Economic Theory (2007) 30: 107–118
DOI 10.1007/s00199-005-0032-z

RESEARCH ARTICLE

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The private value single item bisection auction

Received: 16 October 2004 / Accepted: 26 August 2005 / Published online: 11 November 2005

Abstract In this paper we present a new iterative auction, the bisection auction, that can be used for the sale of a single indivisible object. The bisection auction has fewer rounds than the classical English auction and causes less information to be revealed than the Vickrey auction. Still, it preserves all characteristics the English auction shares with the Vickrey auction: there exists an equilibrium in weakly dominant strategies in which everyone behaves truthfully, the object is allocated in accordance with efficiency requirements to the buyer who has the highest valuation, and the price paid by the winner of the object equals the second-highest valuation.

Keywords Single item auction · Weakly dominant strategy · Extensive form games of incomplete information · Information revelation · Binary search

JEL Classification Numbers C72 · D44

Elena Grigorieva acknowledges support by the Dutch Science Foundation NWO through grant 401-01-101. Jean-Jacques Herings acknowledges support by the Dutch Science Foundation NWO through a VICI-grant. Rudolf Müller acknowledges support by European Commission through funds for the International Institute of Infonomics.

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1 Introduction

A central task of auction design has been to develop mechanisms that have an implementation in weakly-dominant strategies resulting in an efficient allocation. Due to the Revelation Principle, the focus has mainly been on direct revelation mechanisms (see e.g. Mas-Colell, Whinston, and Green 1995). In the private value environment the challenge is considered to be solved since the Vickrey–Clarke–Groves direct mechanism implements the efficient allocation and is incentive-compatible (Clarke 1971; Groves 1973; Vickrey 1961). However, by construction, implementation of an equilibrium strategy in a direct mechanism requires elicitation of complete and exact preference information. It has been recognized that the full revelation of bidders' preferences is not necessarily a desirable feature of a mechanism (Engelbrecht-Wiggans and Kahn 1991; Rothkopf, Tisberg, and Kahn 1990). Bidders might be reluctant to truthfully reveal their full private value if there will be subsequent auctions or negotiations in which the information revealed can be used against them.

Such considerations lead to an interest in auctions where bidders need not reveal their information entirely but only partially. Recent research has begun to examine limited revelation auctions - mechanisms that elicit bidders' valuations only up to some limited precision (Conen and Sandholm 2002; Gilpin and Sandholm 2003; Nisan and Blumrosen 2002; Parkes 2004). In particular, the effect on allocative efficiency of low valuation revelation is studied. For example, Nisan and Blumrosen Nisan and Blumrosen (2002) introduce auctions in the single-object setting where each bidder has a small number of possible bids to choose from, which allows bidders to retain much of their private valuation information. The authors determine auctions of this kind where the loss of economic efficiency incurred relative to unconstrained auctions is mild.

The question that arises is how to design an auction that elicits less information about bidders' valuations than the Vickrey limited information revelation is often believed to be incompatible with these requirements. Nevertheless, the primary contribution of this paper is to present and to analyze an alternative auction format, called the bisection auction, that possesses these properties for the case of selling a single indivisible object under private values.

Suppose a single indivisible object is auctioned. The buyers' valuations are assumed to be integer, randomly drawn from a bounded interval – by default of the form $[0, 2^R)$ for some positive integer R. The bisection auction has R rounds. The price sequence starts at the middle of the initial interval with a price equal to 2^{R-1}. Bidders report their demand at the current price by sealed bids. A *yes*-bid stands for the announcement to be willing to buy at the current price, a *no*-bid for the contrary. As a function of these bids, the auctioneer announces the price of the next round.

In case there are at least two players submitting a *yes*-bid, the price goes up to the middle of the upper half interval, i.e. to the interval $[2^{R-1}, 2^R)$. The players that are allowed to participate actively in the next round are the ones that said *yes*. In case there is at most one player saying *yes*, attention shifts to the lower half interval, i.e. the interval $[0, 2^{R-1})$ and the price goes down to the middle of this interval. The active players in the next round are the ones that said *no*. In case there is a single buyer that submitted a *yes*-bid, this buyer becomes the winner and gets the object. Nevertheless the auction doesn't end, but continues. In this second phase of the auction the transaction price – the highest price at which at least one of the other bidders would say *yes* – is determined.

Iterating this procedure will eventually yield a winner and a price. If in no round there was precisely one player that said *yes* then ties are broken by random assignment to a player who is still active after R rounds. At no point during the auction are the bidders informed that the object has been assigned to the winning bidder, or indeed that any particular bidder has dropped out of the auction. The price is uniquely determined because in each round the length of the current interval goes down by one half. Since the initial interval is of length 2^R, after R rounds the resulting interval is of length 1. And since it is a half-open interval, it contains exactly one integer. This integer is declared to be the price the winner of the auction has to pay for the object.

A related auction format is mentioned in Fujishima et al. (1999) in the context of designing iterative auctions with quick and predictable termination time. In particular Fujishima et al. (1999) write: "In what we call "the binary price-search auction", the auctioneer queries all bidders whether they are willing to pay a given price. If only one bidder answers *yes*, he gets the good at that price. If zero or more than one answer *yes*, another query follows at a lower or higher price, in such a way to converge at logarithmic speed to a price that exactly one bidder will be willing to pay. Unfortunately, this and other "accelerated auctions" are difficult to analyze with the tools of game theory."

The binary price-search auction is very similar to the bisection auction. A subtle difference is that the binary price-search auction stops as soon as the winner has been found. In this paper we present a full game-theoretic analysis of the bisection auction. We show that the bisection auction is strategically equivalent to the Vickrey auction (and hence also to the English auction). It implies that, truth-telling is a weakly dominant strategy in the bisection auction and the equilibrium results in an efficient allocation. While being strategically equivalent to the Vickrey and English auctions the bisection auction outperforms them. It is preferred over the English auction because of its speed and over the Vickrey auction because of its low valuation revelation requirement.

2 The bisection auction

The following example illustrates how the bisection auction works.

Example Suppose there are four bidders, A, B, C, and D, with the following integer private valuations from the interval [0, 16): 11, 7, 15, 9. To determine the winner and the price in this setting the bisection auction takes four rounds and starts with an ask price equal to 8. Suppose that each bidder chooses to respond truthfully and follows a straightforward strategy under which he says *yes* if an ask price is less or equal to his valuation and *no* otherwise. Bidders are not informed about other bidders' choices. The bisection auction proceeds as follows:

Round	Price	Lower bound	Upper bound	Bidder A	Bidder B	Bidder C	Bidder D
1	8	0	16	Yes	No	Yes	Yes
2	12	8	16	No	No	Yes	No
3	10	8	12	Yes	No	Yes	No
4	11	10	12	Yes	No	Yes	No

Since three bidders submitted yes-bids in the first round, the price increases to the middle of the current price and the current upper bound. So the ask price of the second round is 12. These three bidders remain active while bidder B drops out. We allow a drop-out to submit any bid, but consider any bid of a drop-out as a no-bid. Since there is only one yes-bid in the second round we have a winner and we enter what we call the price determination phase. From now on, any bid of the winner, bidder C, is considered as a yes-bid. Players A and D are still active. In the third round, there are two yes-bids so the price increases. Player D drops out. In the fourth round, the auction terminates. Taking into account bids made during the last round we compute the final lower and upper bounds. Since there were two yes-bids the upper bound remains 12 while the lower bound becomes 11. The winner, bidder C, takes the object and pays price 11 which is the smallest Walrasian price for the demand announced by the bidders that participated in this auction.

2.1 Formal representation of the auction

Here we model the auction as a non-cooperative game in extensive form with imperfect information.

The game

(1) A finite set $N = \{1, \dots, n\}$ of players that participate in the game.
(2) A number of rounds R that specifies the duration of the game.
(3) For each player a decision set $A = \{yes, no\}$. This reflects the fact that each player has to make a binary decision in each and every of his information sets.
(4) Every node in the game tree is a vector a that represents the history of decisions chosen by players before the game reached this node. Formally, $a = (a_k)_{k=1}^{r}$ with $1 \leq r \leq R$ where $a_k = (a_{ki})_{i=1}^{n}$ for $k < r$ and $a_r = (a_{ri})_{i=1}^{j}$ for some $j \leq n$. Here a_{ki} is the decision in A by player i in round k.

The length of a node is defined as $l(a) = (r-1)n + j$. The initial node a_0 is a node with length equal to 0. The nodes with length equal to rn are referred to as the start of round $r+1$. The terminal nodes are nodes whose length is equal to Rn.

(5) There is an edge between two nodes a and b if the length of b is equal to the length of a plus one, and for all i and k for which a_{ki} is defined, $a_{ki} = b_{ki}$.

Let X_r denote the set of nodes corresponding to round r. The set X_r partitions into n sets X_{ri}, each X_{ri} being the set of nodes in round r where player i has to make a decision. Formally, $X_{ri} = \{a \mid l(a) = (r-1)n + i - 1\}$. For a node

$a \in X_r$ and $q < r$, let $a^q = (a_k)_{k=1}^q$ denote the part of the history a corresponding to the first q rounds. We associate with every node $a \in X_r$ the set $P(a)$ of players whose decisions made in round $r-1$ equal *yes*, taking into account that decisions of drop-outs are counted as *no*, decisions of a winner as *yes*.

(6) Node a in X_{ri} belongs to the information set $H(a)$ defined as the set of all nodes b from X_{ri} such that for all $k < r$
1. $a_{ki} = b_{ki}$
2. $|P(a^k)| \leq 1$ iff $|P(b^k)| \leq 1$.

(7) We associate with each terminal node τ, a set WIN(τ) that is a set of candidate winner(s) of the game if the game terminates in node τ. If in one of the rounds corresponding to node τ there has been exactly one *yes*-bid, then WIN(τ) consists of the player that submitted this bid. Otherwise, WIN(τ) equals the set of players that never dropped out. In that case, the winner of the game is determined by a lottery among all players i in WIN(τ). The price the winner pays is

$$p(\tau) = 2^{R-1} + \sum_{k=1}^{R-1} \lambda_k 2^{R-k-1} + \beta$$

where for all k

$$\lambda_k = \begin{cases} -1 & \text{if } |P(\tau^k)| \leq 1 \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta = \begin{cases} -1 & \text{if } |P(\tau)| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The payoff of player i with private value v_i in terminal node τ is defined by

$$U_i(\tau) = \begin{cases} v_i - p(\tau) & \text{with probability } \frac{1}{|WIN(\tau)|} \text{ if } i \in WIN(\tau) \\ 0 & \text{else.} \end{cases}$$

A few remarks. To describe the auction more precisely, we partition the set of players N into three subsets of players, $A(a)$ – the set of players that are active in node a, $W(a)$ – the winner set in node a, and $D(a)$ – the set of players that dropped out before the game reached node a. For the initial node a_0 they are defined as follows: $A(a_0) = N$, $W(a_0) = \emptyset$ and $D(a_0) = \emptyset$. We iteratively define them for nodes with arbitrary length. For a node $a = (a_k)_{k=1}^r$ with $l(a) = nr$,

$$W(a) = \begin{cases} W(a^{r-1}) & \text{if } W(a^{r-1}) \neq \emptyset \\ \emptyset & \text{if } W(a^{r-1}) = \emptyset \text{ and } |P(a)| \neq 1 \\ P(a) & \text{if } W(a^{r-1}) = \emptyset \text{ and } |P(a)| = 1 \end{cases}$$

$$D(a) = \begin{cases} D(a^{r-1}) & \text{if } |P(a)| \leq 1 \\ D(a^{r-1}) \cup \{i \in A(a^{r-1}) \mid a_{ri} = no\} & \text{otherwise} \end{cases}$$

$$A(a) = N \setminus (W(a) \cup D(a)).$$

For a node $a = (a_k)_{k=1}^r$ with $l(a) \neq nr$, we define these sets as follows: $W(a) = W(a^{r-1})$, $D(a) = D(a^{r-1})$, and $A(a) = A(a^{r-1})$.

In a similar way as for terminal nodes, we can associate a price with an arbitrary internal node. For a node $a = (a_k)_{k=1}^r$ with $l(a) = nr$ we define its price by

$$p(a) = 2^{R-1} + \sum_{k=1}^{r} \lambda_k 2^{R-k-1}$$

where for all k

$$\lambda_k = \begin{cases} -1 & \text{if } \; |P(a^k)| \leq 1 \\ 1 & \text{otherwise.} \end{cases}$$

Note that for a with $l(a) = nr$ the price $p(a) = p(a^{r-1}) + 2^{R-r-1}$ if $|P(a)| > 1$ and $p(a) = p(a^{r-1}) - 2^{R-r-1}$ if $|P(a)| \leq 1$. Furthermore, for a node $a = (a_k)_{k=1}^r$ with $l(a) \neq nr$ we define $p(a) = p(a^{r-1})$.

In any node a we now have a price. The actions a player can take in this node a, either a *yes* or a *no*, can be interpreted as being his answer to the question:

Are you willing to pay the price $p(a)$ for the object we are selling?

Moreover, these prices can be used to characterize information sets. For any $a, b \in X_{ri}$ from the same information set, $p(a) = p(b)$. Each information set in X_{ri} can be represented by its associated price and the sequence of decisions chosen by player i in the first $r - 1$ rounds.

Observe that the way the auction proceeds depends only on the behavior of active players. The following result shows that the bisection auction leads to a very particular information structure.

Proposition 2.1 *For an information set in which player i is active and for every decision of player i made in this set, there exists exactly one immediate successor information set $H(a)$ in which player i is still active. Moreover, this immediate successor is, in case the player chose* yes, *the one with $|P(a)| > 1$ and the one with $|P(a)| \leq 1$ in the other case.*

This proposition is an immediate consequence of the definition of an information set. Once a player said *no* from the fact that he stays active he can infer that there was nobody or exactly one player with a *yes*-bid, but he can't distinguish between these two possible cases so that he can conclude whether the winner is found already.

3 Playing the game

3.1 Equivalence of strategies and threshold strategies

Let $\mathcal{H}_i$ denote the collection of player i's information sets.

Definition 3.1 *A* strategy *for player i is a function $s_i : \mathcal{H}_i \rightarrow A$.*

First notice that the number of information sets of each player corresponding to round k is equal to 4^{k-1} (unless the number of players equals two, in which case it is 3^{k-1}). The total number of information sets in the game for each player is equal to $\sum_{k=1}^{R} 4^{k-1} = \frac{1}{3}(4^R - 1)$. Thus, the number of possible strategies of a player equals $2^{\frac{1}{3}(4^R-1)}$. We will show that the number of essentially different strategies is much lower, in the order of 2^R to be specific. We denote by $p(H)$ the price corresponding to information set H.

Definition 3.2 *Let t be an integer, called the threshold. The threshold strategy $s_i^t : \mathcal{H}_i \to A$ assigns the action* yes *to $H \in \mathcal{H}_i$ with $p(H) \leq t$ and the action* no *to $H \in \mathcal{H}_i$ with $p(H) > t$.*

We show next that from a strategic perspective players may restrict themselves to using threshold strategies only. First we need the following result.

Lemma 3.3 *Let s_i be a strategy of player i and let r be a round. Then there is a unique information set $H_{r,i}(s_i)$ of player i such that if he is still active in round r he will necessarily be in information set $H_{r,i}(s_i)$. Furthermore, the price of this information set $H_{r,i}(s_i)$ equals*

$$p(H_{r,i}(s_i)) = 2^{R-1} + \sum_{k=1}^{r-1} \lambda_k 2^{R-k-1}$$

where for all k

$$\lambda_k = \begin{cases} -1 & \text{if } \; s_i(H_{k,i}(s_i)) = no \\ 1 & \text{if } \; s_i(H_{k,i}(s_i)) = yes. \end{cases}$$

Lemma 3.3 follows easily as a consequence of Proposition 2.1. It enables us to specify for any strategy its equivalent threshold strategy.

Definition 3.4 *Let s_i be a strategy of player i. The* threshold *$t_i(s_i)$ is defined by $t_i(s_i) = p(H_{R,i}(s_i)) + \beta$ where $\beta = -1$ if $s_i(H_{R,i}(s_i)) = no$ and $\beta = 0$ if $s_i(H_{R,i}(s_i)) = yes$.*

The following result implies that a strategy s_i and the threshold strategy $s_i^{t_i(s_i)}$ are realization-equivalent.

Theorem 3.5 *Let s_i be a strategy and let $\widetilde{s_i}$ be the threshold strategy with threshold $t_i(s_i)$. Let s_{-i} be an arbitrary collection of strategies for players other than i. Let a be the realized history if $s = (s_i, s_{-i})$ is played and let b be the realized history if $\widetilde{s} = (\widetilde{s_i}, s_{-i})$ is played. Then for both histories holds*

1. *For every round r, $A(a^{r-1}) = A(b^{r-1})$, $D(a^{r-1}) = D(b^{r-1})$ and $W(a^{r-1}) = W(b^{r-1})$.*
2. *For every round r where player i is active according to a we have $a^r = b^r$.*
3. *For all $k \leq r$ and all players $j \neq i$, $a_{kj} = b_{kj}$.*
4. *All players $j \neq i$ in all rounds reach the same information sets.*
5. *The payoff of all players are the same in a and b.*

Proof We will first prove statements 1, 2 and 3 by induction.

(A) For round $r = 1$. By definition, $A(a^0) = A(b^0)$, $D(a^0) = D(b^0)$ and $W(a^0) = W(b^0)$.

In order to prove (2) and (3), notice that the set X_{1j} is the only information set of any player j in round 1. Obviously $a_{1j} = b_{1j}$ for all $j \neq i$, so it remains to prove that $a_{1i} = b_{1i}$ or equivalently that $s_i(X_{1i}) = \widetilde{s}_i(X_{1i})$.

If $s_i(X_{1i}) = no$ we have to show that $t_i(s_i) < p(X_{1i})$. By Lemma 3.3 it holds that $\lambda_1 = -1$, so

$$t_i(s_i) = 2^{R-1} + \sum_{k=1}^{R-1} \lambda_k 2^{R-k-1} + \beta \leq p(X_{1i}) - 2^{R-2} + \sum_{k=2}^{R-1} 2^{R-k-1} < p(X_{1i}).$$

Similarly, if $s_i(X_{1i}) = yes$ we can show that $t_i(s_i) \geq p(X_{1i})$.

(B) Now suppose that (1)–(3) are true in round r. We will show that they are also true for $r + 1$. Using (3) of the induction hypothesis it follows easily that $A(a^r) = A(b^r)$, $D(a^r) = D(b^r)$ and $W(a^r) = W(b^r)$, which proves (1).

In order to prove (2) and (3), suppose that according to a player i is active in round $r+1$. Then player i is also active in round r. We know from the induction hypothesis that $a^r = b^r$. It is then clear that $a_{r+1,j} = b_{r+1,j}$ for all players $j \neq i$. The only thing left to show is $a_{r+1,i} = b_{r+1,i}$ or equivalently $s_i(H_{r+1,i}(s_i)) = \widetilde{s}_i(H_{r+1,i}(s_i))$.

If $s_i(H_{r+1,i}(s_i)) = no$, we have to show that $t_i(s_i) < p(H_{r+1,i}(s_i))$. Lemma 3.3 implies $\lambda_{r+1} = -1$. So,

$$t_i(s_i) = p(H_{r+1,i}(s_i)) + \sum_{k=r+1}^{R-1} \lambda_k 2^{R-k-1} + \beta \leq p(H_{r+1,i}(s_i)) \\ -2^{R-r-2} + \sum_{k=r+2}^{R-1} 2^{R-k-1} < p(H_{r+1,i}(s_i)).$$

The other case goes again along the same lines of reasoning.

(4) This follows easily from (2).
(5) Observe that the payoff is a function of the information sets reached in round R and decisions of active players made in the last round. All these are the same.

□

Corollary 3.6 *Any strategy s_i of a player i can be represented by an equivalent threshold strategy $\widetilde{s}_i$.*

In the following, we will restrict our attention to threshold strategies and denote them just by referring to the threshold.

3.2 Playing the game with threshold strategies

In this subsection we show that the winner of the object to be sold is a player with the maximum threshold and the price equals the second-highest threshold, and consequently that truth-telling is a weakly dominant strategy.

Let $t = (t_i)_{i \in N}$ be a profile of thresholds played in the bisection auction. This profile remains fixed during the next few statements. We denote the terminal node where the game ends according to this profile by τ^*. Let $p(r) = p((\tau^*)^{r-1})$ be the price in round r for this realization of the game.

Definition 3.7 *The* maximum threshold $t_{\max}$ *is defined to be the number* $max_{\mathrm{i}\in\mathrm{N}} t_i$.

Definition 3.8 *Let* k *be a player with* $t_k = t_{\max}$. *The* second-highest threshold t_{sec} *is defined to be the number* $max_{\mathrm{i}\in\mathrm{N}/\{\mathrm{k}\}} t_i$.

Theorem 3.9 *Let* $t = (t_i)_{i\in N}$ *be a profile of thresholds played in the bisection auction. The winner of the game is necessarily a player whose threshold equals the maximum threshold.*

Proof According to the definition the set of candidate winner(s) of the game is $\mathrm{WIN}(\tau^*) = W(\tau^*)$ if $W(\tau^*) \neq \emptyset$ and $WIN(\tau^*) = A(\tau^*)$ otherwise.
Case 1. $W(\tau^*) \neq \emptyset$.

Let $W(\tau^*) = \{k\}$. Consider the round r in which player k became the winner. Then $a_{rk} = yes$ and $a_{rj} = no$ for all $j \neq k$. Thus $t_j < p(r)$ for all $j \neq k$ and $t_k \geq p(r)$.
Case 2. $W(\tau^*) = \emptyset$.

Note that all players in $A(\tau^*)$ must have chosen the same action in each round. Therefore they have the same thresholds. A player j who became inactive in some round r must have said *no* in this round while all players in $A(\tau^*)$ said *yes*. But then $t_j < p(r) \leq t_k$ for all $k \in A(\tau^*)$ and for all $j \notin A(\tau^*)$. □

Lemma 3.10 *Let* $t = (t_i)_{i\in N}$ *be a profile of thresholds played in the bisection auction. A player with the second highest threshold remains active till the end of the game.*

Proof Let player j be a player with the second highest threshold. We have to show that $j \in A(\tau^r)$ for all $0 \leq r \leq R$. The case $r = 0$ is trivial. Suppose the statement is true for some r. We show that it is true for $r + 1$. There are three situations concerning the winner set that could possibly occur.

(A) $W(\tau^{r+1}) = \emptyset$. It means that $|P(\tau^{r+1})| \neq 1$. Suppose $|P(\tau^{r+1})| = 0$. Then $A(\tau^{r+1}) = A(\tau^r)$ – all active players remained active, thus player j too. Suppose $|P(\tau^{r+1})| > 1$. Then every player with the highest threshold, as well as any player with the second highest threshold, decides *yes*. In this case all players from $A(\tau^r)$ with a *yes*-bid remain active, including player j.
(B) $W(\tau^r) = \emptyset$ and $W(\tau^{r+1}) \neq \emptyset$. It means that $|P(\tau^{r+1})| = 1$. The only *yes*-bid comes from the, in this case unique, player with the highest threshold, and he becomes the winner. The set of active players is $A(\tau^{r+1}) = A(\tau^r) \,/\, W(\tau^r)$. Thus player j remains active.
(C) $W(\tau^r) \neq \emptyset$. For this case it holds that $|P(\tau^{r+1})| \geq 1$. Suppose $|P(\tau^{r+1})| = 1$. Then all players from $A(\tau^r)$ have made a *no*-bid and remain active, thus player j too. If $|P(\tau^{r+1})| > 1$, then at least one player from $A(\tau^r)$ has made a *yes*-bid and remains active. This surely includes player j because he has the highest threshold among the players that are still active.

□

Lemma 3.11 *Let* $t = (t_i)_{i\in N}$ *be a profile of thresholds played in the bisection auction. The price of the terminal node* τ^* *is equal to the threshold of any player that is active at the end of the game.*

Proof Consider a player from $A(\tau^*)$, say i. First of all we will show that $|t_i - p(R)| \leq 1$. We prove this inductively by showing that $|t_i - p(r)| \leq 2^{R-r}$ for each round r. For $r = 1$ the statement follows easily. Suppose it is true for some r.

(a) $p(r) \leq t_i$. Together with the assumption of the induction hypothesis it gives us $p(r) \leq t_i \leq p(r) + 2^{R-r}$. Moreover, in this case player i has made a *yes*-bid in round r. Since he is active in all rounds, the price went up to $p(r) + 2^{R-r-1}$ in round $r + 1$, so $|t_i - p(r + 1)| \leq 2^{R-r-1}$.
(b) $p(r) > t_i$. For this situation we show in the same way that $p(r) - 2^{R-r} \leq t_i \leq p(r)$ and $p(r+1) = p(r) - 2^{R-r-1}$ from which it follows that $|t_i - p(r+1)| \leq 2^{R-r-1}$.

Thus, $|t_i - p(R)| \leq 1$ for any player who is active at the end of the game.
Case 1. $p(\tau^*) = p(R) - 1$ and so $|P(\tau^*)| \leq 1$.

This last inequality implies that player i said *no* in the last round, so $p(R) > t_i$. Since $|t_i - p(R)| \leq 1$, $p(\tau^*) = t_i$.
Case 2. $p(\tau^*) = p(R)$ and so $|P(\tau^*)| > 1$.
This last inequality implies that player i said *yes* in the last round, in other words, $p(R) \leq t_i$. If, he said *yes* in all rounds, $p(R) = 2^R - 1$, and thus $p(R) = t_i$. Otherwise, if follows from the construction of the bisection auction that the last round in which player i has said *no* has a price equal to $p(R) + 1$. So $p(R) \leq t_i < p(R) + 1$, and thus $p(\tau^*) = t_i$. □

Corollary 3.12 *Let $t = (t_i)_{i \in N}$ be a profile of thresholds played in the bisection auction. The price the winner of the game pays is equal to the second highest threshold.*

Definition 3.13 The *truth-telling strategy* of player i is the threshold strategy for which t is equal to player i's valuation v_i.

Theorem 3.14 *The truth-telling strategy of player i is a weakly dominant strategy.*

Proof From the previous results, specifically Theorem 3.9 and Corollary 3.12, it follows that we can interpret the thresholds from Corollary 3.6 as bids in the Vickrey auction. The bisection auction is therefore strategically equivalent to the Vickrey auction, and telling the truth is a weakly dominant strategy. □

A final immediate consequence of this theorem is the following corollary.

Corollary 3.15 *The truth-telling strategy profile constitutes a symmetric Nash equilibrium. It is the unique perfect Nash equilibrium.*

3.3 Some remarks on generalizations of the bisection auction

The bisection auction assumes that the bidders' valuations are randomly drawn from a probability distribution on the bounded set $[0, 2^R)^n$. There is no problem to extend the bisection auction to the case where the upper bound is equal to infinity. The key insight is that the current construction where after each round the current interval is split in two equally sized intervals can be generalized to any split without changing our main results. In particular, one may start the first round with the interval $[0, \infty)$, which splits in the intervals $[0, u)$ and $[u, \infty)$ for some integer u. As long as all realizations of valuations are assumed to be finite, after a finite number of rounds attention will be restricted to an interval with a finite upper bound.

The rules of the bisection auction presented in the sections above are constructed in such a way that the only information revealed to players after each round is the change of the price. This information policy is less restrictive than it may look at first sight. First of all, in order to make the formal representation of the auction and the proofs of its properties concise and comprehensive we decided not to include the number or identities of the drop-outs in the information revealed. Notice however that revealing which players dropped out does not reveal whether or when the winner has been determined. Thus, revealing which players drop out does not change the weak dominance of truth-telling. Only the formal proof of this property becomes more sophisticated. Information sets of a player are characterized then, in addition to price and own previous actions, by time and the identity of the players that dropped out. So, the strategy space becomes larger and more complicated. But we still can show that for any strategy from this extended space there exists an outcome equivalent threshold strategy, so that our results of this section hold for the auction with such an information policy.

The possibilities offered by for instance the internet render a practical implementation of the bisection auction feasible. Bidders should be informed about the rules of the bisection auction, in particular about how prices respond to the number of *yes*-bids, and about the fact that the price in each round is the only information transmitted to them.

Now we will examine what happens if after each round the bids of all the players are revealed, so not only the drop-outs but also the winner becomes publicly known. The following example shows that truth-telling is not a weakly dominant strategy in the bisection auction under this information policy.

Example Consider the bisection auction under the full information policy with two players. Let player A choose to act according to the following strategy: to say *no* up to the moment the winner is found and *yes* afterwards. Regardless of his valuation the best that player B can do against this strategy is to say *no* in all rounds but the very last one. Player B will get the object for a price equal to zero. □

Nevertheless, it can still be shown that the truth-telling profile constitutes an ex-post Nash equilibrium. Indeed, let's analyze what the best thing is player i can do given that all other players follow the truth-telling strategy. If player i doesn't win, his payoff is zero. If he wins he pays the price equal to the highest valuation of other players, so the price is not influenced by his behaviour during the auction. Thus, any strategy that guarantees winning in case player i's valuation is the highest is a best response. One such strategy is the truth-telling strategy.

4 Conclusions

We proposed a new auction, the bisection auction, and analyzed its equilibrium properties. First, we have proved that in the bisection auction, threshold strategies are sufficient from a strategic point of view. Furthermore, we have shown strategic equivalence of the bisection auction to the Vickrey auction. Using this result we established that the proposed auction is incentive compatible, that is truth-telling is a weakly dominant strategy. Moreover, the equilibrium that results when everyone tells the truth is efficient in the sense that the player with the highest valuation gets the object.

Concerning the revelation of bidders valuation, participation in the bisection auction is less demanding than in the Vickrey auction and usually less demanding than in the English auction. In the Vickrey auction all players need to reveal their valuation, and in the English auction all bidders except the bidder with the highest valuation need to reveal their valuation. In the bisection auction only the bidder with the second highest valuation has to do this, all other bidders only reveal their valuation up to a precision required to determine the winner and the precise value of the second highest valuation. Moreover, the procedure used in the bisection auction guarantees a fast and predictable termination, in contrast to the English auction which is on average relatively slow if it uses unit increments and risks inefficient allocations if it uses larger increments.

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