JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 96, No. 2, pp. 311-335, FEBRUARY 1998

Intersection Theorems with a Continuum of Intersection Points[1]

P. J. J. HERINGS[2] AND A. J. J. TALMAN[3]

Communicated by F. Zirilli

Abstract. In all existing intersection theorems, conditions are given under which a certain subset of a collection of sets has a nonempty intersection. In this paper, conditions are formulated under which the intersection is a continuum of points satisfying some interesting topological properties. In this sense, the intersection theorems considered in this paper belong to a new class. The intersection theorems are formulated on the unit cube and it is shown that both the vector of zeroes and the vector of ones lie in the same component of the intersection. An interesting application concerns the model of an economy with price rigidities. Using the intersection theorems of this paper, it is easily shown that there exists a continuum of zero points in such a model. The intersection theorems treated give a generalization of the well-known lemmas of Knaster, Kuratowski, and Mazurkiewicz (Ref. 1), Scarf (Ref. 2), Shapley (Ref. 3), and Ichiishi (Ref. 4). Moreover, the results can be used to sharpen the usual formulation of the Scarf lemma on the cube.

Key Words. Intersection theorems, zero-point problems, economic equilibrium, connectedness, closed coverings, balancedness.

1. Introduction

In intersection theorems, conditions are given under which a certain subset of a collection of sets has a nonempty intersection. Well-known intersection theorems on the unit simplex are given in Knaster, Kuratowski, and

[1]This research is part of the VF-program "Competition and Cooperation." This work was financially supported by the Cooperation Centre Tilburg and Eindhoven Universities, Netherlands.

[2]Fellow of the Royal Netherlands Academy of Arts and Sciences, Department of Econometrics and CentER, Tilburg University, Tilburg, Netherlands.

[3]Professor of Mathematical Economics, Department of Econometrics and CentER, Tilburg University, Tilburg, Netherlands.

Mazurkiewicz (Ref. 1, KKM lemma), Scarf (Ref. 2, Scarf lemma), Shapley (Ref. 3, KKMS lemma), Ichiishi (Ref. 4, Ichiishi lemma), and Gale (Ref. 5, Gale lemma). Intersection theorems can be used to prove the existence of solutions to mathematical programming problems, economic equilibrium existence problems, and solutions to game theoretic problems. The KKM lemma and the Scarf lemma can be used to prove the Brouwer fixed-point theorem, and also to show the existence of an equilibrium in an exchange economy with or without production. Both the KKMS lemma and the Ichiishi lemma are very useful when showing the nonemptiness of the core of a cooperative game; see Shapley (Ref. 3), Ichiishi (Ref. 4), and Shapley and Vohra (Ref. 6). In Gale (Ref. 5), an intersection theorem is used to show the existence of an equilibrium in an economy with indivisible commodities. In order to prove the existence of a Nash equilibrium in a noncooperative game, it is useful to formulate intersection theorems on the cube or even more general the simplotope; see for example Van der Laan and Talman (Ref. 7).

In all the intersection theorems stated above, conditions are given under which a certain subset of a collection of sets has a nonempty intersection. The sets in the collection form a closed covering of a simplex or a simplotope. In this paper, intersection theorems on the cube are formulated such that the intersection is guaranteed to consist of a continuum of points. Hence, these intersection theorems belong to a new class. The intersection theorems considered in this paper are somewhat related to an intersection theorem on the unit simplex formulated in Friedenfelds (Ref. 8). That intersection theorem generalizes the Scarf lemma. Often it has a continuum of intersection points, although this is not necessarily the case as opposed to the intersection theorems treated in this paper.

Let I_n denote the set of integers $\{1, \ldots, n\}$ for some natural number n. Let

$$Q^n = \{q \in \mathbb{R}^n \mid 0 \leq q_j \leq 1, \forall j \in I_n\}$$

denote the n-dimensional unit cube. Conditions are given on a collection of subsets covering the cube such that certain subsets of this collection have an intersection consisting of a continuum of points. Moreover, the intersection has some interesting topological properties. It will be shown that it has a component, i.e., a maximally connected subset, containing both the vertex being the vector of zeroes and the vertex being the vector of ones. The intersection theorems formulated in this paper generalize the KKM, Scarf, KKMS, and Ichiishi lemmas on the unit simplex and also lead to a strengthening of the usual formulation of the Scarf lemma on the cube. There is a close relationship between the intersection theorems of this paper and the equilibrium existence problem in economies with price rigidities as introduced in Drèze (Ref. 9). The intersection theorems of this paper give a more abstract formulation of the equilibrium existence problem in such an economy and they can be used to show the existence of a continuum of equilibria in the Drèze model.

In Section 2, some mathematical preliminaries are given and some useful results are derived. These results are used in Section 3 to formulate several intersection theorems on the cube that belong to the new class. Using one of the intersection theorems of Section 3, it is possible to strengthen the usual formulation of the Scarf lemma on the cube. In Section 4, it is shown that the Scarf lemma, the KKM lemma, the KKMS lemma, and the Ichiishi lemma follow immediately using the intersection theorems of Section 3. In Section 5, a continuum of equilibria is shown to exist in the Drèze model using the intersection theorems of this paper.

2. Some Preliminaries

In the following, 0^n will denote an n-dimensional vector containing only zeroes and 1^n, a n-dimensional vector of ones. The closure of a subset S of some topological space will be denoted by $\mathrm{cl}(S)$. The convex hull of a subset S of some Euclidean space will be denoted by $\mathrm{co}(S)$. For the remainder of the paper, it will be useful to consider a correspondence $\zeta: Q^n \to \mathbb{R}^n$ satisfying the following assumption.

Assumption (A). The correspondence $\zeta: Q^n \to \mathbb{R}^n$ satisfies:

(A1) ζ is a correspondence with a closed graph satisfying $\zeta(q)$ is nonempty and convex for every $q \in Q^n$ and $\bigcup_{q \in Q^n} \zeta(q)$ is bounded;

(A2) $\forall q \in Q^n$, $\exists z \in \zeta(q)$ such that, for every $j \in I_n$,

$$q_j = 0 \text{ implies } z_j \geq 0,$$

$$q_j = 1 \text{ implies } z_j \leq 0;$$

(A3) $\forall q \in Q^n$, $\forall z \in \zeta(q)$, $\exists p \in \mathbb{R}^n_{++}$ such that $p \cdot z = 0$.

Assumption (A1) guarantees that the correspondence ζ is upper semicontinuous. Assumption (A2) specifies a boundary condition for ζ; and Assumption (A3) is equivalent to the condition that, if $z \in \zeta(q)$ for some $q \in Q^n$, then

$$z_j > 0 \text{ for any } j \in I_n \text{ implies } z_k < 0 \text{ for some } k \in I_n,$$

$$z_j < 0 \text{ for any } j \in I_n \text{ implies } z_k > 0 \text{ for some } k \in I_n.$$

The set Z_ζ of zero points of ζ is defined by

$$Z_\zeta = \{q \in Q^n \mid 0^n \in \zeta(q)\}.$$

Models of economies with price rigidities as introduced in Drèze (Ref. 9) yield excess demand correspondences satisfying Assumption (A) as is shown in Herings (Ref. 10). In Theorem 2.1, interesting properties of the set of zero points of a correspondence ζ satisfying Assumption (A) are given. Since the intersection theorems developed in the next section are all proved using Theorem 2.1, this shows that there is a close relationship between the equilibrium existence problem in economies with price rigidities and the class of intersection theorems to be considered in this paper.

The following definitions and results, which can be found in, for instance, Armstrong (Ref. 11), will be useful later on. A topological space X is said to be connected if it is not the union of two nonempty disjoint, closed sets. A subset of a topological space is connected if it becomes a connected space when given the induced topology. Intuitively, a connected set is a set which is of one piece. The component of an element x in a topological space X is the union of all connected subsets of X containing x. It is not difficult to show that a component is connected and that the component of an element x in a topological space X is the largest connected subset of X containing x.

It will be useful in the proof of Theorem 2.1 to extend a correspondence ζ satisfying Assumption (A) such that it is defined on $\mathbb{R}^n$. For every $q \in Q^n$, choose an element $\hat{z}(q) \in \zeta(q)$ satisfying

$$\hat{z}_j(q) \geq 0, \qquad \text{if } q_j = 0,$$

$$\hat{z}_j(q) \leq 0, \qquad \text{if } q_j = 1.$$

Assumption (A) on ζ guarantees that $\hat{z}(q)$ can be chosen in this way for every $q \in Q^n$. For a nonempty compact set $S \subset \mathbb{R}^n$, define the correspondence $\Pi_S : \mathbb{R}^n \to S$ as the orthogonal projection on S, so

$$\Pi_S(x) = \{s' \in S \mid s' \in \arg\min_{s \in S} \|x - s\|_2\}, \qquad \forall x \in \mathbb{R}^n.$$

It is not difficult to show that Π_S is a continuous function if S is convex; see, for example, Mas-Colell (Ref. 12). Let $\hat{\zeta} : \mathbb{R}^n \to \mathbb{R}_n$ be the correspondence with graph in $\mathbb{R}^n \times \mathbb{R}^n$ given by the set

$$\mathrm{cl}(\{(x, \hat{z}(\Pi_{Q^n}(x))) \mid x \in \mathbb{R}^n\}).$$

Notice that component $j \in I_n$ of the projection function Π_{Q^n} is given by

$$(\Pi_{Q^n}(x))_j = \max\{0, \min\{x_j, 1\}\}, \qquad \forall x \in \mathbb{R}^n.$$

Finally, define the correspondence $\bar{\zeta}: \mathbb{R}^n \to \mathbb{R}^n$ by

$$\bar{\zeta}(x) = \zeta(x), \qquad \forall x \in Q^n,$$
$$\bar{\zeta}(x) = \mathrm{co}(\hat{\zeta}(x)), \qquad \forall x \in \mathbb{R}^n \backslash Q^n.$$

Lemma 2.1. Let a correspondence $\zeta: Q^n \to \mathbb{R}^n$ satisfying Assumption (A) be given. Then, $\bar{\zeta}$ is an upper semicontinuous correspondence, and $\bar{\zeta}(q)$ is nonempty and convex for every $q \in Q^n$. For every $x \in \mathbb{R}^n$, $z \in \bar{\zeta}(x)$ implies $z \in \zeta(\Pi_{Q^n}(x))$. Moreover, for every $x \in \mathbb{R}^n$,

$$z \in \bar{\zeta}(x) \text{ and } x_j < 0 \text{ imply } z_j \geq 0,$$

$$z \in \bar{\zeta}(x) \text{ and } x_j > 1 \text{ imply } z_j \leq 0.$$

Proof. Clearly, $\hat{\zeta}$ is a nonempty-valued correspondence. It follows from the definition of $\hat{\zeta}$ and the boundedness of the set $\bigcup_{q \in Q^n} \zeta(q)$ that the correspondence $\hat{\zeta}$ is upper semicontinuous. Using the definition of $\hat{\zeta}$ and the upper semicontinuity of ζ, it follows that $z \in \hat{\zeta}(x)$ implies $z \in \zeta(\Pi_{Q^n}(x))$. Let $z \in \hat{\zeta}(x)$ for some $x \in \mathbb{R}^n$ with $x_j < 0$ for some $j \in I_n$. Then, there exists a sequence $(x^r, z^r)_{r \in \mathbb{N}}$ such that

$$z^r = \hat{z}(\Pi_{Q^n}(x^r)) \quad \text{and} \quad \lim_{r \to \infty} (x^r, z^r) = (x, z).$$

Since $x_j < 0$, there exists $r' \in \mathbb{N}$ such that $r \geq r'$ implies $x_j^r < 0$, hence $(\Pi_{Q^n}(x^r))_j = 0$, and $z_j^r \geq 0$. Consequently, $z_j \geq 0$. It can be shown in a similar way that $z \in \hat{\zeta}(x)$ and $x_j > 1$ implies $z_j \leq 0$.

Using Todd (Ref. 13, p. 56, Theorem 1.4), the extension to $\mathbb{R}^n$ of the upper semicontinuous correspondence ζ defined on the closed set Q^n, obtained by assigning the empty set to every point $x \in \mathbb{R}^n \backslash Q^n$ is upper semicontinuous, the union of this extension of ζ with the correspondence $\hat{\zeta}$ is upper semicontinuous, and a convexified upper semicontinuous correspondence is upper semicontinuous. So, $\bar{\zeta}$ is upper semicontinuous. Clearly, $\bar{\zeta}(q)$ is nonempty and convex for every $q \in Q^n$. The other statements in Lemma 2.1 follow immediately from the properties derived for $\hat{\zeta}$. □

The following lemma is a special case of Theorem 3 in Mas-Colell (Ref. 14), in the sense that in the lemma convex-valued correspondences are considered, while Mas-Colell treats the more general case of contractible valued correspondences. It is a generalization of Theorem 2 in Browder (Ref. 15, p. 186), where the case with continuous functions is considered.

Lemma 2.2. Let $S \subset \mathbb{R}^n$ be a nonempty, compact, convex set; let the correspondence $\Psi: S \times [0, 1] \to S$ be upper semicontinuous; and let $\Psi(s, t)$

be nonempty and convex, $\forall (s, t) \in S \times [0, 1]$. Then, the set

$$F_\Psi = \{(s, t) \in S \times [0, 1] \mid s \in \Psi(s, t)\}$$

contains a component F_Ψ^c such that

$$(S \times \{0\}) \cap F_\Psi^c \neq \emptyset, \qquad (S \times \{1\}) \cap F_\Psi^c \neq \emptyset.$$

Theorem 2.1 will be proved as an application of Lemma 2.2.

Theorem 2.1. Let a correspondence $\zeta: Q^n \to \mathbb{R}^n$ satisfying Assumption (A) be given. Then, the set Z_ζ has a component Z_ζ^c containing 0^n and 1^n.

Proof. Let Z be a compact, convex set containing $\bigcup_{q \in Q^n} \zeta(q)$. Let the set R be defined by

$$R = \left\{ r \in \mathbb{R}^n \,\middle|\, \sum_{j=1}^{n} r_j = 0,\, r_j \geq -1,\, \forall j \in I_n \right\}.$$

Clearly, the set R is nonempty, compact, and convex. Let the correspondence $\alpha: Z \to R$ be defined by

$$\alpha(z) = \{\bar{r} \in R \mid \bar{r} \cdot z \geq r \cdot z,\, \forall r \in R\}, \qquad \forall z \in Z.$$

Using the maximum theorem [see, for example, Hildenbrand (Ref. 16, p. 30)], it follows immediately that the correspondence α is upper semicontinuous. Let the correspondence $\beta: R \times [1-n, 2] \to Z$ be defined by

$$\beta(r, t) = \bar{\zeta}(r + t1^n), \qquad \forall (r, t) \in R \times [1-n, 2].$$

Then, β is upper semicontinuous because of the upper semicontinuity of $\bar{\zeta}$ and the continuity of the function assigning $r + t1^n$ to $(r, t) \in R \times [1-n, 2]$. Let the correspondence $\Psi: Z \times R \times [1-n, 2] \to Z \times R$ be defined by

$$\Psi(z, r, t) = \beta(r, t) \times \alpha(z), \qquad \forall (z, r, t) \in Z \times R \times [1-n, 2].$$

Then, obviously, Ψ is upper semicontinuous and nonempty valued. The set $Z \times R$ is nonempty, compact, and convex. Clearly, the set $\beta(r, t)$ is convex for every $(r, t) \in R \times [1-n, 2]$. Using the convexity of R and the linearity in r of $r \cdot z$, it follows that the set $\alpha(z)$ is convex for every $z \in Z$. Using Lemma 2.2, it follows that there is a component F_Ψ^c of the set

$$F_\Psi = \{(z, r, t) \in Z \times R \times [1-n, 2] \mid (z, r) \in \Psi(z, r, t)\},$$

such that

$$F_\Psi^c \cap (Z \times R \times \{1-n\}) \neq \emptyset, \qquad F_\Psi^c \cap (Z \times R \times \{2\}) \neq \emptyset.$$

Clearly,

$(z^*, r^*, t^*) \in F^c_\Psi$ implies $(z^*, r^*) \in \Psi(z^*, r^*, t^*) = \bar{\zeta}(r^* + t^{*n}) \times \alpha(z^*)$.

Suppose that $\max_{j \in I_n} z_j^* > 0$. Since $z^* \in \zeta(\Pi_{Q^n}(r^* + t^* 1^n))$, there is by Assumption (A3) some $k \in I_n$ with $z_k^* < 0$. It is easily verified that $r \in \alpha(z)$, for any $z \in Z$ with $z_j > z_{j'}$ and $j, j' \in I_n$, implies $r_{j'} = -1$. Consequently, $r_k^* = -1$. If $t^* < 1$, then $r_k^* + t^* < 0$; and since

$$z^* \in \bar{\zeta}(r^* + t^{*n}),$$

this implies $z_k^* \geq 0$, a contradiction. Consider the case $t^* \geq 1$. By definition of α, there exists $j' \in I_n$ such that $z_{j'}^* = \max_{j \in I_n} z_j^* > 0$ and $r_{j'}^* > 0$. Hence, $r_{j'}^* + t^* > 1$; and since

$$z^* \in \bar{\zeta}(r^* + t^* 1^n),$$

this implies $z_{j'}^* \leq 0$, a contradiction. Consequently, we have that $\max_{j \in I_n} z_j^* \leq 0$. By Assumption (A3), this implies $z^* = 0^n$.

Consider the continuous function $f: Z \times R \times [1-n, 2] \to Q^n$ defined by

$$f(z, r, t) = \Pi_{Q^n}(r + t 1^n), \qquad \forall (z, r, t) \in Z \times R \times [1-n, 2].$$

Due to the fact that the image of a connected set by a continuous function is connected, it holds that $f(F^c_\Psi) \subset Q^n$ is connected. If $q^* \in f(F^c_\Psi)$, then

$$q^* = \Pi_{Q^n}(r^* + t^* 1^n), \qquad \text{for some } (z^*, r^*, t^*) \in F^c_\Psi.$$

Hence,

$$0^n = z^* \in \bar{\zeta}(r^* + t^* 1^n) \subset \zeta(\Pi_{Q^n}(r^* + t^* 1^n)) = \zeta(q^*).$$

Consequently, $f(F^c_\Psi) \subset Z_\zeta$. Next, consider the points $(0^n, r^1, 1-n) \in F^c_\Psi$ and $(0^n, r^2, 2) \in F^c_\Psi$. By definition,

$$f(0^n, r^1, 1-n) = \Pi_{Q^n}(r^1 + (1-n)1^n).$$

Since $r^1 \in R$, it holds for every $j \in I_n$ that $r_j^1 \leq n-1$, and consequently,

$$\Pi_{Q^n}(r^1 + (1-n)1^n) = 0^n.$$

Since $r^2 \in R$, it holds for every $j \in I_n$ that $r_j^2 \geq -1$, and consequently,

$$\Pi_{Q^n}(r^2 + 2 \cdot 1^n) = 1^n.$$

Hence,

$$0^n, 1^n \in f(F^c_\Psi).$$

Therefore, the set Z_ζ has a component Z^c_ζ containing both 0^n and 1^n. □

Theorem 2.1 will turn out to be a very useful tool for proving a number of intersection theorems in the next section. Since Theorem 2.1 is used in the proof of all these intersection theorems, Theorem 2.1 can be seen as a unifying theorem.

3. Intersection Theorems with a Continuum of Intersection Points

In the following, for

$$q\in Q^n, \qquad I^0(q)=\{j\in I_n \mid q_j=0\}, \qquad I^1(q)=\{j\in I_n \mid q_j=1\},$$

$s^0(q)$ denotes the number of elements in the set $I^0(q)$, $s^1(q)$ denotes the number of elements in the set $I^1(q)$, and

$$s(q)=s^0(q)+s^1(q).$$

The jth unit vector in $\mathbb{R}^n$ will be denoted by e^j. In Theorem 3.1, it will be assumed that, if an index $j\in I_n$ is taken, then

$$j+1=1, \qquad \text{if } j=n,$$

$$j-1=n, \qquad \text{if } j=1.$$

Theorem 3.1. Let $C^1,\ldots,C^n$ be closed subsets of Q^n satisfying $\bigcup_{j=1}^n C^j=Q^n$. Moreover, for every $q\in Q^n$, for every $j\in I_n$, $q_j=0$ or $q_{j+1}=1$ implies $q\in C^j$. Then, there exists a connected subset S of Q^n such that $0^n\in S$, $1^n\in S$, and $S\subset\bigcap_{j=1}^n C^j$.

Proof. For every $q\in Q^n$, let the set $J(q)$ be defined by

$$J(q)=\{j\in I_n \mid q\in C^j\}.$$

Let $t(q)$ denote the number of elements in the set $J(q)$. Let the correspondence $\zeta: Q^n\to\mathbb{R}^n$ be defined by

$$\zeta(q)=\operatorname{co}(\{e^j-(1/n)1^n \mid j\in J(q)\}).$$

First, we verify that ζ satisfies Assumption (A). It follows immediately, using the closedness of the sets C^j, $\forall j\in I_n$, that ζ is upper semicontinuous. Clearly, $\zeta(q)$ is nonempty and convex for every $q\in Q^n$ and $\bigcup_{q\in Q^n}\zeta(q)$ is bounded. Hence, Assumption (A1) is satisfied by ζ.

If, for some $j\in I_n$, $q_j=0$, then $q\in C^j$ and hence,

$$e^j-(1/n)1^n\in\zeta(q).$$

If, for some $j \in I_n$, $q_j = 1$, then $q \in C^{j-1}$ and hence,

$$e^{j-1} - (1/n)1^n \in \zeta(q).$$

Three cases have to be considered.

(C1) If $q = 1^n$, then consider $z \in \zeta(q)$ with

$$z = \sum_{j \in I_n} (1/n)(e^j - (1/n)1^n) = 0^n.$$

So, it follows that $z_j \leq 0$ for every $j \in I_n$.

(C2) If $\forall j \in I_n$, $0 \leq q_j < 1$, then define

$$z = \sum_{j \in J(q)} [1/t(q)](e^j - (1/n)1^n).$$

It follows immediately that $z \in \zeta(q)$ and $q_j = 0$ implies $z_j = 1/t(q) - 1/n \geq 0$.

(C3) If $\exists j_1 \in I_n$ with $q_{j_1} = 1$ and $\exists j_2 \in I_n$ with $q_{j_2} < 1$, then choose some $j' \in I_n$ satisfying $q_{j'} = 1$ and $q_{j'-1} \neq 1$. Define

$$z = \sum_{j \in I^0(q)} (1/n)e^j + (1 - s^0(q)/n)e^{j'-1} - (1/n)1^n.$$

Then, it is easily verified that $z \in \zeta(q)$. Moreover,

$$q_j = 0 \text{ implies } z_j \geq 1/n - 1/n = 0,$$

$$q_j = 1 \text{ implies } z_j = -1/n < 0.$$

Cases 1, 2, and 3 show that ζ satisfies Assumption (A2).

Since $\forall q \in Q^n$, $\forall z \in \zeta(q)$, $1^n \cdot z = 0$, also Assumption (A3) is satisfied by ζ.

If $0^n \in \zeta(q^*)$ for some $q^* \in Q^n$, or equivalently $q^* \in Z_\zeta$, then obviously $q^* \in C^j$ for every $j \in I_n$, so $q^* \in \bigcap_{j \in I_n} C^j$. By Theorem 2.1, there is a connected set of points $Z_\zeta^c \subset Z_\zeta$ satisfying $0^n \in Z_\zeta^c$ and $1^n \in Z_\zeta^c$. Consequently, S can be taken equal to Z_ζ^c. □

In Fig. 1, Theorem 3.1 is illustrated for the case $n = 2$. The set $C^1 \cap C^2$ consists of four components, and one of them contains both the points (0, 0) and (1, 1). It should be mentioned that Fig. 1 illustrates a rather nice case, in the sense that the sets C^1 and C^2 have a fairly easy structure. Besides the boundary condition and the requirement that these two sets cover Q^2, the only requirement made is that the sets C^1 and C^2 are closed. Hence, in general much more complicated situations might arise. The above remark is true for all illustrations in the sequel.

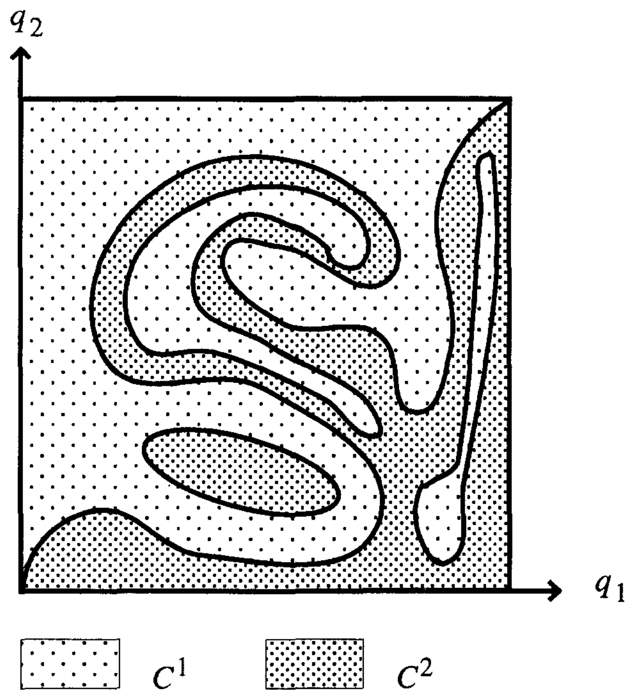

Fig. 1. Illustration of Theorem 3.1, case $n=2$.

It should be noticed that it is possible to replace the boundary condition $q_j=0$ or $q_{j+1}=1$ implies $q\in C^j$ by the more general condition that there exists a permutation $\pi: I_n \rightarrow I_n$ such that there is no nonempty, proper subset J of I_n satisfying $\pi(J)=J$, whereas, for every $j\in I_n$, $q_j=0$ or $q_{\pi(j)}=1$ implies $q\in C^j$. Theorem 3.1 corresponds to the choice $\pi=(2, \ldots, n, 1)$. In Example 3.1, a counterexample is given for any permutation $\pi: I_n\rightarrow I_n$ such that there exists a nonempty, proper subset J of I_n such that $\pi(J)=J$.

Example 3.1. Let $\pi: I_n \rightarrow I_n$ be a permutation, and let J be a nonempty, proper subset of I_n such that $\pi(J)=J$. Let k be an element of $I_n\backslash J$. Let the sets C^j, $\forall j\in I_n$, be defined by

$$C^j=\{q\in Q^n \mid 0\le q_j\le 1/4 \text{ or } 3/4\le q_{\pi(j)}\le 1\}, \qquad \forall j\in I_n\backslash\{k\},$$
$$C^k=Q^n.$$

Notice that the conditions of the more general specification of Theorem 3.1 using the permutation π are satisfied by this choice of the sets C^j, $\forall j\in I_n$.

Suppose that there exists a connected subset S of Q^n such that $0^n \in S$, $1^n \in S$, and $S \subset \bigcap_{j \in I_n} C^j$. Let the continuous function $f: S \to \mathbb{R}$ be defined by

$$f(q) = \max_{j \in J} q_j - 1/2.$$

Notice that

$$f(0^n) = -1/2 < 0, \qquad f(1^n) = 1/2 > 0.$$

Hence, it follows that there is $\bar{q} \in S$ such that $f(\bar{q}) = 0$. Therefore, there is $j^1 \in J$ such that $\bar{q}_{j^1} = 1/2$. Let $j^2 \in J$ be given by $j^2 = \pi(j^1)$. Since $\bar{q} \in C^{j^1}$, it follows that $3/4 \leq \bar{q}_{j^2} \leq 1$, and hence $0 = f(\bar{q}) \geq 1/4$, a contradiction. Consequently, there exists no connected subset S of Q^n such that $0^n \in S$, $1^n \in S$, and $S \subset \bigcap_{j \in I_n} C^j$.

The following theorem generalizes both Theorem 3.1 and the more general specification with the permutation π.

Theorem 3.2. Let $C^1, \ldots, C^n$ be closed subsets of Q^n satisfying $\bigcup_{j=1}^n C^j = Q^n$. Moreover, for every $q \in Q^n \setminus \{1^n\}$, for every $j \in I_n$, $q_j = 0$ implies $q \in C^j$, and $q_j = 1$ implies $q \in C^k$ for some $k \in I_n \setminus I^1(q)$. Then, there exists a connected subset S of Q^n such that $0^n \in S$, $1^n \in S$, and $S \subset \bigcup_{j=1}^n C^j$.

Proof. For every $q \in Q^n$, let the set $J(q)$ be defined by

$$J(q) = \{j \in I_n \mid q \in C^j\},$$

and let the correspondence $\zeta: Q^n \to \mathbb{R}^n$ be defined by

$$\zeta(q) = \mathrm{co}(\{e^j - (1/n)1^n \mid j \in J(q)\}).$$

Similar to the proof of Theorem 3.1, it can be shown that ζ satisfies Assumptions (A1) and (A3). Assumption (A2) remains to be verified. The case with $q = 1^n$ is considered first. For every $0 < \epsilon \leq 1$ and for every $j \in I_n$, it holds that $1^n - \epsilon e^j \in C^j$ using the assumptions of Theorem 3.2. Since C^j is closed for every $j \in I_n$, this implies that $1^n \in \bigcap_{j=1}^n C^j$, and hence $0^n \in \zeta(1^n)$. Next, consider the case with $q \in Q^n \setminus \{1^n\}$ and $I^0(q) \cup I^1(q) \neq \emptyset$. Let $k \in I_n \setminus I^1(q)$ be such that $q \in C^k$. Consider $z \in \zeta(q)$ given by

$$z = \sum_{j \in I^0(q)} [1/s(q)]e^j + [s^1(q)/s(q)]e^k - (1/n)1^n.$$

If $q_j = 0$, then

$$z_j \geq 1/s(q) - 1/n \geq 0.$$

If $q_j = 1$, then

$$z_j = -1/n < 0.$$

Hence, Assumption (A2) is satisfied by ζ. By Theorem 2.1, there exists a connected set $Z_\zeta^c \subset Z_\zeta$ containing both 0^n and 1^n. It is easily seen that $q^* \in Z_\zeta$ implies $q^* \in \bigcap_{j=1}^n C^j$. So, the set Z_ζ^c satisfies all the requirements imposed on the set S. □

For the case $n=2$, Theorem 3.1 is equivalent to Theorem 3.2. For the case $n \geq 2$, Theorem 3.2 is clearly more general. By symmetry considerations, the following dual theorem follows as a corollary to Theorem 3.2.

Theorem 3.3. Let $C^1, \dots, C^n$ be closed subsets of Q^n satisfying $\bigcup_{j=1}^n C^j = Q^n$. Moreover, for every $q \in Q^n \setminus \{0^n\}$, for every $j \in I_n$, $q_j = 1$ implies $q \in C^j$, and $q_j = 0$ implies $q \in C^k$ for some $k \in I_n \setminus I^0(q)$. Then, there exists a connected subset S of Q^n such that $0^n \in S$, $1^n \in S$, and $S \subset \bigcap_{j=1}^n C^j$.

So far, intersection theorems have been considered where a statement is made about the intersection of all the sets covering Q^n. For example, in the KKMS lemma or the Ichiishi lemma [see Shapley (Ref. 3) and Ichiishi (Ref. 4), respectively], a statement is made about the intersection of sets in certain subsets of the collection of sets covering Q^n. Theorem 3.4 is also an intersection theorem in this spirit. Moreover, unlike the other theorems given so far, it is completely symmetric with respect to the assumptions made on the sets in the cover of Q^n.

Theorem 3.4. Let $C^1, \dots, C^n$ and $D^1, \dots, D^n$ be closed subsets of Q^n satisfying $(\bigcup_{j=1}^n C^j) \cup (\bigcup_{j=1}^n D^j) = Q^n$. Moreover, for every $q \in Q^n$, for every $j \in I_n$, $q_j = 0$ implies $q \in C^j$, and $q_j = 1$ implies $q \in D^j$. Then, there exists a connected subset S of Q^n such that $0^n \in S$ and $1^n \in S$. Moreover, $q^* \in S$ implies $q^* \in C^j \cap D^j$ for some $j \in I_n$, or $q^* \in \bigcap_{j=1}^n C^j$, or $q^* \in \bigcap_{j=1}^n D^j$.

Proof. For every $q \in Q^n$, let the sets $J^0(q)$ and $J^1(q)$ be defined by

$$J^0(q) = \{j \in I_n \mid q \in C^j\}, \qquad J^1(q) = \{j \in I_n \mid q \in D^j\}.$$

Notice that $I^0(q) \subset J^0(q)$ and $I^1(q) \subset J^1(q)$. Let the correspondence $\zeta: Q^n \to \mathbb{R}^n$ be defined by

$$\zeta(q) = \mathrm{co}(\{e^j - (1/n)1^n \mid j \in J^0(q)\} \cup \{(1/n)1^n - e^j \mid j \in J^1(q)\}).$$

It follows immediately that ζ satisfies Assumptions (A1) and (A3). Consider an element $q\in Q^n$ satisfying $I^0(q)\cup I^1(q)\neq\emptyset$. Let $z\in\zeta(q)$ be given by

$$z=\sum_{j\in I^0(q)}[1/s(q)](e^j-(1/n)1^n)+\sum_{j\in I^1(q)}[1/s(q)]((1/n)1^n-e^j).$$

If $q_j=0$, then

$$z_j=1/s(q)-s^0(q)/s(q)n+s^1(q)/s(q)n\geq[n-s^0(q)]/s(q)n\geq 0.$$

Similarly, it can be shown that $q_j=1$ implies

$$z_j\leq[s^1(q)-n]/s(q)n\leq 0.$$

Hence, ζ satisfies Assumption (A2).

By Theorem 2.1, there is a connected set of points Z^c_ζ satisfying $0^n\in Z^c_\zeta$, $1^n\in Z^c_\zeta$, and $q^*\in Z^c_\zeta$ implies $0^n\in\zeta(q^*)$. Let 0^n be an element of $\zeta(q^*)$ for some $q^*\in Q^n$. Then there exists, for every $j\in I_n$, $\lambda^j\geq 0$ and $\mu^j\geq 0$ such that

$$\sum_{j=1}^n\lambda^j(e^j-(1/n)1^n)+\sum_{j=1}^n\mu^j((1/n)1^n-e^j)=0^n,$$

where

$$\lambda^j=0,\qquad \text{if } q\notin C^j,$$

$$\mu^j=0,\qquad \text{if } q\notin D^j,$$

$$\sum_{j=1}^n\lambda^j+\sum_{j=1}^n\mu^j=1.$$

Let the numbers λ and μ be given by

$$\lambda=\sum_{j=1}^n\lambda^j,\qquad \mu=\sum_{j=1}^n\mu^j.$$

For every $j\in I_n$, it holds that

$$\lambda^j-(1/n)\lambda=\mu^j-(1/n)\mu.$$

Hence,

$$\lambda^j-\mu^j=(1/n)(\lambda-\mu),$$

being independent of j. Three possibilities can occur. If $\lambda>\mu$, then for every $j\in I_n$, $\lambda^j-\mu^j>0$; hence, for every $j\in I_n$, $\lambda^j>0$, and consequently

$q^* \in \bigcap_{j=1}^n C^j$. If $\lambda = \mu$, then for every $j \in I_n$, $\lambda^j = \mu^j$. Since for some $k \in I_n$, $\lambda^k > 0$ or $\mu^k > 0$, it holds that $q^* \in C^k \cap D^k$ for some $k \in I_n$. If $\lambda < \mu$, then $q^* \in \bigcap_{j=1}^n D^j$. □

Theorem 3.4 is illustrated in Fig. 2 for the case $n = 2$. It is easily verified that the set $(C^1 \cap D^1) \cup (C^2 \cap D^2) \cup (C^1 \cap C^2) \cup (D^1 \cap D^2)$ consists of two components, one of them containing the points (0, 0) and (1, 1).

We will show that at least one point in the set S lies in the intersection of C^k and D^k for some index $k \in I_n$. It is even possible to show that

$$\left(\bigcap_{j=1}^n C^j\right) \cap D^k \neq \varnothing, \qquad \text{for some } k \in I_n,$$

$$\left(\bigcap_{j=1}^n D^j\right) \cap C^k \neq \varnothing, \qquad \text{for some } k \in I_n.$$

Theorem 3.5. Let $C^1, \ldots, C^n$ and $D^1, \ldots, D^n$ be closed subsets of Q^n satisfying $(\bigcup_{j=1}^n C^j) \cup (\bigcup_{j=1}^n D^j) = Q^n$. Moreover, for every $q \in Q^n$, for every $j \in I_n$, $q_j = 0$ implies $q \in C^j$, and $q_j = 1$ implies $q \in D^j$. Then, there exists a connected subset S of Q^n such that $0^n \in S$ and $1^n \in S$. Moreover, $q^* \in S$ implies

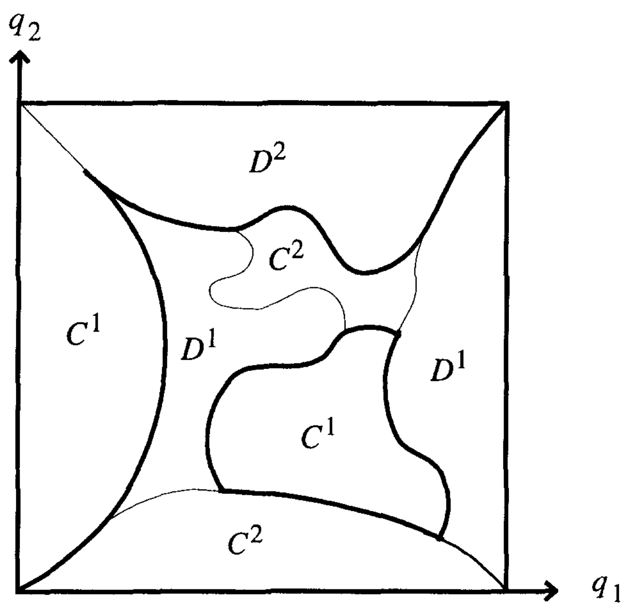

Fig. 2. Illustration of Theorem 3.4, case $n = 2$.

$q^* \in C^j \cap D^j$ for some $j \in I_n$, or $q^* \in \bigcap_{j \in I_n} C^j$, or $q^* \in \bigcap_{j \in I_n} D^j$. Furthermore, there exists s^1, $s^2 \in S$ such that

$$s^1 \in \bigcup_{k=1}^{n} \left(\left(\bigcap_{j=1}^{n} C^j \right) \cap D^k \right), \qquad s^2 \in \bigcup_{k=1}^{n} \left(\left(\bigcap_{j=1}^{n} D^j \right) \cap C^k \right).$$

Proof. By Theorem 3.4, there is a connected subset S of Q^n such that $0^n \in S$, $1^n \in S$, and $q^* \in S$ implies $q^* \in C^j \cap D^j$ for some $j \in I_n$, or $q^* \in \bigcap_{j=1}^{n} 1\ C^j$, or $q^* \in \bigcap_{j=1}^{n} 1\ D^j$, Clearly, S can be chosen such that it is a closed set. Let the sets $J^0(q)$ and $J^1(q)$ be defined as in the proof of Theorem 3.4, and let $j^0(q)$ and $j^1(q)$ denote the number of elements in these sets, respectively. Let the correspondence $\Phi^0: S \to \mathbb{R}$ be defined by

$$\Phi^0(q) = \{j^0(q)\}, \qquad \text{if } j^0(q) > 0,$$
$$\Phi^0(q) = \varnothing, \qquad \text{if } j^0(q) = 0.$$

Let the correspondence $\Phi^1: S \to \mathbb{R}$ be defined by

$$\Phi^1(q) = \{-j^1(q)\}, \qquad \text{if } j^1(q) > 0,$$
$$\Phi^1(q) = \varnothing, \qquad \text{if } j^1(q) = 0.$$

Finally, let the correspondence $\Phi: S \to \mathbb{R}$ be defined by

$$\Phi(q) = \text{co}(\Phi^0(q) \cup \Phi^1(q)), \qquad \forall q \in S.$$

Using Theorem 3.4, it follows easily that Φ is an upper semicontinuous correspondence and $\Phi(q)$ is nonempty and convex for every $q \in S$. It is shown that $\Phi(S)$ is connected, and hence an interval.

Suppose that $\Phi(S)$ is not connected; then, it can be partitioned in two nonempty disjoint sets T^1 and T^2, both being closed in $\Phi(S)$. By Proposition 1 of Hildenbrand (Ref. 16, p. 22), both $\Phi^{-1}(T^1)$ and $\Phi^{-1}(T^2)$ are closed in S. Suppose that

$$\Phi^{-1}(T^1) \cap \Phi^{-1}(T^2) \neq \varnothing;$$

then, there exists an element $q \in S$ and points t^1 and t^2 such that

$$t^1 \in \Phi(q) \cap T^1, \qquad t^2 \in \Phi(q) \cap T^2.$$

Since Φ is convex-valued, it holds that

$$(1-\lambda)t^1 + \lambda t^2 \in \Phi(q), \qquad \text{for every } \lambda \in [0, 1].$$

Hence, there exists a continuous function $f: [0, 1] \to \Phi(S)$ such that $f(0) = t^1$ and $f(1) = t^2$. So, t^1 and t^2 are an element of the same component of $\Phi(S)$, a contradiction. Therefore,

$$\Phi^{-1}(T^1) \cap \Phi^{-1}(T^2) = \varnothing.$$

Clearly,

$$\Phi^{-1}(T^1) \cup \Phi^{-1}(T^2) = S.$$

Hence, S is not connected, a contradiction. Consequently, $\Phi(S)$ is connected.

It holds that $0^n \in S$, $n \in \Phi(0^n)$, $1^n \in S$, and $-n \in \Phi(1^n)$. Therefore, $\Phi(S) = [-n, n]$. Suppose that, for every $k \in I_n$,

$$\left(\bigcap_{j=1}^{n} C^j\right) \cap D^k \cap S = \varnothing.$$

Then,

$$\Phi\left(\left(\bigcap_{j=1}^{n} C^j\right) \cap S\right) = \{n\}$$

and

$$\Phi\left(S \setminus \bigcap_{j=1}^{n} C^j\right) \subset [-n, n-1],$$

so

$$\Phi(S) \subset [-n, n-1] \cup \{n\},$$

a contradiction. Consequently, there is a point $s^1 \in S$ such that

$$s^1 \in \left(\bigcap_{j=1}^{n} C^j\right) \cap D^k, \qquad \text{for some } k \in I_n.$$

Similarly, it can be shown that there is a point $s^2 \in S$ such that

$$s^2 \in \left(\bigcap_{j=1}^{n} D^j\right) \cap C^k, \qquad \text{for some } k \in I_n. \qquad \square$$

Theorem 3.5 strengthens the usual formulation of the analog of the Scarf lemma on the cube [see Freund (Ref. 17)], which claims that, under the conditions of Theorem 3.5, there is $k \in I_n$ such that $C^k \cap D^k \neq \varnothing$.

Next, intersection theorems with a continuum of intersection points generalizing the KKMS lemma and the Ichiishi lemma will be considered. To do this, we first give a definition of a balanced collection of sets. For every nonempty subset T of I_n, define the vector e^T by

$$e_j^T = 1/|T|, \qquad \text{if } j \in T,$$
$$e_j^T = 0, \qquad \text{if } j \notin T,$$

where $|T|$ denotes the number of elements in the set T. Define the vector $e^{\varnothing}$ by

$$e_j^{\varnothing} = 1/n, \qquad \forall j \in I_n.$$

Denote the collection of all subsets of I_n by $\mathscr{T}_n$. Notice that $\varnothing \in \mathscr{T}_n$. Let $\mathscr{B}$ be a nonempty collection of elements of $\mathscr{T}_n$, say $\mathscr{B} = \{T^1, \ldots, T^m\}$. The collection $\mathscr{B}$ is said to be balanced if there exist positive numbers $\lambda^1, \ldots, \lambda^m$ such that

$$\sum_{i=1}^{m} \lambda^i = 1, \qquad \sum_{i=1}^{m} \lambda^i e^{T^i} = (1/n)1^n.$$

This definition of balancedness is slightly more general than the usual one, since the empty set is not excluded as an element of a balanced collection of sets. If only nonempty subsets of I_n are considered, then the definition reduces to the usual one.

Theorem 3.6. Let $\{C^T \mid T \in \mathscr{T}_n\}$ be a collection of closed subsets of Q^n satisfying $\bigcup_{T \in \mathscr{T}_n} C^T = Q^n$. Moreover, for every $q \in Q^n$ with $\varnothing \neq I^0(q) \neq I_n$, $q \in C^T$ for a set $T \in \mathscr{T}_n$ satisfying $I^0(q) \subset T$, and for every $q \in Q^n$ with $\varnothing \neq I^1(q) \neq I_n$, $q \in C^T$ for a set $T \in \mathscr{T}_n$ satisfying $I^1(q) \subset I_n \backslash T$. Then, there exists a connected subset S of Q^n such that $0^n \in S$, $1^n \in S$, and for every $q^* \in S$ there is a balanced collection $\{T^1, \ldots, T^m\}$ of sets in $\mathscr{T}^n$ such that $q^* \in \bigcap_{i=1}^m C^{T^i}$.

Proof. For every $q \in Q^n$, let the set $J(q)$ be defined by

$$J(q) = \{T \in \mathscr{T}_n \mid q \in C^T\}.$$

Let the correspondence $\zeta: Q^n \to \mathbb{R}^n$ be defined by

$$\zeta(q) = \mathrm{co}(\{e^T - (1/n)1^n \mid T \in J(q)\}).$$

It follows immediately that ζ satisfies Assumptions (A1) and (A3).

Consider the point $q = 0^n$. Due to the boundary condition, it holds that, for every $j \in I_n$ and for every $\epsilon \in (0, 1]$, the point $0^n + \epsilon e^j$ belongs to $C^{I_n \backslash \{j\}}$ or to C^{I_n}. Hence, since each C^T is closed, $0^n \in C^{I_n \backslash \{j\}}$ for every $j \in I_n$, or $0^n \in C^{I_n}$. Clearly, both the collection $\{I_n \backslash \{j\} \in \mathscr{T}_n \mid j \in I_n\}$ and the collection $\{I_n\}$ are balanced and therefore $0^n \in \zeta(0^n)$. Similarly, since $1^n - \epsilon e^j$ belongs to $C^{\{j\}}$ or to $C^{\varnothing}$ for every $\epsilon \in (0, 1]$, it holds that $1^n \in C^{\{j\}}$ for every $j \in I_n$, or $1^n \in C^{\varnothing}$.

Hence, $0^n \in \zeta(1^n)$, since both the collection $\{\{j\} \in \mathscr{T}_n \mid j \in I_n\}$ and the collection $\{\varnothing\}$ are balanced.

Consider a point $q \in Q^n \setminus \{0^n, 1^n\}$ with $I^0(q) \cup I^1(q) \neq \varnothing$. Let $T^0 \in \mathscr{T}_n$ be a set satisfying $I^0(q) \subset T^0$ and $q \in C^{T^0}$, and let $T^1 \in \mathscr{T}_n$ be a set satisfying $I^1(q) \subset I_n \setminus T^1$ and $q \in C^{T^1}$. If $T^0 = \varnothing$ or $T^1 = \varnothing$, then clearly $0^n \in \zeta(q)$ and Assumption (A2) is satisfied. Hence, consider the case $T^0 \neq \varnothing$ and $T^1 \neq \varnothing$. Consider $z \in \zeta(q)$ given by

$$z = [|T^0|/n]e^{T^0} + [(n - |T^0|)/n]e^{T^1} - (1/n)1^n.$$

For $j \in I^0(q)$, it holds that

$$z_j \geq (|T^0|/n)(1/|T^0|) - 1/n = 0,$$

and for $j \in I^1(q)$ that

$$z_j \leq (|T^0|/n)(1/|T^0|) - 1/n = 0.$$

Consequently, Assumption (A2) is satisfied.

By Theorem 2.1, there is a connected set of points Z^c_ζ satisfying $0^n \in Z^c_\zeta$, $1^n \in Z^c_\zeta$, and $q^* \in Z^c_\zeta$, implies $0^n \in \zeta(q^*)$. Notice that $0^n \in \zeta(q^*)$ if and only if there exists a balanced collection $\{T^1, \ldots, T^m\}$ of sets in $\mathscr{T}_n$ satisfying that $q^* \in \bigcap_{i=1}^m C^{T^i}$, i.e., $0^n \in \zeta(q^*)$ if and only if there exists $\lambda^{*1}, \ldots, \lambda^{*m} > 0$ with $\sum_{i=1}^m \lambda^{*i} = 1$ such that $q^* \in C^{T^i}$ for every $i \in I_m$ and

$$0^n = \sum_{i=1}^m \lambda^{*i}(e^{T^i} - (1/n)1^n),$$

hence

$$\sum_{i=1}^m \lambda^{*i} e^{T^i} = (1/n)1^n. \qquad \square$$

Since the boundary condition in Theorem 3.6 is not specified for $q = 0^n$ and $q = 1^n$, it is possible that $C^{\varnothing} = \varnothing$ or $C^{I_n} = \varnothing$. It should be noticed that the boundary condition specified in Theorem 3.6 is weaker than the condition that, for every $q \in Q^n \setminus \{0^n, 1^n\}$ with $\varnothing \neq I^0(q) \cup I^1(q)$, $q \in C^T$ for a set $T \in \mathscr{T}_n$ satisfying $I^0(q) \subset T$ and $I^1(q) \subset I_n \setminus T$. Theorem 3.6 is illustrated in Fig. 3. In the illustration, n equals 2. In this low-dimensional case, the only difference from Theorem 3.1 or Theorem 3.2 is the possibility of nonempty sets $C^{\varnothing}$ or $C^{\{1,2\}}$. In the case $n = 2$, the minimal balanced collections of sets are given by $\{C^{\varnothing}\}$, $\{C^{\{1,2\}}\}$, and $\{C^{\{1\}}, C^{\{2\}}\}$. It is easily verified that, in Fig. 3, the union over all balanced collections of sets $\mathscr{B}$ of the intersection of the sets in $\mathscr{B}$ consists of three components, with one component containing both the points 0^n and 1^n. In the case $n \geq 3$, the situation may be much more complicated than in Theorem 3.1 and Theorem 3.2.

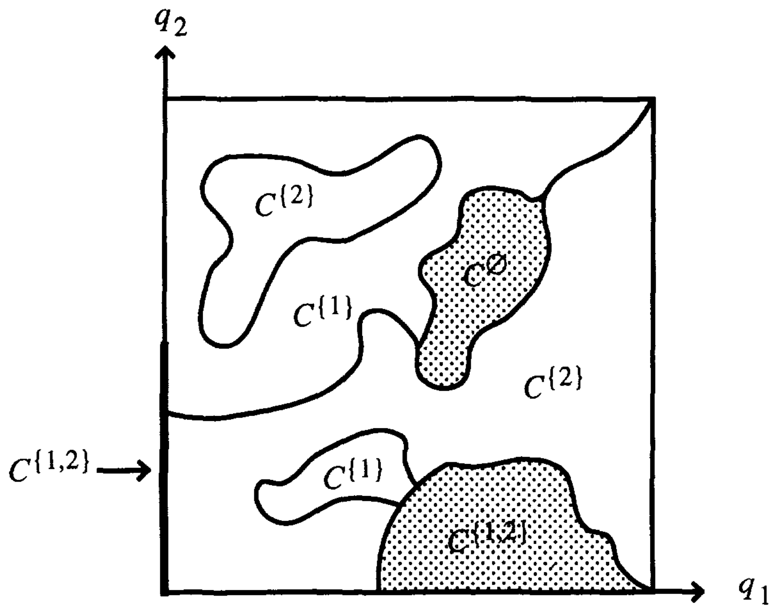

Fig. 3. Illustration of Theorem 3.6, case $n=2$.

By symmetry considerations, Theorem 3.7 follows immediately as a corollary to Theorem 3.6.

Theorem 3.7. Let $\{C^T | T \in \mathscr{T}_n\}$ be a collection of closed subsets of Q^n satisfying $\bigcup_{T\in\mathscr{T}_n} C^T = Q^n$. Moreover, for every $q \in Q^n$ with $\emptyset \neq I^0(q) \neq I_n$, $q \in C^T$ for a set $T \in \mathscr{T}_n$ satisfying $I^0(q) \subset I_n \backslash T$, and for every $q \in Q^n$ with $\emptyset \neq I^1(q) \neq I_n$, $q \in C^T$ for a set $T \in \mathscr{T}_n$ satisfying $I^1(q) \subset T$. Then, there exists a connected subset S of Q^n such that $0^n \in S$, $1^n \in S$, and for every $q^* \in S$ there is a balanced collection $\{T^1, \ldots, T^m\}$ of sets in $\mathscr{T}_n$ such that $q^* \in \bigcap_{i=1}^m C^{T^i}$.

4. Intersection Theorems on the Unit Simplex

In this section, a number of well-known intersection theorems on the $(n-1)$-dimensional unit simplex,

$$S^n = \left\{ p \in \mathbb{R}^n_+ \,\middle|\, \sum_{j=1}^n p_j = 1 \right\},$$

will be shown to follow as corollaries to the theorems of Section 3. Theorem 3.1 leads to the Scarf lemma [see Sperner (Ref. 18), Fan (Ref. 19), and Scarf (Refs. 2, 20)], formulated in Theorem 4.1. In the proof of Theorem 4.1, a cover $\{C^1, \dots, C^n\}$ of S^n satisfying the conditions of Theorem 4.1 is extended in a straightforward way to a cover $\{\hat{C}^1, \dots, \hat{C}^n\}$ of Q^n satisfying the conditions of Theorem 3.1. Then, it follows that there exists a connected set $\hat{S}$ such that $0^n \in \hat{S}$, $1^n \in \hat{S}$, and $\hat{S} \subset \bigcap_{j=1}^n \hat{C}^j$. It will be shown that this connected set $\hat{S}$ has a nonempty intersection with the unit simplex S^n. Finally, it is shown that

$$\bigcap_{j=1}^{n} \hat{C}^j \cap S^n = \bigcap_{j=1}^{n} C^j.$$

Theorem 4.1. Scarf Lemma. Let $C^1, \dots, C^n$ be closed subsets of S^n satisfying $\bigcup_{j=1}^n C^j = S^n$. Moreover, for every $p \in S^n$, for every $j \in I_n$, $p_j = 0$ implies $p \in C^j$. Then $\bigcap_{j=1}^n C^j \neq \varnothing$.

Proof. The case $n=1$ is trivial, so consider the case $n \geq 2$. For every $j \in I_n$, let the set $\hat{C}^j$ be defined by

$$\hat{C}^j = \{q \in Q^n \mid \Pi_{S^n}(q) \in C^j\} \cup \{q \in Q^n \mid q_j = 0\} \cup \{q \in Q^n \mid q_{j+1} = 1\}.$$

It will be shown that the sets $\hat{C}^1, \dots, \hat{C}^n$ satisfy the conditions of Theorem 3.1. Using the closedness of the sets $C^1, \dots, C^n$, the continuity of the function Π_{S^n}, and the property that $\bigcup_{j=1}^n C^j = S^n$, it is easily verified that, for every $j \in I_n$, the set $\hat{C}^j$ is closed and $\bigcup_{j=1}^n \hat{C}^j = Q^n$. If, for $q \in Q^n$, $q_j = 0$ or $q_{j+1} = 1$, then clearly $q \in \hat{C}^j$. So, the sets $\hat{C}^1, \dots, \hat{C}^n$ satisfy the conditions of Theorem 3.1. Hence, there exists a connected subset $\hat{S}$ of Q^n such that $0^n \in \hat{S}$, $1^n \in \hat{S}$, and $\hat{S} \subset \bigcap_{j=1}^n \hat{C}^j$. Let the function $f: \hat{S} \to \mathbb{R}$ be defined by

$$f(s) = \sum_{j=1}^{n} s_j, \qquad \forall s \in \hat{S}.$$

Since the image of a connected set under a continuous function is connected, $f(0^n) = 0$, and $f(1^n) = n$, there is $\hat{s} \in \hat{S}$ such that $f(\hat{s}) = 1$, or equivalently,

$$\hat{s} \in \hat{S} \cap S^n \subset \bigcap_{j=1}^{n} (\hat{C}^j \cap S^n).$$

Clearly,

$$C^j \subset \hat{C}^j \cap S^n, \qquad \forall j \in I_n.$$

Suppose that there exists an element $\hat{q} \in (\hat{C}^k \cap S^n) \setminus C^k$ for some $k \in I_n$. Then, since $\hat{q} \in S^n$ implies that $\Pi_{S^n}(\hat{q}) = \hat{q}$, it holds that $\hat{q}_k = 0$ or $\hat{q}_{k+1} = 1$. Since $\hat{q}_k = 0$ implies $\hat{q} \in C^k$, it holds that $\hat{q}_{k+1} = 1$ and $\hat{q}_k > 0$, yielding a contradiction, since $\hat{q} \in S^n$ with $n \geq 2$. Consequently, $\hat{C}^j \cap S^n = C^j$, $\forall j \in I_n$, and $\hat{s} \in \bigcap_{j=1}^n C^j$. □

Theorem 3.3 leads to the KKM lemma as formulated in Theorem 4.2. In the proof of Theorem 4.2, a cover $\{C^1, \ldots, C^n\}$ of S^n satisfying the conditions of Theorem 4.2 is extended in more or less the same straightforward way as in the proof of Theorem 4.1 to yield a cover $\{\hat{C}^1, \ldots, \hat{C}^n\}$ of Q^n satisfying the conditions of Theorem 3.3. Some notation is introduced first. For $J \in \mathcal{T}_n$, let $Q^n(J)$ denote the set

$$Q^n(J) = \{q \in Q^n \mid q_j = 0, \forall j \in J\}.$$

Define

$$S^n(J) = \{p \in S^n \mid p_j = 0, \forall j \in J\} = S^n \cap Q^n(J).$$

Notice that $Q^n(\varnothing) = Q^n$, $S^n(\varnothing) = S^n$, and $S^n(J) \neq \varnothing$ if and only if J is a proper subset of I_n. Denote the collection of all proper subsets of I_n by $\mathcal{T}'_n$, so $\mathcal{T}'_n = \mathcal{T}_n \setminus \{I_n\}$.

Theorem 4.2. KKM Lemma. Let $C^1, \ldots, C^n$ be closed subsets of S^n satisfying $\bigcup_{j=1}^n C^j = S^n$. Moreover, for every $p \in S^n$ with $I^0(p) \neq \varnothing$, there exists $j \in I_n \setminus I^0(p)$ such that $p \in C^j$. Then, $\bigcap_{j=1}^n C^j \neq \varnothing$.

Proof. For every $j \in I_n$, let the set $\hat{C}^j$ be defined by

$$\hat{C}^j = \bigcup_{J \in \mathcal{T}'_n} \{q \in Q^n(J) \mid \Pi_{S^n(J)}(q) \in C^j\} \cup \{q \in Q^n \mid q_j = 1\}.$$

It will be shown that the sets $\hat{C}^1, \ldots, \hat{C}^n$ satisfy the conditions of Theorem 3.3. Using the closedness of the sets $C^1, \ldots, C^n$, the continuity of the function $\Pi_{S^n(J)}$ for every proper subset J of I_n, and the fact that $\bigcup_{j=1}^n C^j = S^n$, it holds for every $j \in I_n$ that the set $\hat{C}^j$ is closed and that $\bigcup_{j=1}^n \hat{C}^j = Q^n$. Clearly, $q \in Q^n$ and $q_j = 1$ implies $q \in \hat{C}^j$. Consider $q \in Q^n \setminus \{0^n\}$ with $I^0(q) \neq \varnothing$. Then, $q \in Q^n(I^0(q))$ and there exists $k \in I_n \setminus I^0(\Pi_{S^n(I^0(q))}(q)) \subset I_n \setminus I^0(q)$ such that $\Pi_{S^n(I^0(q))}(q) \in C^k$, so $q \in \hat{C}^k$. Consequently, the sets $\hat{C}^1, \ldots, \hat{C}^n$ satisfy the conditions of Theorem 3.3 and there is a connected set $\hat{S}$ such that $0^n \in \hat{S}$, $1^n \in \hat{S}$, and $\hat{S} \subset \bigcap_{j=1}^n \hat{C}^j$. As in the proof of Theorem 4.1, it follows that there is

$$\hat{s} \in \hat{S} \cap S^n \subset \bigcap_{j=1}^n (\hat{C}^j \cap S^n).$$

Clearly, $C^j \subset \hat{C}^j \cap S^n$. Suppose that there exists an element $\hat{q} \in (\hat{C}^k \cap S^n) \setminus C^k$ for some $k \in I_n$. Since $\Pi_{S^n(J)}(\hat{q}) = \hat{q}$ if $\hat{q} \in S^n \cap Q^n(J)$, it follows that $\hat{q}_k = 1$. If $\hat{q}_j = 0$ for every $j \in I_n \setminus \{k\}$, then it follows by the conditions of Theorem 4.2 that $\hat{q} \in C^k$. Hence, $\hat{q}_j > 0$ for some $j \neq k$, giving a contradiction since $\hat{q}_k = 1$ and $\hat{q} \in S^n$. Consequently, $\hat{C}^j \cap S^n = C^j$, $\forall j \in I_n$, and $\hat{s} \in \bigcap_{j=1}^n C^j$. □

In Theorem 4.3, the Ichiishi lemma [see Ichiishi (Ref. 4)] is derived from Theorem 3.6. Denote the collection of all nonempty subsets of I_n by $\mathcal{T}_n^*$, so $\mathcal{T}_n^* = \mathcal{T}_n \setminus \{\varnothing\}$.

Theorem 4.3. Ichiishi Lemma. Let $\{C^T \mid T \in \mathcal{T}_n^*\}$ be a collection of closed subsets of S^n satisfying $\bigcup_{T\in\mathcal{T}_n^*} C^T = S^n$. Moreover, for every $p \in S^n$ with $I^0(p) \neq \varnothing$, there exists $T \in \mathcal{T}_n^*$ such that $p \in C^T$ and $I^0(p) \subset T$. Then, there is a balanced collection $\{T^1, \ldots, T^m\}$ of sets in $\mathcal{T}_n^*$ such that $\bigcap_{i=1}^m C^{T^i} \neq \varnothing$.

Proof. The case where $C^{I_n} \neq \varnothing$ is trivial; hence, consider the case $C^{I_n} = \varnothing$. Let the sets $\hat{C}^{\varnothing}$ and $\hat{C}^{I_n}$ be defined by $\hat{C}^{\varnothing} = \hat{C}^{I_n} = \varnothing$. For every $T \in \mathcal{T}_n \setminus \{\varnothing, I_n\}$, let the set $\hat{C}^T$ be defined by

$$\hat{C}^T = \bigcup_{J\in\mathcal{T}'_n} \{q \in Q^n(J) \mid \Pi_{S^n(J)}(2q) \in C^T\} \cup \{q \in Q^n \mid I_n \setminus T \subset I^1(q)\}.$$

It will be shown that the collection of sets $\{\hat{C}^T \mid T \in \mathcal{T}_n\}$ satisfies the conditions of Theorem 3.6. Clearly, for every $T \in \mathcal{T}_n$, $\hat{C}^T$ is closed, and $\bigcup_{T\in\mathcal{T}_n} \hat{C}^T = Q^n$. Moreover, if $q \in Q^n \setminus \{0^n\}$ with $I^0(q) \neq \varnothing$, then $q \in Q^n(I^0(q))$ and there is $T \in \mathcal{T}_n$ such that

$$I^0(\Pi_{S^n(I^0(q))}(2q)) \subset T, \qquad \Pi_{S^n(I^0(q))}(2q) \in C^T.$$

Hence, $q \in \hat{C}^T$ while

$$I^0(q) \subset I^0(\Pi_{S^n(I^0(q))}(2q)) \subset T.$$

If $q \in Q^n \setminus \{1^n\}$ and $I^1(q) \neq \varnothing$, then $q \in \hat{C}^{I_n \setminus I^1(q)}$. Consequently, the collection $\{\hat{C}^T \mid T \in \mathcal{T}_n\}$ satisfies the conditions of Theorem 3.6 and there is a connected set $\hat{S}$ with the properties stated in Theorem 3.6. As in the proof of Theorem 4.1, it follows that there exists

$$\hat{s} \in \hat{S} \cap \left\{q \in Q^n \,\middle|\, \sum_{j=1}^n q_j = 1/2\right\}.$$

Hence, there is a balanced collection $\{T^1, \ldots, T^m\}$ of sets in $\mathcal{T}_n \setminus \{\varnothing, I_n\}$ such that $\hat{s} \in \bigcap_{i=1}^m \hat{C}^{T^i}$. Since $\hat{s}_j \neq 1$, $\forall j \in I_n$, it holds that for every $i \in I_m$, there exists $J \in \mathcal{T}'_n$ such that $\hat{s} \in Q^n(J)$ and $\Pi_{S^n(J)}(2\hat{s}) \in C^{T^i}$. Since $\hat{s} \in Q^n(J)$ and $\sum_{j=1}^n \hat{s}_j = 1/2$ implies $\Pi_{S^n(J)}(2\hat{s}) = 2\hat{s}$, it holds that $2\hat{s} \in \bigcap_{i=1}^m C^{T^i}$. □

In Theorem 4.3, a cover of S^n with sets in $\mathcal{T}_n^* = \mathcal{T}_n \setminus \{\varnothing\}$ is considered, which is the usual formulation. Clearly, the statement of Theorem 4.3 is still true if a cover with sets in $\mathcal{T}_n$ is considered, since in the case $C^{\varnothing} \neq \varnothing$ Theorem 4.3 is trivially true. It is clear that also Theorem 3.7 can be used to derive the Ichiishi lemma. Similarly, the KKMS lemma can be easily derived from both Theorem 3.6 and Theorem 3.7. In Theorem 4.4, the derivation using Theorem 3.7 will be shown.

Theorem 4.4. KKMS Lemma. Let $\{C^T \mid T \in \mathscr{T}_n^*\}$ be a collection of closed subsets of S^n satisfying $\bigcup_{T\in\mathscr{T}_n^*}^n C^T = S^n$. Moreover, for every $p \in S^n$ with $I^0(p) \neq \varnothing$, there exists $T \in \mathscr{T}_n^*$ such that $p \in C^T$ and $I^0(p) \subset I_n \setminus T$. Then, there is a balanced collection $\{T^1, \ldots, T^m\}$ of sets in $\mathscr{T}_n^*$ such that $\bigcap_{i=1}^m C^{T^i} \neq \varnothing$.

Proof. The proof goes along the same lines as the proof of Theorem 4.3 by using Theorem 3.7 instead of Theorem 3.6 and defining, for every $T \in \mathscr{T}_n \setminus \{\varnothing, I_n\}$,

$$\hat{C}^T = \bigcup_{J \in \mathscr{T}_n} \{q \in Q^n(J) \mid \Pi_{S^n(J)}(2q) \in C^T\} \cup \{q \in Q^n \mid T \subset I^1(q)\}. \qquad \square$$

Recently, many authors provided simple and elementary proofs of the KKMS theorem; see, for instance, Ichiishi (Ref. 21), Shapley and Vohra (Ref. 6), Komiya (Ref. 22), Krasa and Yannelis (Ref. 23), and Zhou (Ref. 24). The proof of Theorem 4.4 provides one more alternative way to show the KKMS theorem.

5. Application to Economies with Price Rigidities

In Section 2, it has been remarked that the total excess demand correspondence ζ of an economy with price rigidities as introduced in Drèze (Ref. 9) satisfies Assumption (A). A zero point of the correspondence ζ corresponds to an equilibrium in such an economy. It is shown that the intersection theorems of Section 3 can be used to show Theorem 2.1, and hence can be used to show the existence of a continuum of equilibria, containing the equilibria induced by the points 0^n and 1^n. This also shows the equivalence between the intersection theorems of Section 3 and Theorem 2.1. Since it is possible to give a constructive proof of any of the intersection theorems of Section 3, one obtains also an alternative way to show the existence of a continuum of equilibria, a result also obtained in Herings (Ref. 25) and Herings, Talman, and Yang (Ref. 26).

Theorem 5.1 gives the result of Theorem 2.1 in the case $\zeta: Q^n \to \mathbb{R}^n$ is a continuous function, denoted by z. Theorem 2.1 can then be derived from Theorem 5.1 in a similar way as the Kakutani fixed-point theorem can be derived from the Brouwer fixed-point theorem. In the proof of Theorem 5.1, the intersection result given in Theorem 3.1 is used, since this should be considered as the least general result in Section 3.

Theorem 5.1. Let a function $z: Q^n \rightarrow \mathbb{R}^n$ satisfying Assumption (A) be given. Then, the set Z_z has a component Z_z^c containing 0^n and 1^n.

Proof. For every $j \in I_n$, let the set C^j be defined by

$$C^j = \{q \in Q^n \mid z_j(q) = \max(\{z_k(q) \mid k \in I_n\})\} \cup \{q \in Q^n \mid q_j = 0 \text{ or } q_{j+1} = 1\}.$$

Due to the continuity of z, the set C^j, $\forall j \in I_n$, is closed. Moreover, the sets $C^1, \ldots, C^n$ satisfy the other conditions of Theorem 3.1. Therefore, there exists a connected set S in Q^n such that $0^n \in S$, $1^n \in S$, and $S \subset \bigcap_{j=1}^n C^j$. It is shown that $q^* \in S$ implies $z(q^*) = 0^n$.

Let some $q^* \in S$ be given. If $I^1(q^*) = \emptyset$, then for every $j \in I_n$, either

$$q_j^* = 0 \quad \text{and} \quad z_j(q^*) \geq 0,$$

or

$$q_j^* > 0 \quad \text{and} \quad z_j(q^*) = \max(\{z_k(q^*) \mid k \in I_n\}) \geq 0,$$

implying in both cases that $z(q^*) = 0^n$ by Assumption (A3). If $\emptyset \neq I^1(q^*) \neq I_n$, then there exists $k \in I_n$ such that $q_k^* = 1$ and $q_{k+1}^* \neq 1$, hence,

$$0 \geq z_k(q^*) = \max(\{z_j(q^*) \mid j \in I_n\}) \geq 0,$$

so $z(q^*) = 0^n$ by Assumption (A3). If $I^1(q^*) = I_n$, then trivially $z(q^*) = 0^n$. □

References

1. Knaster, B., Kuratowski, C., Mazurkiewicz, C., *Ein Beweis des Fixpunktsakres für n-dimensionale Simplexe*, Fundamenta Mathematicae, Vol. 14, pp. 132–137, 1929.
2. Scarf, H., *The Approximation of Fixed Points of a Continuous Mapping*, SIAM Journal on Applied Mathematics, Vol. 15, pp. 1328–1343, 1967.
3. Shapley, L. S., *On Balanced Games without Side Payments*, Mathematical Programming, Edited by T. C. Hu and S. M. Robinson, Academic Press, New York, New York, pp. 261–290, 1973.
4. Ichiishi, T., *Alternative Version of Shapley's Theorem on Closed Coverings of a Simplex*, Proceedings of the American Mathematical Society, Vol. 104, pp. 759–763, 1988.
5. Gale, D., *Equilibrium in a Discrete Exchange Economy with Money*, International Journal of Game Theory, Vol. 13, pp. 61–64, 1984.
6. Shapley, L. S., and Vohra, R., *On Kakutani's Fixed-Point Theorem, the KKMS Theorem, and the Core of a Balanced Game*, Economic Theory, Vol. 1, pp. 108–116, 1991.

7. Van Der Laan, G., and Talman, A. J. J., *Intersection Theorems on the Simplotope*, CentER Discussion Paper 9370, Tilburg University, 1993.
8. Freidenfelds, J., *A Set Intersection Theorem and Applications*, Mathematical Programming, Vol. 7, pp. 199–211, 1974.
9. Drèze, J. H., *Existence of an Exchange Equilibrium under Price Rigidities*, International Economic Review, Vol. 16, pp. 301–320, 1975.
10. Herings, P. J. J., *Equilibrium Existence Results for Economies with Price Rigidities*, Economic Theory, Vol. 7, pp. 63–80, 1996.
11. Armstrong, M. A., *Basic Topology*, Springer Verlag, New York, New York, 1983.
12. Mas-Colell, A., *The Theory of General Economic Equilibrium: A Differentiable Approach*, Cambridge University Press, Cambridge, England, 1985.
13. Todd, M. J., *The Computation of Fixed Points and Applications*, Lecture Notes in Economics and Mathematical Systems, Springer Verlag, Berlin, Germany, Vol. 124, 1976.
14. Mas-Colell, A., *A Note on a Theorem of F. Browder*, Mathematical Programming, Vol. 6, pp. 229–233, 1974.
15. Browder, F. E., *On Continuity of Fixed Points under Deformations of Continuous Mappings*, Summa Brasiliensis Mathematicae, Vol. 4, pp. 183–191, 1960.
16. Hildenbrand, W., *Core and Equilibria of a Large Economy*, Princeton University Press, Princeton, New Jersey, 1974.
17. Freund, R. W., *Combinatorial Theorems on the Simplotope That Generalize Results on the Simplex and Cube*, Mathematics of Operations Research, Vol. 11, pp. 169–179, 1986.
18. Sperner, E., *Neu Beweis für die Invarianz der Dimensionszahl und des Gebietes*, Abhandlungen aus dem Mathematischen Seminar, Universität Hamburg, Vol. 6, pp. 265–272, 1928.
19. Fan, K., *A Covering Property of Simplexes*, Mathematica Scandinavica, Vol. 22, pp. 17–20, 1968.
20. Scarf, H., *The Computation of Economic Equilibria*, Yale University Press, New Haven, Connecticut, 1973.
21. Ichiishi, T., *On the Knaster-Kuratowski-Mazurkiewicz-Shapley Theorem*, Journal of Mathematical Analysis and Applications, Vol. 81, pp. 297–299, 1981.
22. Komiya, H., *A Simple Proof of the KKMS Theorem*, Economic Theory, Vol. 4, pp. 463–466, 1994.
23. Krasa, S., and Yannelis, N. C., *An Elementary Proof of the Knaster-Kuratowski-Mazurkiewicz-Shapley Theorem*, Economic Theory, Vol. 4, pp. 467–471, 1994.
24. Zhou, L., *A Theorem on Open Coverings of a Simplex and Scarf's Core Existence Theorem through Brouwer's Fixed-Point Theorem*, Economic Theory, Vol. 4, pp. 473–477, 1994.
25. Herings, P. J. J., *On the Connectedness of the Set of Constrained Equilibria*, CentER Discussion Paper 9363, Tilburg University, 1993.
26. Herings, P. J. J., Talman, A. J. J., and Yang, Z., *The Computation of a Continuum of Constrained Equilibria*, Mathematics of Operations Research, Vol. 21, pp. 675–696, 1996.