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Bargaining with non-convexities ☆

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ABSTRACT

We consider the canonical non-cooperative multilateral bargaining game with a set of feasible payoffs that is closed and comprehensive from below, contains the disagreement point in its interior, and is such that the individually rational payoffs are bounded. We show that a pure stationary subgame perfect equilibrium having the no-delay property exists, even when the space of feasible payoffs is not convex. We also have the converse result that randomization will not be used in this environment in the sense that all stationary subgame perfect equilibria do not involve randomization on the equilibrium path. Nevertheless, mixed strategy profiles can lead to Pareto superior payoffs in the non-convex case.

1. Introduction

Many problems in economics are complicated by the presence of non-convexities. Scarf (1994) mentions the omnipresence of non-divisibilities in production as an important source of non-convexities in economics. Another example of non-convexities in production is the existence of production technologies with increasing returns to scale. Other important cases of non-convexities result from non-convexities in preferences, even in the presence of lotteries when agents are not expected utility maximizers as is for instance the case in prospect theory, see Kahneman and Tversky (1979), or when randomization is not possible, and non-convexities in the consumption set, for instance caused by the presence of indivisible commodities. Although non-convexities are regarded important, most of the economic literature assumes them away for reasons of intractability.

Non-convexities are frequently studied in the n-person cooperative bargaining literature. There is for instance an extensive literature on the extension of the Nash bargaining solution to non-convex environments (Kaneko, 1980; Conley and Wilkie, 1996; Mariotti, 1997; Zhou, 1997; Xu and Yoshihara, 2006).

On the contrary, the literature on strategic bargaining has not paid much attention to non-convexities, and if so, only for the case with two players. Rubinstein (1982) allows for modest forms of non-convexities. Under his hypothesis there is typically a unique subgame perfect equilibrium. Herrero (1989) considers general non-convexities for the two-player case assuming the set of feasible payoffs to be strictly comprehensive and studies the convergence of pure stationary subgame perfect equilibria to the appropriately defined Nash bargaining solution, but does not prove the existence of such equilibria.

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Conley and Wilkie (1995) also consider a strictly comprehensive set of feasible payoffs and introduce a bargaining protocol that implements their extension of the Nash bargaining solution.

Other notable exceptions are Lang and Rosenthal (2001), In and Serrano (2004), and Shimer (2006). Lang and Rosenthal (2001) study wage bargaining between a union and a single-product firm which needs two types of workers in its production process. The firm bargains with the union over wages for both worker types and is then free to set employment levels to maximize profits. The objective function of the union includes the wage sum and, possibly, employment levels. For a standard Cobb–Douglas production function, it is shown that the resulting set of feasible payoffs is non-convex. In and Serrano (2004) study a bilateral two-issue bargaining procedure with an endogenous agenda. In the procedure, proposals must be made on only one issue at a time, although the proposer can choose which issue to bring to the table. The reduced form of this game, where subgame with a single issue remaining are replaced by the corresponding subgame perfect equilibrium utilities, is one with a non-convex set of feasible payoffs. Shimer (2006) studies a model where a worker is allowed to quit the current job when offered a higher wage by a different employer. Thus when bargaining about the wage, both the worker and the firm take it into account that the employment will terminate as soon as the worker receives a better offer. As a result, the set of expected payoffs feasible for the worker and the firm fails to be convex. In particular, the firm's profit as a function of the wage can be discontinuous at the wage level equal to that of the firm's competitors. All of the above-mentioned papers treat non-convexities in the two-player case only.

Existence of a pure stationary subgame perfect equilibrium in the canonical multilateral bargaining model has only been shown when the set of feasible payoffs is convex. The existence of such an equilibrium has been shown in Banks and Duggan (2000). Merlo and Wilson (1995) consider the n-person cake division problem and obtain the existence of a unique pure stationary subgame perfect equilibrium when the set of feasible payoffs is convex and the proposer selection protocol is deterministic.

We consider the following canonical multilateral bargaining procedure. In each time period, nature randomly selects a player that is allowed to make a proposal. All players respond sequentially to the proposal and either vote in favor or against. As soon as a responder votes against the proposal, the procedure continues in the next period. If all responders vote in favor of the proposal, it is accepted, and the procedure ends. This model is probably the simplest model of multilateral bargaining known in the literature. Merlo and Wilson (1995), Banks and Duggan (2000), Eraslan (2002), Eraslan and Merlo (2002), and Kalandrakis (2006) are some of the many contributions which use a similar or a more general model of multilateral bargaining.

The bargaining game is fully characterized by the set of players, their discount factors, the set of feasible payoffs, and the probability according to which nature selects a particular proposer. The only assumptions we make regarding the set of feasible alternatives are non-substantial technical ones. We normalize the disagreement payoff to be zero and assume the set of feasible payoffs to be closed, comprehensive from below, and the set of non-negative feasible payoffs to be bounded from above. To make the bargaining problem non-trivial, it is moreover assumed that there is an alternative that gives all players a strictly positive payoff.

We show that this entire class of bargaining games has pure stationary subgame perfect equilibria that ensure immediate agreement. This result is surprising as the usual way to deal with non-convexities is to introduce lotteries. For that reason, one might have expected that the equilibria in non-convex bargaining games typically involve mixing. Similarly, one might have expected that non-convexities are a potential source for delay.

We also address the reverse question. Under what conditions are all stationary subgame perfect equilibria of a bargaining game in pure strategies without delay? The answer is that an extremely mild additional assumption assures this: When the set of weakly Pareto optimal alternatives coincides with the set of Pareto optimal ones, all stationary subgame perfect equilibria involve no randomization on the equilibrium path. Equilibria are characterized by the absence of delay.

To derive the first main result, the existence result, we deviate from the usual proof strategy that basically exploits continuity of the best-response correspondences. In our non-convex setting, this correspondence may not be continuous. Instead, we construct an excess utility function that resembles the excess demand function as used in general equilibrium theory.

Let some profile of utilities be given and consider for each player i the (potentially infeasible) proposal player i has to make in order to be consistent with this profile of utilities. Coordinate i of the excess utility function is the degree of feasibility of this proposal. The excess utility function is shown to have a zero point by showing that it is not outward pointing. Next, a zero point is shown to induce a pure stationary subgame perfect equilibrium of the bargaining game.

To prove the second main result, roughly stating that all stationary subgame perfect equilibria are in pure strategies, we proceed in several steps. One of the main steps is to show that in a mixed stationary subgame perfect equilibrium, proposals offering strictly more than the continuation utility to all players are accepted with probability one, whereas proposals offering at least one player strictly less than the continuation utility are accepted with probability zero. The next main step is to argue that for every player there is a unique proposal which maximizes his utility subject to being accepted with probability one and that every mixed stationary subgame perfect equilibrium puts probability one on such a proposal.

Stationary subgame perfect equilibria are efficient in the sense that every proposer selects a weakly Pareto optimal alternative. However, the fact that all equilibria are in pure strategies implies that equilibria may be inefficient in a weaker sense. It is not difficult to construct examples such that the equilibrium utilities are Pareto dominated by the utilities associated with some mixed strategy profiles.

This paper is organized as follows. Section 2 introduces the model and Section 3 provides the existence result. Section 4 proves the converse result and Section 5 concludes.

2. The bargaining game

We consider the bargaining game $\Gamma = (N, V, \delta, \theta)$. There is a set N of n individuals that have to select a single payoff vector in the set of feasible payoffs V, a non-empty subset of $\mathbb{R}^N$. The vector $\delta = (\delta_i)_{i \in N}$ consists of the players' discount factors and $\theta = (\theta_i)_{i \in N}$ denotes the vector of probabilities according to which players are selected as a proposer. Individuals negotiate about the alternative to be selected using the procedure defined as follows.

In every time period t, starting with $t = 0$, nature selects player i to be the proposer with probability θ_i. The selected player makes a proposal, a point in V. After observing the proposal of player i, all players (including the proposer) vote sequentially on the proposal, the order of their responses being history independent. Each player can either accept or reject the proposal. If the proposal is unanimously accepted, it is implemented. As soon as the first rejection occurs, period $t + 1$ begins, with nature selecting a new proposer, and so on.

The utility to player i of agreement on $v \in V$ in time period t is given by $\delta_i^t v_i$. The utility of perpetual disagreement is equal to 0 for all players. We denote the zero vector in $\mathbb{R}^N$ by 0^N. Players have von Neumann–Morgenstern utility functions.

If v and v' are two vectors in $\mathbb{R}^N$, then we write $v \leq v'$ to mean $v_i \leq v'_i$ for each $i \in N$, and we write $v \ll v'$ to mean $v_i < v'_i$ for each $i \in N$. The set V is said to be comprehensive from below if whenever $v' \in V$, $v \in \mathbb{R}^N$ and $v \leq v'$, then $v \in V$. Our assumptions are as follows.

(A) The set V is closed, comprehensive from below, and it contains the point 0^N in its interior. The set $V_+ = V \cap \mathbb{R}^N_+$ is bounded.
(B) For all $i \in N$, the discount factor δ_i belongs to $[0, 1)$.

We have put our assumptions directly on the set of feasible payoffs V. In other non-cooperative models, the primitives consist of an agreement space X and players with utility functions defined on X. The set V is then given by the image of X under the profile of utility functions $(u^i)_{i \in N}$, so both approaches are closely related.

Our assumptions on V are satisfied provided that the agreement space and the utility functions meet the following requirements:

[S0] the agreement space X takes the form of the product $Y \times M$, where Y is a compact (possibly finite) set, and the utility functions u^i are continuous,
[S1] $M = -\mathbb{R}^N_+$ and $u^i(y, m) \to -\infty$ if $m_i \to -\infty$,
[S2] $u^i(y, m) = u^i(y, m')$ if $m_i = m'_i$, and
[S3] there is a $(y, m) \in Y \times M$ such that $u^i(y, m) > 0$ for each $i \in N$.

The role of the set M is to give every player the possibility to sacrifice an arbitrary amount of utility in order for the set V to be comprehensive from below. There are clearly many alternative specifications for M that would make the set V comprehensive from below. Also, we can weaken our comprehensiveness assumption to hold only on $\mathbb{R}^N_+$, so whenever $v' \in V$, $v \in \mathbb{R}^N_+$ and $v \leq v'$, then $v \in V$. In that case our assumptions on V would be satisfied if S1 is replaced by

[S1′] $M = \{m \in \mathbb{R}^N_+ \mid \sum_{i \in N} m_i \leq c\}$, and $u^i(y, m) = 0$ if $m_i = 0$,

where $c > 0$ is the total available amount of some divisible commodity. When the set Y is finite or when the utility functions u^i are not concave, then the resulting set V is typically not convex.

In specific models, like the ones studied by Lang and Rosenthal (2001) or Shimer (2006), our assumptions on V can easily be verified to hold true. In the set-up of In and Serrano (2004), where there are two issues but bargaining takes place on one issue at a time, the reduced game has typically a non-convex set of feasible payoffs. The reduced game would satisfy our assumptions on V, also if there are more than two players.

A *pure stationary strategy* of an individual i with $\theta_i > 0$ specifies a *proposal* x^i and an *individual acceptance set* A^i. A stationary strategy of an individual i with $\theta_i = 0$ specifies only the individual acceptance set A^i. At every history where player i is selected as a proposer, he makes the proposal x^i and at every history where player i has to respond, he accepts the proposal currently on the table if and only if it belongs to A^i. A strategy profile is a *pure stationary subgame perfect equilibrium* (pure SSPE) if it is pure, stationary, and if it induces a Nash equilibrium in every subgame. A pure strategy profile has the *no-delay* property if $x^i \in \cap_{j \in N} A^j$ for all $i \in N$ with $\theta_i > 0$.

Theorem 2 in Banks and Duggan (2000) states that, if the set V satisfies the assumptions (A) and (B) and is convex, then there exists a pure stationary subgame perfect equilibrium which satisfies the no-delay property. We demonstrate that it suffices to make assumptions (A) and (B). It follows in particular that convexity assumptions are not needed. Under the modest additional assumption that weakly Pareto efficient and Pareto efficient points coincide, we also obtain the converse result that every stationary subgame perfect equilibrium uses pure strategies on the equilibrium path and does not involve delay.

3. Existence of stationary subgame perfect equilibria in pure strategies

In this section we present a system of equations which is such that a solution to the system induces a pure SSPE.

Define the function $g : \mathbb{R}^N \to \mathbb{R}$ by the equation

$$g(v) = \sup\{\lambda \in \mathbb{R} \mid v + (\lambda, \ldots, \lambda) \in V\}.$$

It is well-known that under Assumption (A) the function g has the following properties, where ∂V denotes the boundary of V:

[G0] $v \in V$ if and only if $g(v) \geq 0$, and $v \in \partial V$ if and only if $g(v) = 0$,
[G1] g is continuous,
[G2] $g(0^N) > 0$,
[G3] $g(\hat{v}) \geq 0$ and $v \leq \hat{v}$ implies $g(v) \geq 0$,
[G4] $g^{-1}(\mathbb{R}_+) \cap \mathbb{R}^N_+$ is bounded.

Let $N^* = \{i \in N \mid \theta_i > 0\}$ be the set of players with a strictly positive probability to become the proposer. For $i \in N^*$, we define $\alpha_i = (1 - \delta_i + \theta_i\delta_i)/\theta_i$ and we define the function $p^i : \mathbb{R}^N \to \mathbb{R}^N$ by

$$p^i_j(u) = \begin{cases} \alpha_i u_i, & \text{if } j = i, \\ \delta_j u_j, & \text{if } j \in N \setminus \{i\}. \end{cases}$$

The idea behind the function p^i is as follows. Suppose the game Γ admits a pure SSPE with the property that each proposal is immediately accepted and each proposer leaves every responder indifferent between accepting and rejecting the proposal. This means that $x^i_j = \delta_j u_j$ for $j \neq i$, where u denotes the vector of equilibrium utilities. Player i is the proposer with probability θ_i so his expected utility satisfies the equation $u_i = \theta_i x^i_i + (1 - \theta_i)\delta_i u_i$. By rearranging terms we obtain that $x^i_i = \alpha_i u_i$ if $i \in N^*$. The same rearrangement of terms shows that $u_i = 0$ if $i \in N \setminus N^*$. In what follows we shall demonstrate that there does exist an SSPE such that $x^i = p^i(u)$ for every $i \in N^*$ and $u_i = 0$ for every $i \in N \setminus N^*$.

Let $\bar{v} \in \mathbb{R}^N_{++}$ be an upper bound on V_+, so every $v \in V_+$ satisfies $v \leq \bar{v}$.

We define the function $z : [0^N, \bar{v}] \to \mathbb{R}^N$ as follows:

$$z_i(u) = \begin{cases} g(p^i(u)), & \text{if } i \in N^*, \\ -u_i, & \text{if } i \in N \setminus N^*. \end{cases}$$

Consider some $i \in N^*$. If $z_i(u) < 0$, then there is no payoff vector in the set of feasible payoffs that gives player i a payoff of $\alpha_i u_i$ and players $j \neq i$ a payoff of $\delta_j u_j$, so u_i should be lowered. If $z_i(u) > 0$, then there is a payoff vector in V that gives player i a payoff of strictly more than $\alpha_i u_i$ and players $j \neq i$ a payoff of $\delta_j u_j$. In this case it is possible to increase u_i. The function z is therefore related to an excess demand function as used in general equilibrium theory and we can think of z as an *excess utility function*.

To find equilibria in pure strategies, we are looking for solutions to the system of equations $z(u) = 0^N$. Notice that the system of equations $z(u) = 0^N$ is different from the usual one used to demonstrate the existence of bargaining equilibria (see Merlo and Wilson (1995), Banks and Duggan (2000), Kalandrakis (2004, 2006)), where typically each player is maximizing his utility subject to meeting the reservation values. Our assumptions do not imply that the system of equations employed in the usual approach is continuous.

A zero point u^* of z induces a stationary subgame perfect equilibrium of Γ as follows. For $i \in N^*$ we define

$$x^{*i} = p^i(u^*), \tag{3.1}$$

and for $i \in N$ we set

$$A^{*i} = \{v \in V \mid v_i > \delta_i u^*_i\} \cup \{v \in V \mid v_i = \delta_i u^*_i \text{ and } \forall j \in N^* \setminus \{i\},\ v_j \leq \alpha_j u^*_j\}. \tag{3.2}$$

Whenever a player i has to propose, he makes the proposal x^{*i}. Whenever a player i has to respond, he accepts proposals that offer him strictly more utility than $\delta_i u^*_i$ or that offer him exactly $\delta_i u^*_i$ and do not offer more than $\alpha_j u^*_j$ for players $j \in N^* \setminus \{i\}$. In this construction we exploit the freedom we have in case a player is indifferent between accepting and rejecting a proposal and make acceptance conditional on the utilities as proposed to all the players and not only the utility proposed to the responder himself. The alternative specification where we would define the acceptance set of Player i to be equal to $A^{*i} = \{v \in V \mid v_i \geq \delta_i u^*_i\}$ is in general not compatible with equilibrium existence. The specification in Eq. (3.2) prevents an optimizing proposer from asking more than $\alpha_i u^*_i$ for himself, since such a proposal would be rejected by all responders. When the set V contains points that are only weakly Pareto optimal as is for instance the case in Example 3.7, the proposal that maximizes the proposer's utility subject to offering the other players their reservation values would yield the proposer strictly higher utility than $\alpha_i u^*_i$, and this may be incompatible with equilibrium. Equilibrium play may require a proposer to settle for a weakly Pareto optimal proposal that is not Pareto optimal.

Theorem 3.1. *Given a zero point u^* of the function z, define the strategy profile (x^*, A^*) by Eqs. (3.1) and (3.2). Then (x^*, A^*) is a pure stationary subgame perfect equilibrium of the game Γ. For every $i \in N^*$, it holds that $x^{*i} \in \partial V$.*

Proof. It is well-known that the game Γ has the one-shot deviation property, meaning that if there is a subgame where a player has some profitable deviation from a stationary strategy profile, then there must also be a subgame where this player has a profitable one-shot deviation, i.e. a single deviation by this player at the root of the subgame.

First observe that u^* is the vector of expected payoffs of the strategy (x^*, A^*) since

$$\sum_{i \in N^*} \theta_i x^{*i} = \sum_{i \in N^*} \theta_i p^i(u^*) = u^*.$$

Since $z(u^*) = 0$, it holds in particular that for every $i \in N^*$, $z_i(u^*) = 0$, so $g(p^i(u^*)) = 0$. Consider some $i \in N^*$. By Eq. (3.1) and Property G0 it holds that $x^{*i} = p^i(u^*) \in \partial V$.

We verify that no player has a profitable one-shot deviation. Suppose at some history h at time period t, player $i \in N^*$ is proposer and makes proposal $x^i \in V$, potentially different from x^{*i}. Notice that x^{*i} is accepted leading to the expected utility $\delta_i^t \alpha_i u_i^*$ for player i. If $x^i \notin \cap_{j \in N} A^{*j}$, then it will be rejected, leading to an expected utility of $\delta_i^{t+1} u_i^* < \delta_i^t \alpha_i u_i^*$. If $x^i \in \cap_{j \in N} A^{*j}$ it will be accepted, leading to a utility of $\delta_i^t x_i^i$. We argue that in the latter case $x_i^i \leq \alpha_i u_i^*$, so the expected utility to player i of proposing x^i is less than or equal to the expected utility of proposing x^{*i}. Since $x^{*i} \in \partial V$, $x^i \in V$, and V is comprehensive from below, there is at least one $j \in N$ such that $x_j^i \leq x_j^{*i}$. If $j = i$, then $x_i^i \leq x_i^{*i} = \alpha_i u_i^*$ and we are done. If $j \neq i$, then $x_j^i \leq x_j^{*i} = \delta_j u_j^*$. Since $x^i \in A^{*j}$, we have $x_j^i = \delta_j u_j^*$ and for every $k \in N^* \setminus \{j\}$, $x_k^i \leq \alpha_k u_k^*$, so in particular $x_i^i \leq \alpha_i u_i^*$.

Suppose at some history h at time period t, player $i \in N$ is responder to a proposal $v \in A^{*i}$. In equilibrium, player i accepts. The expected utility to player i is $\delta_i^t v_i$ if the other players accept and $\delta_i^{t+1} u_i^*$ if some other player rejects, so the expected utility weakly exceeds $\delta_i^{t+1} u_i^*$ in any case. Would player i deviate to a rejection, then his expected utility equals $\delta_i^{t+1} u_i^*$. Consider a proposal $v \notin A_i^*$, so in particular $v_i \leq \delta_i u_i^*$. In equilibrium, player i rejects, leading to expected utility $\delta_i^{t+1} u_i^*$. A deviation to acceptance leads to expected utility $\delta_i^t v_i$ if others accept and $\delta_i^{t+1} u_i^*$ if some other player rejects. In either case, expected utility is bounded above by $\delta_i^{t+1} u_i^*$. $\square$

In order to prove that the game Γ has a pure SSPE, we show that the function z has a zero point.

Definition 3.2. Let $\underline{a}, \bar{a}$ in $\mathbb{R}^m$ be such that $\underline{a} \ll \bar{a}$. The function $f : [\underline{a}, \bar{a}] \to \mathbb{R}^m$ is *outward pointing at* $a \in [\underline{a}, \bar{a}]$ if $f(a) \neq 0$ and, for $k = 1, \ldots, m$, $a_k = \underline{a}_k$ implies $f_k(a) \leq 0$, $\underline{a}_k < a_k < \bar{a}_k$ implies $f_k(a) = 0$, and $a_k = \bar{a}_k$ implies $f_k(a) \geq 0$. A function is *outward pointing* if it is outward pointing at some point of $[\underline{a}, \bar{a}]$.

We remark that the function f is outward pointing at a point a of the set $[\underline{a}, \bar{a}]$ if and only if $f(a)$ is a non-zero element of the normal cone of $[\underline{a}, \bar{a}]$ at the point a.

We show next that the excess utility function z is not outward pointing.

Lemma 3.3. *The excess utility function z is continuous and not outward pointing.*

Proof. The continuity of z follows immediately from its definition and property G1, which states that g is continuous.

Trivially, z is not outward pointing at any u in the interior of $[0^N, \bar{v}]$.

Suppose z is outward pointing at $u \in [0^N, \bar{v}]$ such that $u_i = \bar{v}_i$ for some $i \in N$. Fix any such player i. Then $z_i(u) \geq 0$. Since clearly it is not the case that $u_i = \bar{v}_i \leq 0$, we have $i \in N^*$ and so $g(p^i(u)) \geq 0$, hence $p^i(u) \in V$. Since the vector $p^i(u)$ is non-negative, it is an element of V_+.

Assume first that the cardinality of N^* is equal to 1 so $N^* = \{i\}$. Then $\theta_i = 1$, and $\theta_j = 0$ if $j \neq i$. If there is $j \in N \setminus N^*$ with $u_j > 0$, then $z_j(u) = -u_j < 0$, which contradicts that z is outward pointing at u. If $u_j = 0$ for every $j \neq i$, then $z_i(u) = g(p^i(u)) = g(u) \leq 0$ and $z_j(u) = -u_j = 0$ if $j \neq i$. By the previous paragraph $z_i(u) \geq 0$, so in fact $z(u) = 0$, and z is not outward pointing at u.

Assume next that the cardinality of N^* is greater than or equal to 2, so that $\alpha_i > 1$. Then $p_i^i(u) = \alpha_i u_i > u_i = \bar{v}_i$ which contradicts our choice of the vector $\bar{v}$ and the fact that $p^i(u) \in V_+$.

Suppose z is outward pointing at 0^N. For each $i \in N^*$ it holds that $z_i(0^N) = g(p^i(0^N)) = g(0^N) > 0$, a contradiction.

Finally, suppose z is outward pointing at u such that $0^N \leq u \ll \bar{v}$, with $0 = u_i$ for some $i \in N$ and $0 < u_j$ for some $j \in N$. Since z is outward pointing, it holds that $z_j(u) = 0$. It follows that $j \in N^*$ and $g(p^j(u)) = 0$, which means that $p^j(u) \in V$. For each $i \in N^*$ with $u_i = 0$ it holds that $p^i(u) \leq p^j(u)$, so $p^i(u) \in V$ and consequently $z_i(u) = g(p^i(u)) \geq 0$. This shows that $z_i(u) \geq 0$ for all $i \in N$ with $u_i = 0$. Since z is outward pointing at u it follows that $z_i(u) = 0$ for every i such that $u_i = 0$. We find that $z(u) = 0$, a contradiction to z being outward pointing at u. $\square$

Hartman and Stampacchia (1966) have shown that for any continuous function on a compact, convex subset C of $\mathbb{R}^m$ mapping into $\mathbb{R}^m$, there is a point $c \in C$ such that the function value at c belongs to the normal cone of C at c. Eaves (1971) calls such a point a stationary point. For the case where C is equal to $[\underline{a}, \bar{a}]$, a function that is not outward pointing as defined in Definition 3.2 has the property that every stationary point is a zero point. Continuous functions on $[\underline{a}, \bar{a}]$ that are not outward pointing therefore have a zero point. To make the paper self-contained, we present a simple proof of this fact next.

Lemma 3.4. *Let $\underline{a}, \bar{a}$ in $\mathbb{R}^m$ be such that $\underline{a} \ll \bar{a}$ and let the continuous function $f^0 : [\underline{a}, \bar{a}] \to \mathbb{R}^m$ be not outward pointing. Then f^0 has a zero point.*

Proof. Let $C = [\underline{a} - 1^m, \bar{a} + 1^m]$. We define the function $f^1 : C \to \mathbb{R}^m$ by

$$f^1(a) = \lambda(a)(\pi_{[\underline{a},\bar{a}]}(a) - a) + (1 - \lambda(a)) f^0(\pi_{[\underline{a},\bar{a}]}(a)), \quad a \in C,$$

where $\pi_{[\underline{a},\bar{a}]}$ is the orthogonal projection function on $[\underline{a}, \bar{a}]$, a function that is continuous, and

$$\lambda(a) = \|\pi_{[\underline{a},\bar{a}]}(a) - a\|_\infty, \quad a \in C,$$

so the function $\lambda : C \to [0, 1]$ measures the distance in infinity norm from the point a to its projection. The function λ is continuous and has the property that it is equal to 1 on the boundary of C, equal to 0 on $[\underline{a}, \bar{a}]$, and strictly in between 0 and 1 everywhere else. We define the function $f^2 : C \to C$ by

$$f^2(a) = \pi_C(a + f^1(a)), \quad a \in C.$$

The function f^2 is a continuous function defined on a non-empty, compact, convex set, so has a fixed point, say a^*, by Brouwer's fixed point theorem.

Suppose a^* belongs to the boundary of C, i.e. $\lambda(a^*) = 1$. Then

$$a^* = f^2(a^*) = \pi_C(a^* + f^1(a^*)) = \pi_C(a^* + \pi_{[\underline{a},\bar{a}]}(a^*) - a^*) = \pi_{[\underline{a},\bar{a}]}(a^*) \neq a^*,$$

a contradiction. It follows that a^* is not in the boundary of the set C, i.e. it belongs to its interior. From this it follows that

$$a^* = \pi_C(a^* + f^1(a^*)) = a^* + f^1(a^*),$$

so $f^1(a^*) = 0^m$. Using the definition of f^1 it then follows that

$$f^0(\pi_{[\underline{a},\bar{a}]}(a^*)) = -\frac{\lambda(a^*)}{1 - \lambda(a^*)}(\pi_{[\underline{a},\bar{a}]}(a^*) - a^*).$$

Now if a^* is not an element of the set $[\underline{a}, \bar{a}]$, then $0 < \lambda(a^*) < 1$ and the vector $a^* - \pi_{[\underline{a},\bar{a}]}(a^*)$ is a non-zero element of the normal cone of $[\underline{a}, \bar{a}]$ at the point $\pi_{[\underline{a},\bar{a}]}(a^*)$. But this means that f^0 is outward pointing at $\pi_{[\underline{a},\bar{a}]}(a^*)$, a contradiction. We conclude that $a^* \in [\underline{a}, \bar{a}]$. Thus $\lambda(a^*) = 0$ and $\pi_{[\underline{a},\bar{a}]}(a^*) = a^*$, so a^* is a zero point of the function f^0. □

Corollary 3.5. *The excess utility function z has a zero point.*

Corollary 3.6. *The game Γ admits a pure stationary subgame perfect equilibrium.*

Example 3.7. Consider the set of feasible payoffs

$$V = \{v \in \mathbb{R}^2 \mid v_1 \leq 2,\ v_2 \leq 2,\ \min\{v_1, v_2\} \leq 1\},$$

which is depicted in Fig. 1. An upper bound on V_+ is given by $\bar{v} = (2, 2)$. Players are selected as proposer with equal probability, so $\theta_1 = \theta_2 = 1/2$.

We describe all subgame perfect equilibria selected by the excess utility function z. Since $\alpha_i = 2 - \delta_i$, the excess utility function z is then defined by

$$z_1(u) = g(p^1(u)) = g((2 - \delta_1)u_1, \delta_2 u_2), \quad u \in [0, \bar{v}],$$
$$z_2(u) = g(p^2(u)) = g(\delta_1 u_1, (2 - \delta_2)u_2), \quad u \in [0, \bar{v}].$$

The relevant part of the boundary of V consists of four types of line segments, characterized by $0 \leq v_1 \leq 1$ and $v_2 = 2$, $v_1 = 1$ and $1 \leq v_2 \leq 2$, $1 \leq v_1 \leq 2$ and $v_2 = 1$, and $v_1 = 2$ and $0 \leq v_2 \leq 1$. Since in equilibrium both $p^1(u)$ and $p^2(u)$ belong to such a line segment, there are potentially sixteen types of equilibria, where a type of equilibrium corresponds to a particular combination of line segments to which the proposals of the two players belong. When we solve for $z(u) = 0$ for each of the sixteen resulting systems of equations, and taking into account that $0 \leq \delta_1, \delta_2 < 1$, we find equilibrium utility levels u_1^* and u_2^*. From these, we can derive the equilibrium proposals x^{*1} and x^{*2} and the equilibrium acceptance sets A^{*1} and A^{*2}. The equilibrium proposals are displayed in Table 1. It turns out that only six out of the potential sixteen types of equilibrium can actually occur, labeled by A to F. The conditions on the discount factors that lead to a particular equilibrium type are displayed in Table 1 and depicted in Fig. 2.

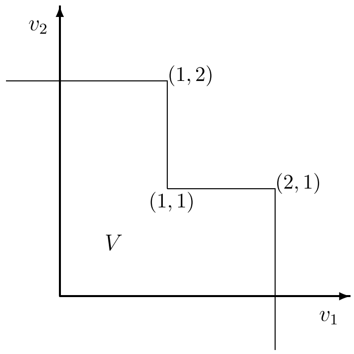

Fig. 1. The set of feasible payoffs.

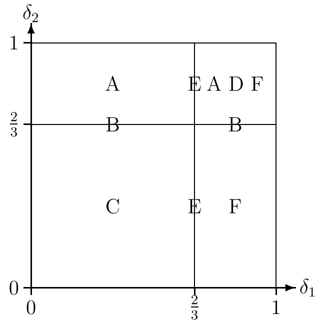

Fig. 2. Equilibrium regions.

Table 1
A summary of the possible equilibrium proposals.

	Discount factors	x^{*1}	x^{*2}	
A	$0 \le \delta_1 < 1,\ \frac{2}{3} < \delta_2 < 1$	$(1, \frac{2\delta_2}{2-\delta_2})$	$(\frac{\delta_1}{2-\delta_1}, 2)$	
B	$0 \le \delta_1 < 1,\ \delta_2 = \frac{2}{3}$	$((2-\delta_1)u_1^*, 1)$	$(\delta_1 u_1^*, 2)$	$\frac{1}{2-\delta_1} \le u_1^* \le \min\{\frac{2}{2-\delta_1}, \frac{1}{\delta_1}\}$
C	$0 \le \delta_1 < \frac{2}{3},\ 0 \le \delta_2 < \frac{2}{3}$	$(2, \frac{2\delta_2}{2-\delta_2})$	$(\frac{2\delta_1}{2-\delta_1}, 2)$	
D	$\delta_1 > \frac{2}{3},\ \delta_2 > \frac{2}{3}$	$(\frac{2-\delta_1}{\delta_1}, 1)$	$(1, \frac{2-\delta_2}{\delta_2})$	
E	$\delta_1 = \frac{2}{3},\ 0 \le \delta_2 < 1$	$(2, \delta_2 u_2^*)$	$(1, (2-\delta_2)u_2^*)$	$\frac{1}{2-\delta_2} \le u_2^* \le \min\{\frac{2}{2-\delta_2}, \frac{1}{\delta_2}\}$
F	$\frac{2}{3} < \delta_1 < 1,\ 0 \le \delta_2 < 1$	$(2, \frac{\delta_2}{2-\delta_2})$	$(\frac{2\delta_1}{2-\delta_1}, 1)$	

Table 1 in conjunction with Fig. 2 shows that equilibria are unique for discount factors in Regions A, C, and F in case at least one player has a discount factor below 2/3 and no player has a discount factor equal to 2/3. When the discount factor of at least one player is exactly equal to 2/3, we are in Regions B or E, and there are infinitely many equilibria and infinitely many possible equilibrium utilities. Finally, when both players have a discount factor above 2/3, equilibria of type A, D, and F co-exist. Fig. 3 illustrates a typical combination of proposals for the various types of equilibria. Fig. 4 illustrates the three possible equilibria when $\delta_1 = \delta_2 = 5/6$.

A striking feature of Example 3.7 is that the equilibrium proposals are typically not Pareto optimal, but only weakly so. The only exception are cases where $\delta_1 = 2/3$ or $\delta_2 = 2/3$, when there is a continuum of equilibria, and only one equilibrium in the continuum involves Pareto optimal proposals.

The most extreme equilibrium proposals for player 1 are $x^{*1} = (2, 0)$, which occurs when $\delta_2 = 0$ and $0 \le \delta_1 < 2/3$, and $x^{*1} = (1, 2-\varepsilon)$ when $\delta_2 = (4-2\varepsilon)/(4-\varepsilon)$ and $0 \le \delta_1 < 1$. Notice that in the latter equilibrium, player 1 may offer more to player 2 than to himself, even if player 1 is more patient than player 2.

Another interesting feature of this example is that comparative statics may be counterintuitive. Consider the symmetric equilibrium corresponding to Region D when both players have discount rates exceeding 2/3. In this region it holds that increasing patience worsens the bargaining position of a player. When discount rates converge to 1, the equilibrium proposals of both players converge to $(1, 1)$, a payoff that is weakly dominated for both players by alternative payoffs.

The example also demonstrates that equilibrium proposals are not even weakly Pareto optimal when compared to lotteries over feasible payoffs. When players become sufficiently patient, their equilibrium proposals become arbitrarily close to $(1, 1)$. A lottery that selects payoff $(2, 1)$ with probability 1/2 and payoff $(1, 2)$ with probability 1/2 gives strictly higher utility to both players than both equilibrium proposals.

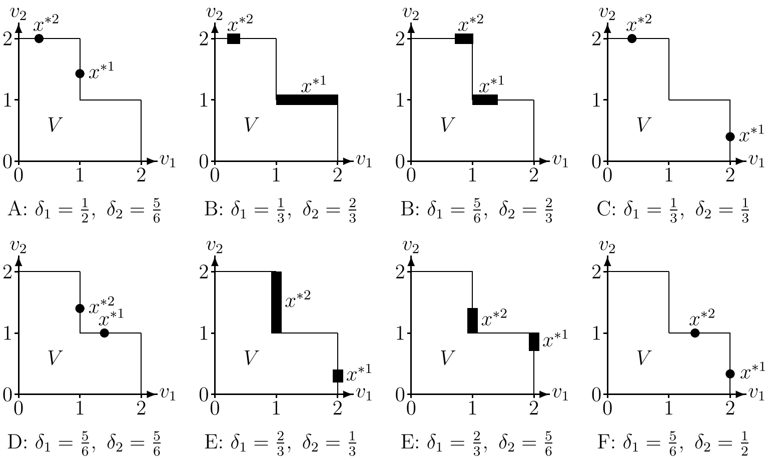

Fig. 3. Equilibrium proposals.

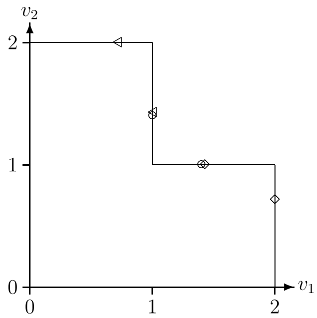

Fig. 4. $\delta_1 = \frac{5}{6}$, $\delta_2 = \frac{5}{6}$: A ◁, D ∘, F ⋄.

Not every stationary subgame perfect equilibrium is in the form expressed by Eqs. (3.1) and (3.2). Indeed, for every δ_1 and δ_2 greater than 1/2 there exists a stationary subgame perfect equilibrium where both players propose one of the two Pareto-optimal points, say the point $(1, 2)$. In such an equilibrium the proposals are $x^1 = x^2 = (1, 2)$ and the acceptance sets are $A_1 = \{v \in V \mid v_1 \geq \delta_1\}$ and $A_2 = \{v \in V \mid v_2 \geq 2\delta_2\}$. The pair (x, A) is indeed a stationary subgame perfect equilibrium provided that $1/2 < \delta_1 < 1$ and $1/2 < \delta_2 < 1$. In the next section we will identify a simple assumption under which every stationary subgame perfect equilibrium satisfies Eqs. (3.1) and (3.2).

4. A characterization of all stationary subgame perfect equilibria

Banks and Duggan (2000) show that if the set V satisfies assumptions (A) and (B) and, moreover, is strictly convex, then every no-delay stationary subgame perfect equilibrium is in pure strategies.

In this section we only use assumptions (A) and (B) and employ the additional assumption that each weakly Pareto-efficient point of V_+ is Pareto efficient. Under this assumption we show that in every stationary subgame perfect equilibrium of the game Γ, each player chooses a pure strategy on the equilibrium path, leading to a proposal which is immediately accepted by all the players. Furthermore, the vector of utilities in any stationary subgame perfect equilibrium of Γ is a zero of the function z as defined in Section 3.

Let $\mathcal{B}$ be the set of Borel subsets of V. A *stationary strategy* for player $i \in N^*$ can be summarized by a pair (μ^i, τ^i) where $\mu^i : \mathcal{B} \to [0, 1]$ is a probability measure and $\tau^i : V \to [0, 1]$ is a $\mathcal{B}$-measurable function. The number $\mu^i(B)$ is the probability for player i to choose a proposal from the Borel set B, and $\tau^i(v)$ is the probability for player i to accept the proposal v. For a player $i \in N \setminus N^*$, the stationary strategy specifies only the function τ^i. Given a strategy profile (μ, τ), let $\tau(v) = \tau^1(v) \times \cdots \times \tau^n(v)$ denote the probability that the proposal v is unanimously accepted. Let u_i denote the expected utility to player i. Conditional on receiving the proposal v, player i's expected utility is equal to $[\tau(v)v_i + (1 - \tau(v))\delta_i u_i]$. Thus u_i satisfies the equation

$$u_i = \sum_{j \in N^*} \theta_j \int_V [\tau(v)v_i + (1 - \tau(v))\delta_i u_i] d\mu^j(v). \tag{4.1}$$

Notice that Eq. (4.1) admits exactly one solution.

Example 4.1. In addition to its pure strategy subgame perfect equilibria, the game considered in Example 3.7 also possesses stationary subgame perfect equilibria where both the proposers and the responders play mixed actions. Below we construct a subgame perfect equilibrium where player 1 proposes several points of the form $(x, 1)$ with positive probability. To support this as an equilibrium strategy we choose the functions τ^2 in such a way that the expected utility for player 1 from proposing each point $(x, 1)$ is the same for every x in some non-degenerate interval. Player 2 plays a mixed response to each proposal $(x, 1)$ of player 1 because his equilibrium continuation utility is 1, so he is indifferent between accepting and rejecting each such proposal. Similarly, player 2 proposes several points of the form $(1, x)$ with positive probability, and player 1 plays a mixed response to each proposal $(1, x)$ of player 2.

Fix $\delta_i = \delta \in (2/3, 1)$ and let

$$S^1 = \{v \in V \,|\, (2-\delta)/\delta \le v_1 \le 2 \text{ and } v_2 = 1\},$$
$$S^2 = \{v \in V \,|\, (2-\delta)/\delta \le v_2 \le 2 \text{ and } v_1 = 1\}.$$

Let μ^1 and μ^2 be arbitrarily chosen probability distributions on $\mathcal{B}$ such that $\mu^i(S^i) = 1$. Define the functions τ^1 and τ^2 as

$$\tau^1(v) = \begin{cases} \dfrac{2(1-\delta)}{\delta(v_2-1)}, & \text{if } v \in S^2, \\ 1, & \text{if } v_1 > 1, \\ 0, & \text{otherwise}, \end{cases}$$
$$\tau^2(v) = \begin{cases} \dfrac{2(1-\delta)}{\delta(v_1-1)}, & \text{if } v \in S^1, \\ 1, & \text{if } v_2 > 1, \\ 0, & \text{otherwise}. \end{cases}$$

This strategy profile results in an expected utility of $u_i = 1/\delta$ to both players. To see this, we verify next that $u_i = 1/\delta$ solves Eq. (4.1). Thus assume that the expected utility to each player i is indeed $1/\delta$. Then the expected utility to player 1 following the proposal v is given by

$$\tau(v)v_1 + (1-\tau(v))\delta u_1 = \begin{cases} (2-\delta)/\delta, & \text{if } v \in S^1, \\ 1, & \text{if } v \in S^2. \end{cases}$$

Since player i's proposal is chosen from S^i with probability 1, the right-hand side of Eq. (4.1) is $(2-\delta)/2\delta + 1/2 = 1/\delta$. Thus $u_1 = 1/\delta$ indeed solves Eq. (4.1). Of course, the argument for player 2 is symmetric.

Now, each proposal in S^i gives player i the same expected utility, namely $(2-\delta)/\delta$ and no proposal outside S^i gives a higher expected utility. Thus choosing, possibly at random, a proposal from S^i is sequentially rational for each player i. Furthermore, player j is indifferent between accepting and rejecting a proposal $v \in S^i$ of player $i \neq j$, since either way player j obtains an expected utility of 1. Thus player j can play a mixed response to any proposal from S^i. This establishes that the joint strategy above is a subgame perfect equilibrium.

Recall that a point $v \in V$ is said to be *weakly Pareto-efficient* if there is no point $x \in V$ such that $x_i > v_i$ for each $i \in N$. It is Pareto-efficient if there is no point $x \in V$ such that $x_i \ge v_i$ for each $i \in N$ with one inequality being strict. We employ the following additional assumption:

(C) Each weakly Pareto-efficient point of V_+ is Pareto efficient.

Assumption (C) is widely used in the literature and is often referred to as the condition of "non-levelness" of the relevant part of the boundary of the set V.

We now proceed to show that if the payoff set satisfies the assumptions (A), (B), and (C), then in any stationary subgame perfect equilibrium of the game Γ, (i) each proposer plays a pure strategy, i.e. chooses some proposal with probability 1, and (ii) the equilibrium proposal is accepted by each responder with probability 1. The intuition for the result can be summarized as follows: Given a vector of equilibrium utilities there is only one proposal that maximizes the proposer's utility subject to meeting the reservation utility of all the responders. Thus the proposer has a unique best response. And the responders have to accept the equilibrium proposals with probability one since otherwise the proposer will not have a well-defined best response. Notice that Theorem 4.2 leaves open the possibility that the responders cast a vote at random when responding to an out of equilibrium proposal.

Theorem 4.2. *Let the strategy profile $\sigma = (\mu, \tau)$ be an SSPE of Γ. Then for each $i \in N^*$ there exists a proposal x^i in V such that $\mu^i(\{x^i\}) = 1$ and $\tau^j(x^i) = 1$ for all $j \in N$. Furthermore, the equilibrium utility u induced by σ satisfies $z(u) = 0$.*

Proof. For $i \in N$, let $r_i = \delta_i u_i$. Define the sets D and B by

$$D = \bigcap_{i \in N} \{v \in V \mid v_i \ge r_i\} \text{ and } B = \bigcap_{i \in N} \{v \in V \mid v_i > r_i\}.$$

STEP 0: It holds that $u_i \geq 0$ for each $i \in N$.

Indeed, rejecting any proposal yields a player a utility of zero irrespective of the strategies of other players. Thus, in any Nash equilibrium of the game Γ the utility to any player is at least zero.

STEP 1: If $v \in V$ is such that $\tau(v) > 0$, then $v \in D$.

Suppose $\tau(v) > 0$ and consider a history h at time period 0 after which player i has to respond to the proposal v. Notice that according to the strategy profile σ all players accept v with strictly positive probability. A rejection of v by player i yields him a utility of r_i, while accepting it yields a utility of v_i with some positive probability and r_i with the complementary probability. Since, according to the subgame perfect equilibrium strategy profile σ, player i accepts v with positive probability, we must have the inequality $v_i \geq r_i$.

STEP 2: If $v \in B$, then $\tau(v) = 1$.

Suppose without loss of generality that the players respond in the sequence $1, \dots, n$. Consider a history h at time period 0 after which player n has to respond to the proposal v. All players preceding player n have accepted the proposal v, because otherwise player n is not requested to cast his vote by definition of the game Γ. Accepting v by player n yields him a utility of v_n, while rejecting it leads to a utility of $\delta_n u_n = r_n$. Thus player n has to accept v with probability 1. We conclude that $\tau_n(v) = 1$.

Suppose that $\tau_{i+1}(v) = \cdots = \tau_n(v) = 1$ for some i. Consider a history h at time period 0 after which player i has to respond to the proposal v. According to the definition of the game Γ, the players preceding i in the response sequence have all accepted the proposal v. The players $i+1, \dots, n$ will accept the proposal v with probability 1 by the induction hypothesis, if player i accepts it. Thus accepting v by player i yields a utility of v_i, while rejecting it gives a utility of $\delta_i u_i = r_i$. We conclude that $\tau_i(v) = 1$.

STEP 3: The set B is non-empty.

Suppose B is an empty set. We show next that then $D \subset \{r\}$. For suppose the set D contains a point v other than r. Then the point r is not Pareto-efficient and, by Assumption (C), it is not weakly Pareto-efficient. It follows that there is a point $v \in V$ such that $v_i > r_i$ for each $i \in N$. But such a point v is an element of the set B, a contradiction. This establishes that the set D is either empty or it contains the point r alone.

For each $i \in N$ and each $v \in V$, it holds that $\tau(v)v_i + (1-\tau(v))\delta_i u_i = \delta_i u_i$, since it follows from Step 1 that if $\tau(v) > 0$, then $v = r$, whereas the equality is trivial when $\tau(v) = 0$. Using Eq. 4.1 we find that $u_i = \delta_i u_i$ for each $i \in N$. We conclude that $u = 0$. But 0 belongs to the interior of V by Assumption (A), which implies that $B = \cap_{i\in N}\{v \in V \mid v_i > 0\}$ is a non-empty set, a contradiction.

STEP 4: The set D equals the closure of B.

Take $v \in D$ and an open neighborhood O of v. We must show that the intersection $O \cap B$ is non-empty. Consider the point $y = (1-\lambda)v + \lambda r$ for some $\lambda \in (0,1)$ chosen small enough such that y lies in the set O. Since $r_i \leq y_i \leq v_i$ for each $i \in N$ and since V is comprehensive from below by Assumption (A), $y \in D$. The point y is not Pareto-efficient. Indeed, if $y = v$, then $y = r$, which is not Pareto-efficient because the set B is non-empty. And if y is not equal to v, it is dominated by v. Hence, by Assumption (C), the point y is not weakly Pareto-efficient. Thus there is a point $x \in V$ such that $y_i < x_i$ for all $i \in N$. Consider the point $x(\epsilon) = \epsilon x + (1-\epsilon)y$ for $0 < \epsilon < 1$. Since $r_i \leq y_i < x_i(\epsilon) \leq x_i$ for each $i \in N$ and since V is comprehensive from below, $x(\epsilon) \in B$. And since $y \in O$, one can choose ϵ small enough so that $x(\epsilon) \in O$.

STEP 5: For $i \in N^*$, the set $X^i = \arg\max\{v_i | v \in D\}$ contains a single element, x^i. The point x^i lies on the boundary of V and $x^i_j = r_j$ for each $j \in N \setminus \{i\}$ and $x^i_i > r_i$.

The set X^i is non-empty, because D is a compact set. Take any point $x \in X^i$ and suppose $r_j < x_j$ for some $j \in N \setminus \{i\}$. Define the point v by the equation

$$v_k = \begin{cases} x_k, & \text{if } k \in N \setminus \{j\}, \\ r_j, & \text{if } k = j. \end{cases}$$

Thus $v_k \leq x_k$ with strict inequality for $k = j$. Since V is comprehensive from below by Assumption (A), $v \in V$. Furthermore, the point v is not Pareto-efficient being dominated by x. Hence by Assumption (C) it is not weakly Pareto-efficient. Therefore, there exists a point $y \in V$ such that $v_k < y_k$ for each $k \in N$. Since $r_k \leq x_k = v_k < y_k$ for each $k \in N \setminus \{j\}$ and $r_j = v_j < y_j$, the point y is an element of D. Furthermore $x_i < y_i$, contradicting the choice of x in X^i.

We have thus shown that, for each $x \in X^i$, for each $j \in N \setminus \{i\}$, $x_j = r_j$. It now follows at once that X^i contains a single element.

If x^i were not a boundary point of V, there would be a point $y \in V$ such that $x_k < y_k$ for all $k \in N$. Such a point y would then be in the set D, contradicting the fact that x^i is an element of X^i. Finally, $x^i_i > r_i$ for otherwise $x^i = r$, while r is not an element of the boundary of V.

STEP 6: For $i \in N^*$, it holds that $\mu^i(\{x^i\}) = 1$ and $\tau(x^i) = 1$.

Let $q_i(v) = \tau(v)v_i + (1-\tau(v))r_i$ be the utility to player i from proposing the point $v \in V$ and let $Q^i = \arg\max\{q_i(v)| v \in V\}$ and $q_i = \max\{q_i(v)|v \in V\}$. Since μ^i is part of an SSPE, we have that $\mu^i(Q^i) = 1$. In particular, Q^i is a non-empty set. By Step 2, any point v of B is accepted with probability 1. Hence $q_i \geq v_i$ for each $v \in B$. Since by Step 4 the set D is the closure of B, we must have $q_i \geq v_i$ for each $v \in D$, and therefore $q_i \geq x^i_i$. On the other hand, if $v \in V \setminus D$ then $\tau(v) = 0$ by

Step 1, so $q_i(v) = r_i < x_i^i$. And if $v \in D \setminus \{x^i\}$ then $v_i < x_i^i$ by Step 5, so $q_i(v) < x_i^i$. Thus $q_i(v) < x_i^i \le q_i$ for all $v \in V \setminus \{x^i\}$ and hence $Q^i \subset \{x^i\}$. Thus $\mu(\{x^i\}) = 1$ and $Q^i = \{x^i\}$ and $q_i = q(x^i)$. Now if $\tau(x^i) < 1$ then $q_i = q_i(x^i) < x_i^i \le q_i$, a contradiction. Hence $\tau(x^i) = 1$.

Step 7: The vector of ex ante expected utilities u satisfies $z(u) = 0$.

By Step 5 we have $x_j^i = \delta_j u_j$ for each $i \in N^*$ and each $j \in N$ such that $j \neq i$. Eq. (4.1) now reads

$$u_i = \sum_{j \in N^*} \theta_j x_i^j = \theta_i x_i^i + (1 - \theta_i)\delta_i u_i.$$

Thus if $i \in N^*$ then $x_i^i = \alpha_i u_i$ and therefore $x^i = p^i(u)$. And, also by Step 5, the point x^i lies in the boundary of the set V, so $z(p^i(u)) = 0$. And if $i \in N \setminus N^*$, then $u_i = 0$, since this is the only solution to $u_i = \delta u_i$. In either case $z_i(u) = 0$. □

5. Conclusion

The presence of non-convexities poses many problems in economic modeling. Such problems are usually resolved by the use of lotteries, which convexify the problem. In this paper we argue that the canonical model of non-cooperative bargaining does not involve such difficulties. Even when the set of feasible payoffs is not convex, there exists a pure stationary subgame perfect equilibrium satisfying the no-delay property. The only assumptions on the set of feasible payoffs that are needed for this result are that the set of feasible payoffs is closed, comprehensive from below, and that its restriction to the individually rational payoffs is bounded.

When we impose the mild additional requirement that the weak Pareto optimal payoffs in the set of feasible payoffs coincide with the Pareto optimal ones, we also obtain the reverse result that all subgame perfect equilibria in stationary strategies use pure strategies on the equilibrium path and lead to absence of delay. When players bargain in a non-convex environment, it is not only the case that equilibria without randomization exist, but even stronger, there are no equilibria where randomization is used. Nevertheless, it is easy to construct examples where stationary mixed strategy profiles would lead to Pareto improvements of the equilibrium utilities.

Here we have restricted ourselves to the canonical model of non-cooperative bargaining. It is natural to examine to what extent our main results are also valid in extensions of this basic model, allowing for more general proposer selection and cake processes as studied for instance in Merlo and Wilson (1995). We have studied the classical case with unanimous acceptance of proposals. A generalization of our results to the case with a general set of decisive coalitions as in Banks and Duggan (2000) will fail to hold, since in such a setting pure stationary subgame perfect equilibria may fail to exist even when the sets of feasible payoffs are convex. The assumption of unanimous approval seems therefore to be crucial for the existence of pure stationary subgame perfect equilibria and the absence of randomization in general environments.

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