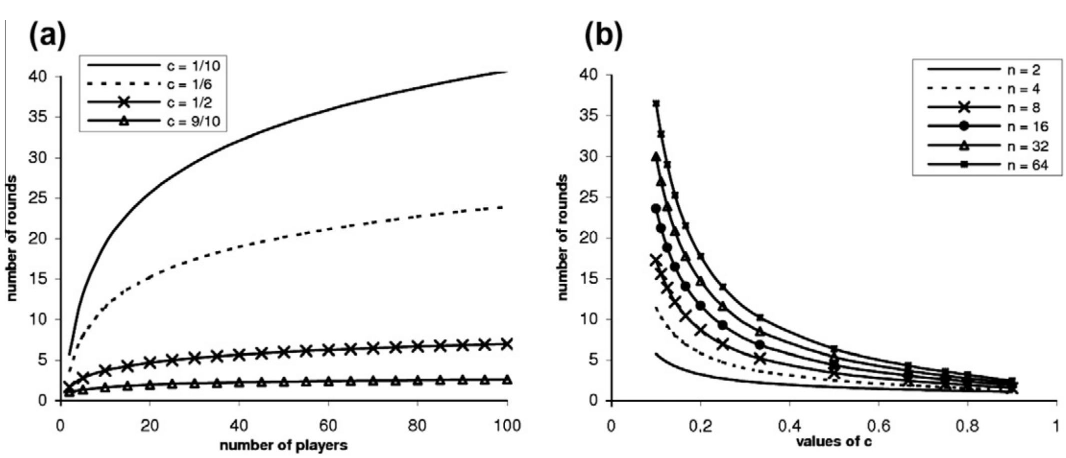

Fig. 1. The expected number of rounds (a) for different fixed values of c; (b) for different fixed numbers of players.

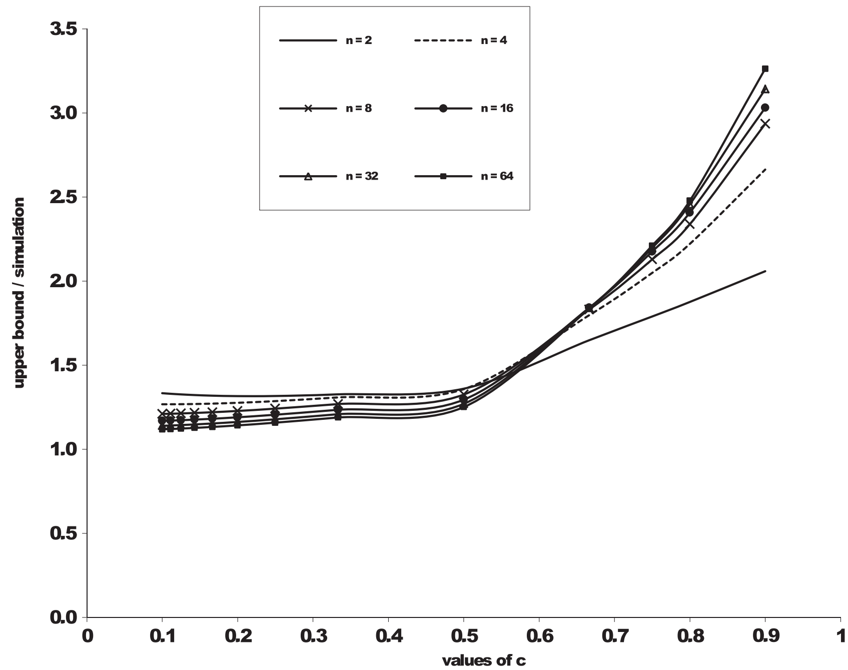

Fig. 2. The ratio between the upper bound from Theorem 5.4 and the results from Table 1.

which completes the proof. □

Now we find bounds on D_n and E_n.

Lemma 5.2. *For any* $n \geqslant 2, D_n < e^{\frac{1-c}{c^2}}$.

Proof. We have to show that $\ln D_n < \frac{1-c}{c^2}$. Let us define $\lambda = \frac{1}{1-c}$. Notice that since $0 < c < 1$ it holds that $\lambda > 1$.

We have

$$\ln D_n = \ln\left(\prod_{k=1}^{n} \frac{\lambda^k}{\lambda^k - 1}\right) = \sum_{k=1}^{n}[\ln \lambda^k - \ln(\lambda^k - 1)] \leqslant \sum_{k=1}^{n} \partial_x \ln x_{|x=\lambda^k-1}$$
$$= \sum_{k=1}^{n} \frac{1}{\lambda^k - 1} < \sum_{k=1}^{n} \frac{1}{\lambda^k - \lambda^{k-1}} = \frac{1}{\lambda - 1}\sum_{k=1}^{n} \frac{1}{\lambda^{k-1}} < \frac{1}{\lambda - 1}\sum_{k=0}^{\infty} \frac{1}{\lambda^k}$$
$$= \frac{\lambda}{(\lambda - 1)^2} = \frac{1-c}{c^2},$$

where the first inequality holds because of the concavity of ln. □

Lemma 5.3. *For any* $n \geqslant 2$ *and any* $c \leqslant \frac{1}{2}$, $E_n < 1 + \log_a n$, *with base* $a = \frac{1}{1-c}$.

Proof. The proof is by induction on n. The basis of the induction holds since $\frac{1+c(1-c)}{c(2-c)} < 1 + \log_a 2$ for any $c \leqslant \frac{1}{2}$. Suppose $E_k < 1 + \log_a k$ for any $2 \leqslant k \leqslant n-1$. Using the induction hypothesis,

$$E_n = 1 + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} E_k$$
$$< 1 + (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k}(\log_a k + 1)$$
$$< 1 + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k}\log_a k + \sum_{k=1}^{n-1}\binom{n}{k}(1-c)^k c^{n-k}$$
$$< 2 + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k}\log_a k.$$

Since the logarithm with base $a = \frac{1}{1-c}$ is concave, we know that if $\lambda_k \geqslant 0$ and $\sum_{k=0}^{n}\lambda_k = 1$, then

$$\sum_{k=0}^{n}\lambda_k \log_a(x_k) \leqslant \log_a\left(\sum_{k=0}^{n}\lambda_k x_k\right).$$

So let us take $\lambda_k = \binom{n}{k}(1-c)^k c^{n-k}$ for $k = 0,\ldots,n$ and take $x_0 = x_n = 1$, $x_k = k$ for any $1 \leqslant k \leqslant n-1$.

Then

$$E_n < 2 + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k}\log_a k$$
$$= 2 + \sum_{k=0}^{n}\binom{n}{k}(1-c)^k c^{n-k}\log_a(x_k)$$
$$\leqslant 2 + \log_a\left[\sum_{k=0}^{n}\binom{n}{k}(1-c)^k c^{n-k} x_k\right]$$
$$= 2 + \log_a\left[\sum_{k=1}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} k + c^n + (1-c)^n\right]$$
$$\leqslant 2 + \log_a\left[\sum_{k=1}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} k + n(1-c)^n\right]$$
$$= 2 + \log_a\left[\sum_{k=0}^{n}\binom{n}{k}(1-c)^k c^{n-k} k\right]$$
$$= 2 + \log_a[(1-c)n] = 1 + \log_a n.$$

The last inequality holds since for any $c \leqslant \frac{1}{2}$ and any $n \geqslant 2$ it holds that $c^n + (1-c)^n \leqslant 2(1-c)^n \leqslant n(1-c)^n$. □

A final immediate consequence of Lemmas 5.1–5.3 is the following theorem.

Theorem 5.4. *For any* $c \leqslant \frac{1}{2}$ *and any* $n \geqslant 2$, $e_c(n) \leqslant e^{\frac{1-c}{c^2}}\left(\log_{\frac{1}{1-c}} n + 1\right)$.

Since $e_c(n) < e_{\bar{c}}(n)$ when $c > \bar{c}$, the upper bound for $\bar{c} = \frac{1}{2}$ is also valid for any $c > \frac{1}{2}$.

We have shown that the expected number of rounds of the c-fraction auction is bounded from above by a function that is logarithmic in the number of players. Furthermore, a comparison of the bound with the computed results shows that for a fixed value of c the ratio between the bound and the computed result is approximately constant as a function of n, implying that the bound has approximately the correct order of magnitude. Fig. 2 shows the

Table 2
The expected number of queries $b_c(n)$ in the c -fraction auction.

n	c										
	1/10	1/8	1/6	1/4	1/3	1/2	2/3	3/4	5/6	7/8	9/10
2	11.474	9.467	7.455	5.429	4.400	3.333	2.750	2.533	2.343	2.254	2.202
3	21.790	17.779	13.759	9.718	7.674	5.571	4.442	4.029	3.666	3.496	3.396
4	32.027	26.013	19.988	13.935	10.879	7.752	6.094	5.495	4.972	4.727	4.582
5	42.217	34.200	26.171	18.109	14.044	9.897	7.717	6.938	6.264	5.949	5.762
6	52.375	42.356	32.323	22.254	17.181	12.017	9.320	8.365	7.545	7.162	6.936
7	62.511	50.490	38.454	26.378	20.298	14.120	10.908	9.778	8.815	8.368	8.104
8	72.630	58.607	44.568	30.487	23.401	16.211	12.484	11.180	10.078	9.568	9.268
9	82.735	66.711	50.669	34.583	26.492	18.291	14.051	12.575	11.333	10.763	10.427
10	92.830	74.804	56.761	38.670	29.575	20.363	15.611	13.962	12.582	11.952	11.582
20	193.465	155.430	117.372	79.251	60.124	40.845	31.016	27.653	24.893	23.679	22.985
30	293.842	235.802	177.735	119.597	90.451	61.132	46.258	41.203	37.070	35.264	34.248
40	394.111	316.068	237.995	159.843	120.684	81.336	61.430	54.691	49.201	46.802	45.457
50	494.320	396.274	298.196	200.035	150.865	101.495	76.563	68.144	61.307	58.319	56.644
60	594.492	476.443	358.361	240.192	181.014	121.626	91.673	81.574	73.394	69.825	67.820
70	694.637	556.587	418.501	280.325	211.140	141.736	106.766	94.989	85.468	81.320	78.989
80	794.763	636.711	478.622	320.440	241.249	161.832	121.847	108.394	97.531	92.809	90.152
90	894.874	716.820	538.729	360.542	271.345	181.916	136.918	121.790	109.586	104.290	101.311
100	994.973	796.918	598.825	400.633	301.431	201.992	151.981	135.180	121.634	115.766	112.466

ratio between the upper bound from Theorem 5.4 and the results from Table 1 for up to 64 bidders and different values for c. The ratio increases for large $c > \frac{1}{2}$, as we use the bound for $c = \frac{1}{2}$, but is almost constant for $c \leqslant \frac{1}{2}$, independent of n.

5.2. The expected number of queries

Let $b_c(k)$ be the expected number of queries of the auction with k active players, given that the decision of the active player with the lowest ranking is *yes* in the current round, and let $b^*_c(k)$ be the expected number of queries given that this decision is *no*. Notice that in a round with k active players, k queries are performed. Following the same argument as we used for determining the formula for the expected number of rounds, we find that for any $n \geqslant 2$

$$b_c(n) = n + \sum_{k=1}^{n-1} \binom{n-1}{k} (1-c)^k c^{n-1-k} \left[(1-c)b_c(k+1) + c b^*_c(k+1)\right]. \quad (3)$$

Again, notice that when player $i_{r+1} = i_r$ says *no*, r player j_{r+1} says *yes* with certainty, which causes player i_{r+1} to drop out of the auction. Thus, $b^*_c(2) = 2$ and for all $k \geqslant 2$ it holds that $b^*_c(k+1) = 1 + b_c(k)$.

Recall that $P^n_k = \binom{n}{k}(1-c)^k c^{n-k}$. Using the facts that $b^*(2) = 2$ and $b^*(k+1) = b(k) + 1$ for all $k \geqslant 2$, we get from (3) that

$$\begin{aligned} b_c(n) &= n + \sum_{k=1}^{n-1} P^{n-1}_k \left[(1-c)b_c(k+1) + c b^*_c(k+1)\right] = n + (1-c) \\ &\times \sum_{k=1}^{n-2} P^{n-1}_k b_c(k+1) + (1-c)P^{n-1}_{n-1} b_c(n) + c\sum_{k=2}^{n-1} P^{n-1}_k b^*_c(k+1) \\ &+ cP^{n-1}_1 b^*_c(2) = n + (1-c)\sum_{k=1}^{n-2} P^{n-1}_k b_c(k+1) + (1-c)^n b_c(n) \\ &+ c\sum_{k=2}^{n-1} P^{n-1}_k [b_c(k) + 1] + 2(n-1)(1-c)c^{n-1} = n + (1-c)^n b_c(n) \\ &+ 2(n-1)(1-c)c^{n-1} + (1-c)\sum_{k=2}^{n-1} P^{n-1}_{k-1} b_c(k) + c\sum_{k=2}^{n-1} P^{n-1}_k b_c(k) \\ &+ c\sum_{k=2}^{n-1} P^{n-1}_k = n + (1-c)^n b_c(n) + 2(n-1)(1-c)c^{n-1} \\ &+ c - c^n - (n-1)(1-c)c^{n-1} + \sum_{k=2}^{n-1}\left[(1-c)P^{n-1}_{k-1} + cP^{n-1}_k\right] b_c(k) \\ &= n + (1-c)^n b_c(n) + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n-1} P^n_k b_c(k). \end{aligned}$$

Rewriting yields, for any $n \geqslant 2$,

$$\left[1-(1-c)^n\right] b_c(n) = n + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} b_c(k). \quad (4)$$

Now notice that since in the first round player i_1 says *yes* with certainty, the expected number of queries in the auction of n players is equal to $b_c(n)$. Thus using (4) we can compute the expected number of queries performed in the auction with n players. Substituting in $n = 2$ yields $b_c(2) = \frac{2+2c(1-c)}{c(2-c)}$. All other values can be determined recursively. Table 2 presents the computational results for different values of c in the auction with up to 100 players. Again, the c-fraction auction needs only very few queries to allocate the object. For instance, with 100 bidders and $c = 1/2$, the auction needs less than 202 queries on average, surprisingly little if one realizes that 100 is the absolute minimum with 100 bidders.

Fig. 3(a) demonstrates that for a fixed value of c, the expected number of queries increases in the number of players participating in the auction. Fig. 3(b) shows that for a fixed number of players, the expected number of queries decreases as c becomes larger.

We show next that the expected number of queries is bounded from above by a function that is linear in the number of players. To prove this we introduce some notation and several lemmas.

Define $B_2 = \frac{2+2c(1-c)}{c(2-c)}$ and, for any $n > 2$,

$$B_n = n + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n-1} \binom{n}{k} (1-c)^k c^{n-k} B_k.$$

Recall that $D_n = \prod_{k=1}^{n} \frac{1}{1-(1-c)^k}$.

Lemma 5.5. *For any $n \geqslant 2$, $b_c(n) \leqslant B_n \cdot D_n$.*

Proof. The proof is identical to the proof of Lemma 5.1 if we replace $e_c(k)$ by $b_c(k)$ and E_k by B_k for all $2 \leqslant k \leqslant n$. □

From Lemma 5.2 we know that for any $n \geqslant 2$, $D_n \leqslant e^{\frac{1-c}{c^2}}$. We find now a bound on B_n.

Lemma 5.6. *For any $n \geqslant 2$, $B_n \leqslant \left(\frac{2}{c} + \frac{1}{2}\right)(n+1)$.*

Proof. The proof is by induction on n. The basis of the induction holds since it can be easily shown that $B_2 < 3\left(\frac{2}{c} + \frac{1}{2}\right)$. Now suppose that $B_k \leqslant \left(\frac{2}{c} + \frac{1}{2}\right)(k+1)$ for any $2 \leqslant k \leqslant n-1$. Using the induction hypothesis,

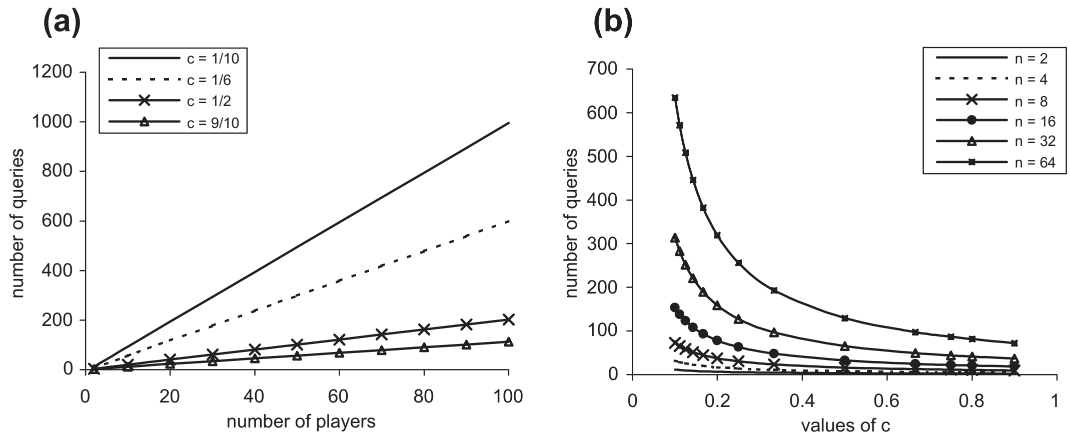

Fig. 3. The expected number of queries (a) for different fixed values of c; (b) for different fixed numbers of players.

$$
\begin{aligned}
B_n &= n + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} B_k \\
&\leqslant n + (n-1)(1-c)c^{n-1} + c - c^n + \sum_{k=2}^{n-1}\left(k\right)(1-c)^k c^{n-k}\left(\frac{2}{c}+\frac{1}{2}\right) \\
&\quad \times (k+1) \leqslant 2n + c + \left(\frac{2}{c}+\frac{1}{2}\right)\sum_{k=0}^{n}\binom{n}{k}(1-c)^k c^{n-k} k + \left(\frac{2}{c}+\frac{1}{2}\right) \\
&\quad \times \sum_{k=0}^{n}\binom{n}{k}(1-c)^k c^{n-k} = 2n + c + \left(\frac{2}{c}+\frac{1}{2}\right)(1-c)n \\
&\quad + \left(\frac{2}{c}+\frac{1}{2}\right) = \left(\frac{2}{c}+\frac{1}{2}\right)(n+1) + c\left(1-\frac{n}{2}\right) \leqslant \left(\frac{2}{c}+\frac{1}{2}\right)(n+1).
\end{aligned}
$$

The last inequality holds since $n \geqslant 2$. □

A final immediate consequence of Lemmas 5.2, 5.5, 5.6 is the following theorem.

Theorem 5.7. *For any integer* $n \geqslant 2$, $b_c(n) \leqslant e^{\frac{1-c}{c^2}}\left(\frac{2}{c}+\frac{1}{2}\right)(n+1)$.

We have shown that the expected number of queries is bounded from above by a function that is linear in the number of players. Again, a comparison of the bound with the computed results suggests that this bound is not tight. It can be easily checked that for a fixed value of c the ratio between the bound and the computed result is approximately constant as a function of n, implying that the bound is likely to have the correct order of magnitude.

6. Efficiency of the auction

In this section we investigate the efficiency of the c-fraction auction when the bluff equilibrium is played. In particular, in the following two subsections, we compute the probability of inefficient allocation and the expected loss of welfare.

6.1. The probability of inefficient allocation

We derive a recursive formula for the probability of inefficient allocation and give an upper bound for this probability using the recursive formula. We denote the probability that the auction with n players terminates in an inefficient allocation by $P_c(n)$.

A first important observation is that the c-fraction auction with k players having valuations drawn from F conditional on these valuations being greater than or equal to $F^{-1}(c)$ has exactly the same structure as the original c-fraction auction with k players having valuations drawn from F. For both cases, the probability of an inefficient allocation under the bluff equilibrium is the same.

Consider the case where the valuation of all players is smaller than $F^{-1}(c)$. The probability of this event is c^n. In this case player i_1 is the only player saying *yes*, rand therefore receives the object. The auction is only efficient if the player with the lowest ranking has the highest valuation, which happens with probability $1/n$. Thus this case contributes $((n-1)/n)c^n$ to $P_c(n)$. Next consider the case where $k \geqslant 1$ players have valuations larger than or equal to $F^{-1}(c)$ and $n-k$ players have valuations smaller than $F^{-1}(c)$, which happens with probability $\binom{n}{k}c^{n-k}(1-c)^k$. For $k = 1$ the auction is efficient, so this case adds zero to $P_c(n)$. Consider the case where $k > 1$. Either player i_1 has a value below $F^{-1}(c)$, responds *no* in round 2, and inefficiency among the remaining k bidders takes place with probability $P_c(k)$. Or player i_1 has a value greater than or equal to $F^{-1}(c)$, in which case the auction starts in round 2 with k bidders having a value greater than or equal to $F^{-1}(c)$ and inefficiency takes place with probability $P_c(k)$. We find that

$$P_c(n) = \frac{n-1}{n}c^n + \sum_{k=2}^{n}\binom{n}{k}c^{n-k}(1-c)^k P_c(k).$$

A direct evaluation of the recursive formula yields that $P_c(2) = \frac{1}{2}\cdot\frac{c}{2-c}$. For $n > 2$, rewriting leads to

$$[1-(1-c)^n]P_c(n) = \frac{n-1}{n}c^n + \sum_{k=2}^{n-1}\binom{n}{k}c^{n-k}(1-c)^k P_c(k). \tag{5}$$

A direct computation of this expression for different combinations of n and c gives the values that are plotted in Fig. 4.

The recursive formula can also be used to derive the following upper bound on $P_c(n)$.

Theorem 6.1. *For all* $n \in \mathbb{N}$, $P_c(n) < c$.

Proof. The proof is by induction on n. The basis of the induction holds since $P_c(1) = 0$ and $P_c(2) = \frac{1}{2}\cdot\frac{c}{2-c} < c$. Suppose that $n \geqslant 3$ and $P_c(k) < c$ for all $1 \leqslant k \leqslant n-1$. Then

$$
\begin{aligned}
P_c(n) &= \frac{1}{1-(1-c)^n}\left[\frac{n-1}{n}\cdot c^n + \sum_{k=2}^{n-1}\binom{n}{k}c^{n-k}(1-c)^k\cdot P_c(k)\right] \\
&< \frac{1}{1-(1-c)^n}\left[c^n + c\sum_{k=2}^{n-1}\binom{n}{k}c^{n-k}(1-c)^k\right] \\
&= \frac{1}{1-(1-c)^n}\left[c^n + c(1-c^n-n(1-c)c^{n-1}-(1-c)^n)\right] \\
&= \frac{c(1-(1-c)^n)}{1-(1-c)^n} + \frac{c^n-c^{n+1}-n(1-c)c^n}{1-(1-c)^n} \\
&= c + \frac{c^n(1-c)(1-n)}{1-(1-c)^n} < c.
\end{aligned}
$$

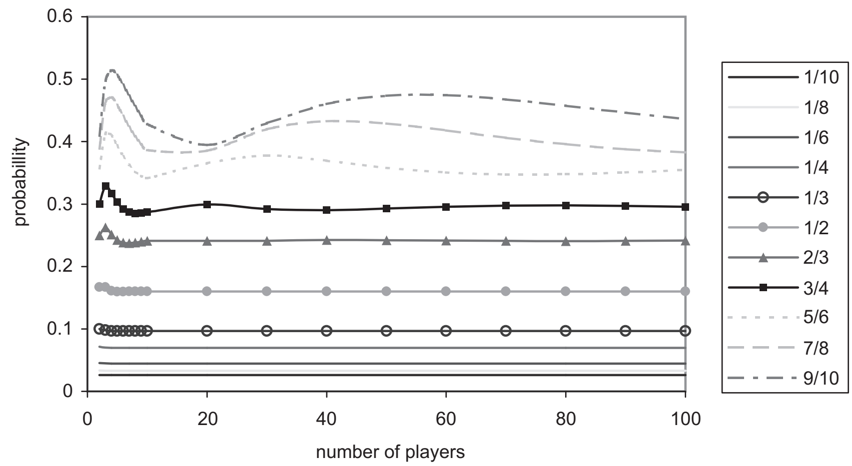

Fig. 4. The probability of inefficient allocation.

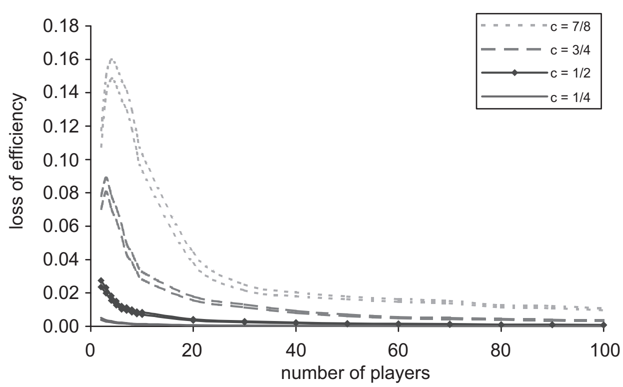

Fig. 5. The expected loss of welfare, 99% confidence interval.

The first inequality holds by the induction assumption and the fact that $\frac{n-1}{n} < 1$. The last inequality holds since $n \geqslant 3$. □

This theorem shows in particular that by choosing an appropriate fraction c in the auction we can make the probability of inefficiency as small as we like, independent of the number of players! Also Fig. 4 shows that the probability of inefficient allocation is quite independent from the number of players.

When we assume $c < \frac{1}{2}$, it can be shown by the same chain of arguments that $P_c(n) \leqslant \frac{1}{2}c$ for all $n \in \mathbb{N}$.

6.2. The expected loss of welfare

The welfare of an auction is equal to the valuation of the winner of the auction. Given v, the maximum welfare is $\max\{v_i | i \in N\}$. The expected loss of welfare, denoted by $L_c(n)$, is the expected value of the difference between the maximum welfare and the valuation of the winner. To estimate the value of $L_c(n)$, we simulated the c-fraction auction and ran it for valuations uniformly and independently drawn from the interval $[0,1)$. For each combination of the value c and the number of players n, we ran 10,000 trials. Fig. 5 shows the 99% confidence interval for the expected loss of welfare. It is interesting to notice that the maximum expected loss does not occur when the number of players is minimal.

For a distribution function F, we define $\gamma(c) \in \mathbb{R}_+$ by

$$\gamma(c) = \sup_{r \in \mathbb{N}} \{F^{-1}(1-(1-c)^r) - F^{-1}(1-(1-c)^{r-1})\},$$

so $\gamma(c)$ measures the maximal difference between the query price q_r and the payment p_r that can occur in an auction. We restrict ourselves to distributions for which $\gamma(c)$ is finite. It is easily verified that for the uniform distribution $\gamma(c)$ is equal to c and for the exponential distribution with parameter λ we have $\gamma(c) = -(\ln(1-c))/\lambda$.

Theorem 6.2. *For all $n \in \mathbb{N}$, $L_c(n) < \gamma(c)c$.*

Proof. Let r be the round in which the winner of the auction is found. The winner said *yes* in round r, so has a valuation at least equal to p_r. Since all other players say *no* in round r or before, they have a valuation strictly less than q_r. The welfare loss is therefore bounded above by $q_r - p_r$, rand therefore by $\gamma(c)$. Hence, $L_c(n) \leqslant \gamma(c) \cdot P_c(n)$. Applying the result of Theorem 6.1 completes the proof. □

Many distributions have the feature that $\lim_{c \downarrow 0} \gamma(c) = 0$, r for instance the exponential distribution with parameter λ, rand any distribution with compact support like the uniform distribution. For such distributions, by choosing an appropriate fraction c in the auction, we can limit the expected loss of welfare to an arbitrarily chosen level, independent of the number of players!

6.3. Tradeoff between efficiency and running time

From the analysis in this and the previous section, we derive the following relation between the value of c, the level of efficiency, and the running time. For a fixed number of players, a smaller fraction c leads to a lower expected loss of welfare and a lower probability of inefficient allocation. But at the same time it leads to a higher expected number of rounds and queries. Thus, increasing running time is a price that we have to pay for increasing the level of welfare. Depending on the priorities of the auctioneer, he may trade off welfare against running time.

Fig. 6 shows the relationship between the expected running time and the probability of an inefficient allocation for several values of n. These relations are built on computational results based on the recursive formulas 2, 4, and 5. Fig. 6(a) shows for every n = 5, 10, 20, rand 30, r a part of the curve $\{(e_c(n), P_c(n) | c \in (0,1)\}$. In accordance with our bounds, it shows that for a fixed probability of inefficiency (vertical axis), we need a number of rounds (horizontal axis) that is logarithmically increasing in the number of bidders, using each time roughly the same c. Fig. 6(b) shows for every n = 5, 10, 20 and 30 a part of the curve $\{(b_c(n), P_c(n) | c \in (0,1)\}$. For a fixed probability of inefficiency, the number of rounds is now increasing linearly in the number of bidders.

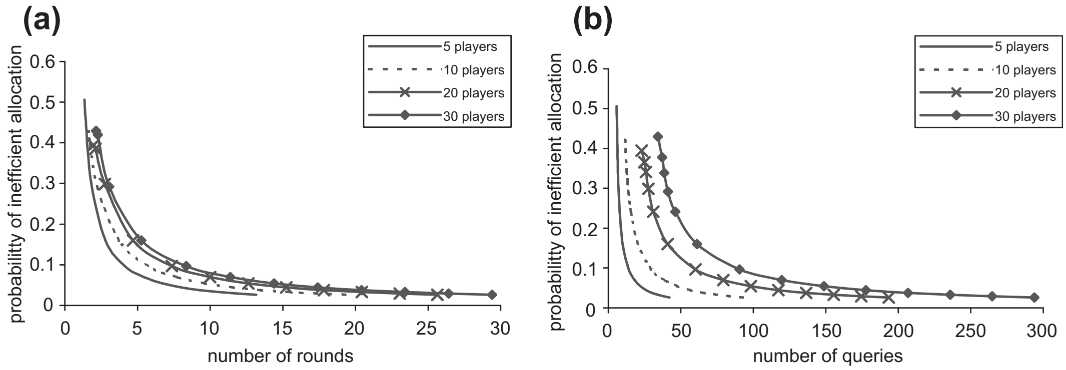

Fig. 6. The tradeoff between (a) the probability of inefficient allocation and the number of rounds; (b) the probability of inefficient allocation and the number of queries.

7. Concluding remarks

We have shown that *c*-fraction auctions provide an easy way of trading off efficiency versus running time of a single item auction. Bluff strategies form an ex-post equilibrium of these auctions. David et al. [9] proposed a slightly different ascending price query auction. Truthful reports to the queries form an ex-post equilibrium in that auction. We expect that choosing increments in their auction in the same way as they are chosen in the *c*-fraction auction provides similar bounds on the number of rounds, number of queries, and efficiency losses. Setting increments dynamically according to the *c*-fraction rule is thus an easy to implement method that leads to auctions that dominate rules-of-thumb approaches like fixed increments or fixed-percentage increments, as for example described in McAfee et al. [17], in all relevant dimensions.

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