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Stochastic bankruptcy games

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Abstract We study bankruptcy problems where the estate and the claims have stochastic values and we allow these values to be correlated. We associate a transferable utility game with uncertainty to a stochastic bankruptcy problem and use the Weak Sequential Core as a solution concept for such games. We test the stability of a number of well known division rules in this stochastic setting and find that all of them are unstable, with the exception of the stochastic extension of the Constrained Equal Awards rule which leads to a Weak Sequential Core element.

Keywords Transferable utility games · Uncertainty · Weak Sequential Core · Bankruptcy games

JEL Classification C71 · C73

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1 Introduction

The classical bankruptcy problem originating in the Talmud has spurred the attention of many game theorists. In the original Talmudic version, there is a dying man who has three wives. Upon his death his estate is to be divided among the wives who have claims over the estate, with the sum of these claims exceeding the worth of the estate. The question is how the estate should be divided over the wives, taking their claims into account. The game theoretic relevance of this issue was first noticed by O'Neill (1982), who transforms the problem into a cooperative game. Later, Aumann and Maschler (1985) also analyzed the problem in a game theoretic set-up and related the Talmudic solution to the nucleolus. Since then the bankruptcy problem triggered a wide range of literature with many applications, interpretations and extensions. Many allocation rules have been suggested as a solution; a thorough inventory of them can be found in Thomson (2003).

Almost the entire literature on bankruptcy games assumes that the value of the estate, as well as the value of the claims, are deterministic. However, in many applications these values are stochastic as they depend on currently unknown future market values. For instance, if a bankrupt firm is liquidated, its assets are sold to third parties. The revenues involved are uncertain at the point in time where the firm is declared bankrupt. Creditors of the firm may hold claims on particular parts of the firm, for instance if some of the assets of the firm serve as collateral for their loans. The value of the claims is therefore uncertain as well. Moreover, the value of a claim is likely to be correlated with the value of other claims, as well as with the value of the estate.

We are only aware of two other papers in the literature on bankruptcy games that deal with uncertainty. In Hougaard and Thorlund-Petersen (2001) the value of the estate is allowed to be uncertain, whereas the claims have deterministic values. The paper investigates how stochastic dominance relations between the stochastic variables describing the estate, influence the expected payoffs to agents under various bankruptcy rules. In Branzei et al. (2003) the value of the estate is deterministic, but a claim is modeled as an interval of possible values without attaching a particular probability distribution. Such interval bankruptcy problems are next translated into cooperative interval games and in particular the Shapley value for such games is studied.

In this paper we introduce the class of stochastic bankruptcy problems. A stochastic bankruptcy problem is characterized by a set of states of nature, estates that have state-contingent values, agents with claims on the estate, where the values of the claims are state-contingent as well, and utility functions of the agents, that map state-contingent profiles of payoffs into utilities. We associate to each stochastic bankruptcy problem a transferable utility game with uncertainty, or briefly TUU-game, as introduced in Habis and Herings (2011). A TUU-game is a two-period cooperative game. In period 0, agents may decide to cooperate or not, facing uncertainty about the state of nature in period 1. In period 1, one state of nature materializes and a state-dependent TU-game is played.

In a classical, static cooperative game it is implicitly assumed that the players can make fully binding agreements on the allocation of payoffs. When such an assumption is made in our stochastic setting, then coalitions can make fully binding state-contingent allocations of payoffs in period 0, and the game becomes formally equivalent to a non-transferable utility game. We, on the other hand, study the case where agents cannot make such fully binding agreements. Instead, agents will not stick to their agreements concerning the future if after the resolution of uncertainty, they are better off when deviating. Hence, we only allow for *self-enforcing* agreements.

An agreement is said to be self-enforcing if there is no coalition of players with a credible deviation in some period. All deviations by singleton coalitions that lead to higher utility are credible. More generally, credible deviations are inductively defined by the requirement that a deviation is credible in some period if there is no further credible deviation by any sub-coalition, now or in the future. In static transferable utility games, the set of deviations and the set of credible deviations coincide (Ray 1989).

In our setting, credibility plays a crucial role and leads to the concept of the Weak Sequential Core, introduced by Kranich et al. (2005) for finite deterministic sequences of TU-games, by Predtetchinski et al. (2006) for two-period exchange economies with incomplete markets, and by Habis and Herings (2011) for TUU-games. Moreover, Habis and Herings (2011) give a characterization of the Weak Sequential Core, and show its non-emptiness if all the state-contingent games played in period 1 are convex. Since Curiel et al. (1987) show that bankruptcy games are convex, the Weak Sequential Core of stochastic bankruptcy games is non-empty.

Similarly to core-compatibility in a static problem, we use the Weak Sequential Core to test the robustness to uncertainty of the most important bankruptcy rules as suggested in the literature; the Proportional rule, the Adjusted Proportional rule, the Constrained Equal Awards rule, the Constrained Equal Losses rule, and the Talmud rule. It is well-known that in deterministic bankruptcy game, all rules lead to elements of the classical core, and are therefore stable. When extended to a stochastic setting, however, each of them is unstable, with the exception of the Constrained Equal Awards rule.

The outline of the paper is as follows. We specify the TUU model in Sect. 2 and give the formal definition of the Weak Sequential Core in Sect. 3. Section 4 introduces stochastic bankruptcy problems, Sect. 5 formulates stochastic bankruptcy games and studies the stability of extensions of deterministic bankruptcy rules to the stochastic setting. Section 6 concludes.

2 Preliminaries

We consider a cooperative game with two time periods, $t \in T = \{0, 1\}$. In period 1, one state of nature s out of a finite set of states of nature S occurs. We define the state of nature for period 0 as state 0, so the set of all states is $S' = \{0\} \cup S$. In period 1, the players are involved in a *cooperative game with transferable utility* (TU-game), where the game itself is allowed to be state-dependent. Period 0 serves as a point in time prior to the resolution of uncertainty.

The TU-game Γ_s played in state $s \in S$ is a pair (N, v_s), where $N = \{1, 2, \dots, n\}$ is the set of players and $v_s : 2^N \to \mathbb{R}$ is a characteristic function which assigns to each coalition $C \subseteq N$ its *worth* $v_s(C)$, with the convention that $v_s(\emptyset) = 0$. Player $i \in N$ evaluates his payoffs by a utility function $u^i : \mathbb{R}^S \to \mathbb{R}$, which assigns to every profile of payoffs $x^i \in \mathbb{R}^S$ a utility level $u^i(x^i)$ and is assumed to be continuous and state-separable, i.e. $u^i(x^i) = \sum_{s \in S} u^i_s(x^i_s)$, where $u^i_s(x^i_s)$ is monotonically increasing. An example are von Neumann–Morgenstern utility functions, which are obtained when all players hold objective probabilities ρ_s for the occurrence of state s, and $u^i_s(x^i_s) = \rho_s w^i(x^i_s)$ for some state-independent function w^i.

A TU-game with uncertainty is defined by Habis and Herings (2011) as follows.

Definition 2.1 A *TU-game with uncertainty* (*TUU-game*) Γ is a tuple (N, S, v, u) where $v = (v_s)_{s \in S}$ and $u = (u^i)_{i \in N}$.

When the cardinality of S is one, a TUU-game reduces to a TU-game. In the absence of uncertainty, all monotonic transformations of utility functions are equivalent, and it is without loss of generality to take $u^i(x^i) = x^i$.

The central question in a TUU-game is how the worth $v_s(N)$ of the grand coalition is distributed among its members in every state $s \in S$. A distribution of state-contingent worths, represented by a matrix $x = (x^1, \ldots, x^n) \in \mathbb{R}^{S \times N}$, is called an *allocation*. The state s component $x_s = (x^1_s, \ldots, x^n_s) \in \mathbb{R}^N$ of an allocation is referred to as the allocation in state s. The total worth obtained by coalition C in state s is $x_s(C) = \sum_{i \in C} x^i_s$. An allocation for a coalition C is a matrix $x^C = (x^i)_{i \in C} \in \mathbb{R}^{S \times C}$, with a state s component $x^C_s \in \mathbb{R}^C$. The restriction of a TUU-game Γ to coalition C is denoted by (Γ, C).

3 The Weak Sequential Core WSC(Γ)

We are interested in the stability of allocations in a TUU-game Γ. In general, the allocation $\bar{x}$ is stable if there is no state $s' \in S'$ and coalition $C \subseteq N$ such that C has a profitable deviation from $\bar{x}$ at s'. There are various ways in which the notion of profitable deviation might be formulated. Here we concentrate on the Weak Sequential Core, introduced in Kranich et al. (2005) for finite deterministic sequences of TU-games, in Predtetchinski et al. (2006) for two-period exchange economies with incomplete markets, and in Habis and Herings (2011) for TUU-games.

Full commitment may be a strong and unrealistic assumption in the presence of time and uncertainty. Once the state of nature is known, there are typically players which have no incentives to stick to the previously arranged allocation of payoffs. The Weak Sequential Core applies to the case with absence of commitment and looks for agreements which are self-enforcing.

Following Habis and Herings (2011), we first define what allocations are feasible for coalitions at different states, then we formalize the notion of credible deviations and finally we define the Weak Sequential Core of a TUU-game. We start with feasibility at future states.

Definition 3.1 Let some allocation $\bar{x}$ be given. The allocation x^C is *feasible* for coalition C at state $s \in S$ given $\bar{x}$ if

$$x^C_{-s} = \bar{x}^C_{-s},$$
$$x^C_s(C) \leq v_s(C).$$

The first condition requires that the members of a coalition take allocations outside state s as given. According to the second condition, in state s the members of a coalition can redistribute at most their worth.

We turn next to feasibility at state 0.

Definition 3.2 The allocation x^C is *feasible* for coalition C at state 0 if

$$x^C(C) \leq v(C).$$

Notice that Definition 3.2 requires the allocation to be feasible for coalition C in every state; $\sum_{i \in C} x_s^i \leq v_s(C)$ must hold for all states s in period 1.

We continue by defining deviations as feasible allocations that improve the utility of every coalition member.

Definition 3.3 Let some allocation $\bar{x}$ be given. A coalition C can *deviate* from $\bar{x}$ at state $s' \in S'$ if there exists a feasible allocation x^C for C at s' such that

$$u^i(x^i) > u^i(\bar{x}^i), \text{ for all } i \in C.$$

The allocation x^C in Definition 3.3 is referred to as a *deviation*. Notice that Definition 3.3 applies both to deviations at $t = 0$ and deviations at $t = 1$. Definition 3.3 can be extended in an obvious way to define deviations from an allocation x^C by a sub-coalition D of C.

Since deviations should be self-enforcing, we use the notion of a credible deviation following the approach developed in Ray (1989) for the static case. Ray (1989) shows that in a static environment the set of deviations coincides with the set of credible deviations. This is no longer true in our setting.

Credible deviations are defined recursively and by backwards induction. At any future state, any deviation by a singleton coalition is credible. A 2-player coalition has a credible deviation at a future state if there is no singleton sub-coalition with a credible counter-deviation at that state. A credible deviation at a future state for an arbitrary coalition is then defined by recursion.

Definition 3.4 Let some allocation $\bar{x}$ be given. Any deviation x^C from $\bar{x}$ at state $s \in S$ by a singleton coalition C is *credible*. A deviation x^C from $\bar{x}$ at state s by coalition C is *credible* if there is no sub-coalition $D \subsetneq C$ such that D has a credible deviation from x^C at state s.

At state 0, again, any deviation by a singleton coalition is credible. A 2-player coalition has a credible deviation at state 0 if there is no singleton sub-coalition with a credible counter-deviation at any state, current or future. A credible deviation at state 0 by an arbitrary coalition is then defined by recursion.

Definition 3.5 Let some allocation $\bar{x}$ be given. Any deviation x^C from $\bar{x}$ at state 0 by a singleton coalition is *credible*. A deviation x^C from $\bar{x}$ at state 0 by coalition C is *credible* if there is no sub-coalition $D \subsetneq C$ and state $s' \in S'$ such that D has a credible deviation from x^C at s'.

Definition 3.6 The *Weak Sequential Core* $\mathrm{WSC}(\Gamma)$ of the game Γ is the set of feasible allocations $\bar{x}$ for the grand coalition from which no coalition ever has a credible deviation.

The Weak Sequential Core of a TUU-game with $|S| = 1$ coincides with the Core of the resulting TU-game.

Definition 3.7 The *Core* $\mathrm{C}(N, v)$ of a TU-game (N, v) is the collection of allocations $\bar{x} \in \mathbb{R}^N$ such that $\bar{x}(N) = v(N)$ and there is no coalition C such that $\bar{x}(C) < v(C)$.

Habis and Herings (2011) provides the following characterization of the Weak Sequential Core by means of the classical core of suitably chosen subgames.

Theorem 3.8 *The following two statements are equivalent:*

(a) $\bar{x} \in \mathrm{WSC}(\Gamma)$,
(b) $\bar{x}$ *is such that* $\bar{x}_s \in \mathrm{C}(\Gamma_s)$ *for all* $s \in S$, *and there is no* $C \subset N$ *and allocation* x^C *such that* $x^C_s \in \mathrm{C}(\Gamma_s, C)$ *for all* $s \in S$, *and* $u^i(x^i) > u^i(\bar{x}^i)$ *for all* $i \in C$.

For an allocation to belong to the Weak Sequential Core of the TUU-game Γ, the allocation should belong to the Core of the TU-game Γ_s in every state $s \in S$. Moreover, no coalition should be able to pick an element x^C_s of the Core of the game restricted to C in every state s, and in doing so improve utility in an ex ante sense.

Definition 3.9 A TU-game (N, v) is *convex* if for all $C \subset N$ and for all $S \subsetneq T \subset N \backslash C$ it holds that $v(S \cup C) - v(S) \leq v(T \cup C) - v(T)$.

Habis and Herings (2011) proves that The Weak Sequential Core is non-empty if all state-contingent TU-games in the TUU-game are convex.

Theorem 3.10 *Let the TUU-game* Γ *be such that* Γ_s *is convex for all* $s \in S$. *Then* $\mathrm{WSC}(\Gamma) \neq \emptyset$.

Notice that this result holds under very weak assumptions with respect to the utility functions u. In particular, concavity of the utility functions is not required.

4 Stochastic bankruptcy problems

Bankruptcy problems originate in a fundamental paper by O'Neill (1982). The problem is based on a Talmudic example, where a man dies, leaving behind an estate, E, which is worth less than the sum of the claims of his wives. The question is how the estate should be divided.

A bankruptcy problem is defined as a pair (E, d), where $d = (d^1, \ldots, d^n)$ is a vector of claims, and $\sum_{i \in N} d^i \geq E \geq 0$. Following O'Neill (1982), the problem can be transformed into a TU-game. The characteristic function $v^{E,d}$ is defined to be

$$v^{E,d}(C) = \max\{E - \sum_{i \in N \backslash C} d^i, 0\}, \quad C \subset N, \tag{1}$$

so the worth of a coalition C in the TU-game $(N, v^{E,d})$ is that amount of the estate which is not claimed by the complement of C. It has been shown by Curiel et al. (1987) that $v^{E,d}$ is convex.

A *rule* is a function that associates with each (E, d) an allocation $x \in \mathbb{R}^N$ such that $\sum_{i\in N} x^i = E$ and $0 \leq x \leq d$. A thorough inventory of the rules can be found in Thomson (2003). The best-known rule is the *Proportional rule* (P) which allocates the estate proportional to the claims. The *Adjusted Proportional rule* (AP) selects the allocation at which each claimant i receives his minimal right $\max\{E - \sum_{j\neq i} d^j, 0\}$, then each claim is revised down accordingly and truncated at what is left of the estate after all minimal rights have been paid, and finally, the remainder of the estate is divided proportionally to the revised claims. The *Constrained Equal Awards rule* (CEA) is in the spirit of equality; it assigns equal amounts to all claimants subject to no one receiving more than his claim. More formally, we have the following.

Definition 4.1 (Constrained Equal Awards rule) For $i' \in N$, for each (E, d), $\mathrm{CEA}^{i'}(E, d) = \min\{d^{i'}, \alpha\}$, where $\alpha \leq \max_{i\in N} d^i$ is chosen so that $\sum_{i\in N} \min\{d^i, \alpha\} = E$.

The *Constrained Equal Losses rule* (CEL), as opposed to the CEA rule, is focusing on losses claimants incur, and makes these losses equal, with no one receiving a negative amount. The recommendation of the Talmud, later formalized in Aumann and Maschler (1985) as the *Talmud rule* (T), is a combination of the CEA rule and the CEL rule, depending on the relation of the half-claims and the value of the estate. The *Piniles' rule* (Piniles 1861) is an application of the CEA rule to the half-claims in two different ways, again depending on the relation of the half-claims and the value of the estate. It coincides with the Talmud rule for the examples that we consider in this paper. The *Constrained Egalitarian rule* (Chun et al. 2001) also gives a central role to the half-claims, and guarantees that the awards are ordered as the claims are. In our examples it also coincides with the Talmud rule. The *Random Arrival rule* (RA) takes all the possible orders of claimants arriving one at a time, compensates them fully until money runs out, and takes the arithmetic average over all orders of arrival.

Many rules are related to the solutions of bankruptcy games. The AP rule corresponds to the τ-value (Curiel et al. 1987), the CEA rule to the Dutta–Ray solution (Dutta and Ray 1989), the T rule to the nucleolus (Aumann and Maschler 1985), and the RA rule to the Shapley value (O'Neill 1982). Any rule generates an allocation in the Core of the bankruptcy game. Indeed, let $\bar{x}$ be the allocation that the rule associates to the bankruptcy problem (E, d). It obviously holds that $\bar{x}(N) = v(N) = E$. Moreover, we have

$$v(C) = \max\{0, E - \sum_{i\in N\setminus C} d^i\} \leq \max\{0, E - \sum_{i\in N\setminus C} \bar{x}^i\} = \max\{0, \sum_{i\in C} \bar{x}^i\} = \bar{x}(C),$$

so no coalition can improve upon $\bar{x}$.

In the original *estate division problem* of the Talmud, a man has 3 wives whose marriage contracts specify that upon his death they should receive 100, 200, and 300, respectively. When the man dies, his estate is found to be worth 100, 200, or 300 in three different scenarios.

Table 1 Characteristic function of estate division

$v^{E,d}$	∅	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$v^{100,d}$	0	0	0	0	0	0	0	100
$v^{200,d}$	0	0	0	0	0	0	100	200
$v^{300,d}$	0	0	0	0	0	100	200	300

Table 2 Estate allocation

Player	Estate	T	P	AP	CEA	CEL	RA
$d^1 = 100$	100	33 1/3	16 2/3	33 1/3	33 1/3	0	33 1/3
	200	50	33 1/3	40	66 2/3	0	33 1/3
	300	50	50	50	100	0	50
$d^2 = 200$	100	33 1/3	33 1/3	33 1/3	33 1/3	0	33 1/3
	200	75	66 2/3	80	66 2/3	50	83 1/3
	300	100	100	100	100	100	100
$d^3 = 300$	100	33 1/3	50	33 1/3	33 1/3	100	33 1/3
	200	75	100	80	66 2/3	150	83 1/3
	300	150	150	150	100	200	150

The characteristic function of the resulting TU-games is shown in Table 1.

Table 2 summarizes the outcomes of the rules described before when applied to these TU-games.

In many bankruptcy situations, both the value of the estate and the value of the claims are uncertain. For instance, in case a firm is liquidated, its assets are sold at prices which are uncertain at the point in time where bankruptcy is declared, which leads to the estate having a stochastic value. The creditors of the firm holds claims that in general depend on the prices paid for a firm's assets as well, for instance when some of these claims are backed up by collateral. The values of the claims are therefore stochastic as well, and in general correlated with each other as well as with the value of the estate.

A *stochastic bankruptcy problem* is defined as a tuple (S, E, d, u), where S is a finite set of states of nature, $E = (E_s)_{s\in S} \in \mathbb{R}^S$ is the value of the estate, $d = (d_s)_{s\in S} \in \mathbb{R}^{S\times N}$ where $d_s = (d_s^1, \ldots, d_s^n) \in \mathbb{R}^N$ is the state-dependent vector of claims, and $u = (u^i)_{i\in N}$ are the continuous, state-separable, and monotonic utility functions of the claimants, where $u^i : \mathbb{R}^S \to \mathbb{R}$. A *rule* for a stochastic bankruptcy problem is a function that associates with each (S, E, d, u) an allocation $x \in \mathbb{R}^{S\times N}$ such that for all $s \in S$, $\sum_{i\in N} x_s^i = E_s$ and $0 \leq x_s \leq d_s$.

The importance of uncertainty in bankruptcy situations has been recognized by previous contributions, in particular Hougaard and Thorlund-Petersen (2001) and Branzei et al. (2003). In Hougaard and Thorlund-Petersen (2001) the case with a stochastic estate and deterministic claims is considered and for various rules the comparative statics of the expected payoff to a player are analyzed when the value of the estate

Table 3 Expected utilities

Player	ST	SP	SAP	SCEA	SCEL	SRA
$d^1 = 100$	42407	32037	39374	61481	0	37315
$d^2 = 200$	63866	61481	65274	61481	45833	66204
$d^3 = 300$	76366	88333	77774	61481	125833	78704

changes in a stochastically dominant way. Branzei et al. (2003) study the case with a deterministic estate and uncertain claims, where an uncertain claim is modeled by the interval of values it can take, without attaching a specific probability distribution. Their analysis then proceeds by defining and studying a particular cooperative interval game.

Any rule f for a bankruptcy problem that is not subject to uncertainty can be extended to a rule Sf for a stochastic bankruptcy problem by applying f state by state. For instance, the CEA rule extends to the SCEA rule by setting $\text{SCEA}_s(S, E, d, u) = \text{CEA}(E_s, d_s)$ for all $s \in S$. Similarly, we obtain definitions of the ST rule, the SP rule, the SAP rule, the SCEL rule, and the SRA rule. It is not difficult to define rules for stochastic bankruptcy problems that are not obtained as extensions of a particular rule for a deterministic bankruptcy problem. For instance, it could be that in two states with identical estate and claims, in one state the CEL rule is applied and in another the CEA rule.

As an example, consider the estate division problem of the Talmud, where as before the claims of the three wives are fixed to 100, 200, and 300, respectively, but the exact value of the estate is uncertain, and the possible values 100, 200, and 300 are each equally likely. After the uncertainty regarding the estate's value is resolved in period 1, one of the original three scenarios of the problem arises. Suppose the wives evaluate the payoffs with the utility function,

$$u^i(x^i) = \sum_{s \in S} \frac{1}{3}(1000x_s^i - (x_s^i)^2),$$

whenever $0 \leq x_s^i \leq 300$ for $s \in S$. Outside this domain the utility function can be anything, as long as it is continuous, state-separable, and monotonically increasing. The estate allocation of the various rules is still described by Table 2, but with a new interpretation. For instance, under the ST rule, at $t = 0$, the wive with a claim of 100 expects a payoff of 33 1/3 with probability 1/3 and a payoff of 50 with probability 2/3. The expected utilities of the wives are depicted in Table 3.

5 Stability of rules

We are interested in the question to what extent rules lead to allocations that are self-enforcing in the presence of uncertainty regarding the value of the estate and the size of the claims. More precisely, we want to examine under what conditions rules lead

Table 4 Credible deviations

Player	E	ST	SP	SAP	SCEL	SRA
$d^1 = 100$	100	25	0	25	0	29
	200	40	33	35	0	29
	300	70	70	65	0.01	60
$d^2 = 200$	100	25	30	25	0	34
	200	75	63	80	49	83
	300	110	108	110	101.13	100
$d^3 = 300$	100	50	70	50	100	37
	200	85	104	85	151	88
	300	120	122	125	198.86	140

to allocations in the Weak Sequential Core of a suitably defined stochastic bankruptcy game.

We first extend the approach of O'Neill (1982) to the stochastic case. We transform a stochastic bankruptcy problem (S, E, d, u) into the TUU-game $\Gamma = (N, S, v, u)$. A natural way to associate a TUU-game with a stochastic bankruptcy problem is to define the worth of a coalition C in state s to be what is left of the estate E_s after each member i of the complementary coalition $N \setminus C$ is paid his complete claim d_s^i,

$$v_s(C) = \max\{E_s - \sum_{i \in N \setminus C} d_s^i, 0\}, \quad s \in S,\ C \subset N.$$

In this way, after the resolution of uncertainty in period 1, we obtain a particular TU-game Γ_s. However, from the point of view of period 0, it is uncertain which TU-game is going to be played. Since the players are not assumed to be risk neutral, the game played ex ante, before the resolution of uncertainty, does not correspond to a TU-game. Moreover, there are two points in time where coalitions may form and deviate: period 0 before the resolution of uncertainty and period 1 after the relevant state of nature is known.

The observation that bankruptcy games are convex (Curiel et al. 1987) combined with Theorem 3.10 leads to the following result.

Theorem 5.1 *The Weak Sequential Core of a stochastic bankruptcy game* $\Gamma = (N, S, v, u)$ *is non-empty.*

We have already argued that, in the absence of uncertainty, a rule leads to an allocation in the Core of the bankruptcy game. Theorem 3.8 now implies that for a stochastic bankruptcy game blocking is not possible after the resolution of uncertainty. However, it might be possible to block the allocation described by a rule ex ante.

We demonstrate that in our example none of the rules mentioned before leads to an allocation in the Weak Sequential Core, with the exception of the SCEA rule. We do this by showing that for all rules mentioned before, with the exception of the SCEA

Table 5 Expected utilities following credible deviation

Player	ST	SP	SAP	SCEL	SRA
$d^1 = 100$	42625	32337	39642	3	37573
$d^2 = 200$	63883	61489	65292	45834	66318
$d^3 = 300$	76958	88467	78217	125838	78762

rule, the grand coalition has a credible deviation at state 0. Table 4 lists credible deviations x and Table 5 the implied utilities. It can be readily verified that $x_s \in \mathrm{C}(\Gamma_s)$ for all $s \in S$ and $u^i(x^i) > u^i(\bar{x}^i)$ for all $i \in C$, where $\bar{x}$ is the allocation prescribed by a particular rule. Theorem 3.8 then implies that the allocations $\bar{x}$ prescribed by the various rules do not belong to the Weak Sequential Core of the stochastic bankruptcy game.

We show next that the SCEA rule belongs to the Weak Sequential Core of the stochastic bankruptcy game quite generally.

Theorem 5.2 *Assume all players have a differentiable and concave von Neumann–Morgenstern utility function w. Then the allocation prescribed by the SCEA rule belongs to the Weak Sequential Core of the stochastic bankruptcy game $\Gamma = (N, S, v, u)$.*

Proof Let $\bar{x}$ be the allocation following from the SCEA rule. We have already argued that $\bar{x}_s \in \mathrm{C}(\Gamma_s)$ holds for all $s \in S$. By Theorem 3.8 it remains to be shown that there is no x^C such that $x_s^C \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$, and $u^i(x^i) > u^i(\bar{x}^i)$ for all $i \in C$.

Consider a stochastic bankruptcy problem with set of players $C \subset N$, estate in state s equal to $\max\{E_s - \sum_{i \in N \setminus C} d_s^i, 0\}$ and claims equal to d_s^i for $i \in C$. The corresponding stochastic bankruptcy game is denoted by $(C, S, v^C, (u^i)_{i \in C})$. Let $\bar{y}^C$ be the allocation resulting from the SCEA rule.

Note that for $D \subset C$, the worth of coalition D in game v_s^C coincides with its worth in the original game, since

$$\begin{aligned} v_s^C(D) &= \max\{v_s^C(C) - \sum_{i \in C \setminus D} d_s^i, 0\} \\ &= \max\{\max\{E_s - \sum_{i \in N \setminus C} d_s^i, 0\} - \sum_{i \in C \setminus D} d_s^i, 0\}, \end{aligned}$$

where either (a) $E_s - \sum_{i \in N \setminus C} d_s^i > 0$, and so $v_s^C(D) = \max\{E_s - \sum_{i \in N \setminus D} d_s^i, 0\} = v_s(D)$, or (b) $E_s - \sum_{i \in N \setminus C} d_s^i \leq 0$, and so $v_s^C(D) = 0 = v_s(D)$.

We have that $\bar{y}_s^C \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$. We show next that $\bar{y}^C$ maximizes the sum of the players utilities over allocations x^C with $x_s^C \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$.

Consider the following constrained maximization problem,

$$\max_{x^C} \sum_{i \in C} u^i(x^i)$$

$$\text{s.t.} \quad \sum_{i \in C} x_s^i = v_s(C), \quad s \in S, \tag{2}$$

$$\sum_{i \in D} x_s^i \geq v_s(D), \quad s \in S, \ \emptyset \neq D \subsetneq C, \tag{3}$$

where condition (2) is required for ex post efficiency and inequality (3) is a no-blocking condition. A solution to the maximization problem maximizes the sum of the players' utilities among those allocations that belong to $\mathrm{C}(\Gamma_s, C)$ for all $s \in S$.

We form the Lagrangian,

$$\mathcal{L}(x, \lambda, \mu) = \sum_{i \in C} \sum_{s \in S} \rho_s w(x_s^i) - \sum_{s \in S} \mu_s (\sum_{i \in C} x_s^i - v_s(C)) - \sum_{s \in S} \sum_{D \subsetneq C} \lambda_s^D (\sum_{i \in D} x_s^i - v_s(D)),$$

where ρ_s is the common probability attached to the occurrence of state s. The first-order conditions, which are necessary and sufficient for a maximum, are given by

$$\rho_s w'(x_s^i) - \mu_s - \sum_{D \subsetneq C | D \ni i} \lambda_s^D = 0, \quad s \in S, \ i \in C, \tag{4}$$

$$\sum_{i \in C} x_s^i - v_s(C) = 0, \quad s \in S, \tag{5}$$

$$\lambda_s^D (\sum_{i \in D} x_s^i - v_s(D)) = 0, \quad s \in S, \emptyset \neq D \subsetneq C, \tag{6}$$

$$\sum_{i \in D} x_s^i - v_s(D) \geq 0, \quad s \in S, \emptyset \neq D \subsetneq C, \tag{7}$$

$$\lambda_s^D \leq 0, \quad s \in S, \emptyset \neq D \subsetneq C. \tag{8}$$

We will show that together with an appropriate choice of λ and μ, $\bar{y}^C$ satisfies these first-order conditions. Conditions (5) and (7) hold since $\bar{y}_s^C \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$. To show that the remaining conditions hold as well, we introduce two subsets of players for each state, and distinguish two cases. For $s \in S$, let $I_s = \{i \in C | \bar{y}_s^i = d_s^i\}$ be the set of those agents whose claim is completely paid in state s.

1. $I_s = \emptyset$
   For all $\emptyset \neq D \subsetneq C$ we set $\lambda_s^D = 0$, thereby satisfying conditions (6) and (8). Since $I_s = \emptyset$, it holds for all $i \in C$ that $\bar{y}_s^i < d_s^i$. By the definition of the SCEA rule, $\bar{y}_s^i$ is independent of i. It follows that $\rho_s w'(\bar{y}_s^i)$ is also independent of i, thus we can define $\mu_s = \rho_s w'(\bar{y}_s^i)$ for all $i \in C$ to satisfy condition (4).

2. $I_s \neq \emptyset$

Let $C = \{i_s^1, \ldots, i_s^c\}$, where $d_s^{i_s^1} \leq d_s^{i_s^2} \leq \ldots \leq d_s^{i_s^c}$ and c denotes the cardinality of C. Then, using the definition of the SCEA rule, for some $k_s \geq 1$, $I_s = \{i_s^1, \ldots, i_s^{k_s}\}$. For $1 < j \leq k_s + 1$ we define $D_s^j = \{i_s^j, \ldots, i_s^c\}$, so $C \setminus D_s^j \subset I_s$ and $C \setminus D_s^{k_s+1} = I_s$. We define $\mu_s = \rho_s w'(\bar{y}_s^{i_s^1})$, i.e. the marginal utility of the player with the lowest claim in state s. For $1 < j \leq k_s + 1$ we define $\lambda_s^{D_s^j} = \rho_s w'(\bar{y}_s^{i_s^j}) - \rho_s w'(\bar{y}_s^{i_s^{j-1}})$. By the definition of the SCEA rule it holds that $\bar{y}_s^{i_s^j} \geq \bar{y}_s^{i_s^{j-1}}$, so $\lambda_s^{D_s^j} \leq 0$. For other coalitions D we set $\lambda_s^D = 0$. It follows that condition (8) is satisfied. The definition of the SCEA rule and Eq. (1) imply that

$$\sum_{i \in D_s^j} \bar{y}_s^i = v_s(C) - \sum_{i \in C \setminus D_s^j} \bar{y}_s^i = \max\{E_s - \sum_{i \in N \setminus C} d_s^i, 0\} - \sum_{i \in C \setminus D_s^j} d_s^i. \quad (9)$$

Since $\sum_{i \in D_s^j} \bar{y}_s^i \geq 0$,

$$\max\{E_s - \sum_{i \in N \setminus C} d_s^i, 0\} - \sum_{i \in C \setminus D_s^j} d_s^i = \max\{E_s - \sum_{i \in N \setminus D_s^j} d_s^i, 0\} = v_s(D_s^j). \quad (10)$$

It follows from Eqs. (9) and (10) that $\sum_{i \in D_s^j} \bar{y}_s^i - v_s(D_s^j) = 0$, so condition (6) is satisfied.

It only remains to show that condition (4) is satisfied as well. All coalitions D that contain player i_s^1 have $\lambda_s^D = 0$, so for player i_s^1 this is immediate. Consider player $i_s^{j'}$ for $1 < j' \leq k_s$. The only coalitions D such that $i_s^{j'} \in D$ and $\lambda_s^D \neq 0$ are of the form $\{i_s^j, \ldots, i_s^c\}$, for $1 < j \leq j'$. Equation (4) reduces to

$$\rho_s w'(\bar{y}_s^{i_s^{j'}}) - \rho_s w'(\bar{y}_s^{i_s^1}) - \sum_{j=2}^{j'} (\rho_s w'(\bar{y}_s^{i_s^j}) - \rho_s w'(\bar{y}_s^{i_s^{j-1}})) = 0.$$

Finally, consider $i \in C \setminus I_s$. Note that all such players receive the same payoff in state s, and equal to $\bar{y}^{i_s^{k_s+1}}$. Since player i is part of all the coalitions D_s^j, we have that Eq. (4) reduces to

$$\rho_s w'(\bar{y}_s^{i_s^{k_s+1}}) - \rho_s w'(\bar{y}_s^{i_s^1}) - \sum_{j=2}^{k_s+1} (\rho_s w'(\bar{y}_s^{i_s^j}) - \rho_s w'(\bar{y}_s^{i_s^{j-1}})) = 0.$$

Thus $\bar{y}^C$ satisfies all the first-order conditions. It follows that there is no x^C such that $x_s^C \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$, and $u^i(x^i) > u^i(\bar{y}^i)$ for all $i \in C$.

We show next that $u^i(\bar{x}^i) \geq u^i(\bar{y}^i)$ for all $i \in C$. For all $s \in S$, it follows from the definition of a rule that $\sum_{i \in C} \bar{y}_s^i = v_s(C)$ and $v_s(C) \leq \sum_{i \in C} \bar{x}_s^i$. Using the definition

Table 6 Estate allocation

Player	Estate	SSD
$d^1 = 100$	100	100
	200	100
	300	100
$d^2 = 200$	100	0
	200	100
	300	200
$d^3 = 300$	100	0
	200	0
	300	0

of the SCEA rule it follows that for all $s \in S$ and $i \in C$, $\bar{y}^i_s \leq \bar{x}^i_s$. Since the utility function is monotonically increasing, we have that $u^i(\bar{x}^i) \geq u^i(\bar{y}^i)$ for all $i \in C$. Therefore, there is no x^C such that $x^C_s \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$, and $u^i(x^i) > u^i(\bar{x}^i)$ for all $i \in C$, thereby showing part (b) of Theorem 3.8. □

Conditional on the realization of the state, one may impose certain desiderata on the rule. For instance, *equal treatment of equals* would impose that if, conditional on the state, agents have equal claims, then equal awards should be assigned. *Invariance under claims truncation* would require that if in some state s a particular claim d^i_s exceeds the estate E_s, then truncation of the claim by E_s should not lead to a different award. Finally, *composition up* would require if in some state s part E'_s of the estate E_s is divided, and next the remainder $E_s - E'_s$ taking into account the revised claims, then this leads to the same awards as when E_s is divided directly. It has been shown in Dagan (1996) that equal treatment of equals, invariance under claims truncation, and composition up axiomatize the CEA rule, so it is immediate that assuming these axioms conditional on each state characterizes the SCEA rule. In the presence of uncertainty, it follows therefore that the SCEA rule satisfies a number of attractive properties, and additionally is enforceable in the sense that it can neither be blocked ex ante nor ex post in a credible way.

The SCEA rule is not the only rule that belongs to the Weak Sequential Core of a stochastic bankruptcy game. Consider, for instance, a *serial dictator rule* (SSD) where in every state, first player 1 is allowed to take his share, next player 2, and so on. The award to player i' in state s is therefore given by $\min\{d^{i'}_s, \max\{E - \sum_{i<i'} d^i_s, 0\}\}$. In our example, this rule would lead to the allocation presented in Table 6. It is readily verified that this allocation belongs to the Weak Sequential Core.

Habis and Herings (2011) prove Theorem 3.10 in a constructive way, demonstrating that under the conditions of the theorem allocations generated by marginal vectors, where the same permutation of players is used in each state, belong to the Weak Sequential Core. It follows that the SSD rule belongs to the Weak Sequential Core. It can be easily verified that the Serial Dictator rule satisfies invariance under claims truncation and composition up. However, it violates one of the most basic axioms, equal treatment of equals, even when formulated in stronger ways where two agents with equal claims in every state should be assigned equal awards, or where equal awards should be assigned when all agents have the same claims in all states. It is clearly a drawback of Theorem 3.10 that its proof is based on the construction of allocations that are not very appealing. Theorem 5.2 on the other hand presents a rule with sound axiomatic foundations, the SCEA rule, that generates an allocation in the Weak Sequential Core. Surprisingly, among the rules with convincing axiomatic underpinnings, only the most egalitarian one leads to Weak Sequential Core elements.

In cooperative game theory, it is standard to motivate allocation rules by providing axiomatizations. For bankruptcy games, many appealing axiomatizations exist, leading to many possible rules. Theorem 5.2 implies that if one cares about enforceability of rules in the absence of commitment and in the presence of uncertainty, and one would like to use a rule with a convincing axiomatization, one should use the SCEA rule.

6 Conclusion

In this paper we have introduced uncertainty into bankruptcy games. Since in most applications the values of assets in a bankruptcy situation, as well as the claims of the players involved, are uncertain, this is a natural and important extension. In this paper we consider allocations, which are stable in the absence of commitment possibilities. Stability would mean here that no coalition has a credible deviation at any decision node. These considerations lead us to apply the concept of the Weak Sequential Core to stochastic bankruptcy games.

A rule assigns an allocation to each bankruptcy game, where the allocation divides the estate in such a way that every player gets a non-negative amount not exceeding his claim. For deterministic bankruptcy games, it is well-known that a rule leads to an allocation in the core. For a stochastic bankruptcy game, a rule assigns a state-contingent allocation with the property that the state-contingent estate is divided in such a way that every player gets a non-negative amount not exceeding his state-contingent claim. It is straightforward to apply the well-known allocation rules as suggested in the literature on deterministic bankruptcy games to the stochastic setting. We demonstrate that none of the rules typically proposed for bankruptcy situations belongs to the Weak Sequential Core, with the exception of the stochastic extension of the Constrained Equal Awards rule.

Our main result assumes von Neumann–Morgenstern utility functions to support the SCEA rule. It is conceivable that alternative theories of decision making under uncertainty (for instance Choquet expected utility, prospect theory, or maximin expected utility) would support other rules. A first step to address such issues would be to extend the TUU model to alternative theories of decision making under uncertainty, where in particular Choquet expected utility and maximin expected utility are not subsumed by the current specification of the model and the current definition of the Weak Sequential Core.

Our model of stochastic bankruptcy games assumes that all uncertainty is resolved in period 1. Complex bankruptcy situations do typically involve a more gradual resolution of uncertainty. This would call for an extension of our model to a multi-period set-up. New conceptual issues arise in this case, like the intertemporal formation of coalitions, a topic that has not yet received a lot of attention in the literature in general.

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