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Universally converging adjustment processes—a unifying approach

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Abstract

Both in game theory and in general equilibrium theory, there exists a number of universally converging adjustment processes. In game theory, these processes typically serve the role of selecting a Nash equilibrium. Examples are, the tracing procedure of Harsanyi and Selten or the equilibrium selection procedure proposed by McKelvey and Palfrey. In general equilibrium, the processes are adjustment rules by which an auctioneer can clear all markets. Examples are the processes studied by Smale, Kamiya, van der Laan and Talman, and Herings. The underlying reasons for convergence have remained rather mysterious in the literature, and convergence of different processes has seemed unrelated. This paper shows that convergence of all these processes relies on Browder's fixed point theorem.

JEL classification: C62; C63; C68; C72

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1. Introduction

Both in game theory and in general equilibrium theory, there exists a number of adjustment processes that are universally convergent. A universally convergent adjustment process in game theory is an adjustment process that converges to a Nash equilibrium for almost all games. A universally convergent adjustment process in general equilibrium theory is an adjustment process that converges to a Walrasian equilibrium for almost all economies. In game theory, these processes typically serve the role of selecting a Nash equilibrium.

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Examples are the tracing procedure of Harsanyi and Selten (1988) or the equilibrium selection procedure proposed by McKelvey and Palfrey (1995). In general equilibrium, the processes are adjustment rules by which an auctioneer can clear all markets. Examples are the processes studied by Smale (1976), van der Laan and Talman (1987), Kamiya (1990) and Herings (1997).

There are several reasons to be interested in universally convergent adjustment processes. They give players in a game the opportunity to coordinate on a uniquely determined Nash equilibrium and an auctioneer in an economy to determine a competitive equilibrium price system. In a more decentralized setting, they give rational agents in an economy the possibility to coordinate on current and future prices. Such processes can be used as a tool to compute equilibria, which is also helpful for comparative statics exercises or policy recommendations, see Judd (1997) and Eaves and Schmedders (1999).

In game theory, multiplicity of Nash equilibria seems to be the rule rather than the exception. This poses serious problems for Nash equilibrium to be used as a solution concept for games. One way out is to develop a theory that selects a unique equilibrium for any game form, and to suppose that all players adopt that theory. An attempt to make such a theory can be found in Harsanyi and Selten (1988). That theory relies heavily on the tracing procedure as introduced in Harsanyi (1975). The tracing procedure is a strategy adjustment procedure by which players can adopt initial beliefs about the play of their opponents and turn them into uniquely determined beliefs consistent with Nash equilibrium. The surprising aspect of the tracing procedure is that convergence to a Nash equilibrium takes place for almost any game for almost any initial beliefs, so the tracing procedure is universally convergent.

Quantal response equilibria as introduced in McKelvey and Palfrey (1995), are statistical versions of Nash equilibria, where each player's payoff is subject to random error. The concept of equilibrium is consistent in the sense that all players maximize their utility given the choices made by the others, and the utility maximizing behavior of a player, together with the error structure, leads to the mixed strategy against which the others optimize. Quantal response equilibria are quite successful in describing the behavior of participants in experiments. McKelvey and Palfrey (1995) also consider a procedure similar to the tracing procedure to select a Nash equilibrium. Start with the quantal response equilibrium where choices are completely determined by the error terms, and follow the path of quantal response equilibria that results when the error terms vanish. McKelvey and Palfrey show that for almost all games, a unique Nash equilibrium is selected in this way. Again, universal convergence of the procedure is obtained.

The simplest price adjustment process studied in general equilibrium theory is the Walrasian tatonnement process. It is well-known that it may not converge to a competitive equilibrium, see Scarf (1960) for some examples. The work of Sonnenschein (1972, 1973), Mantel (1974) and Debreu (1974), basically claiming that any continuous function satisfying homogeneity of degree 0 and Walras' law is the excess demand function of an economy, makes clear that it is possible to construct many examples where Walrasian tatonnement does not converge and displays highly irregular dynamic behavior. The work of Saari and Simon (1978) and Saari (1985) implies that simple adaptations of the Walrasian tatonnement process will not have better convergence properties. Still, at least three universally convergent price adjustment processes are known in the literature, Smale's global Newton method introduced in Smale (1976), the process of Kamiya (1990), and the process proposed in van der Laan and Talman (1987) for which universal convergence has been shown in Herings (1997).

The global Newton method of Smale provides a price adjustment rule that does converge to a competitive equilibrium for almost any economy, so universal convergence is the case. But it does not converge for any initial price system. Only when the initial price system is chosen such that the prices of some commodities are sufficiently close to zero, convergence to a competitive equilibrium can be shown. From the work of Keenan (1981), it follows that there may exist an open set of starting price systems for which Smale's process does not converge to some competitive equilibrium price system.

Another universally convergent price adjustment process has been presented in Kamiya (1990). Under rather weak conditions on the total excess demand function, convergence to a competitive equilibrium price system is guaranteed for almost every starting price system. Although the boundary conditions of Kamiya are weak, they are not derived from assumptions on primitive concepts.

An alternative price adjustment process has been proposed in van der Laan and Talman (1987). For this process, universal convergence has been shown in Herings (1997). Under standard conditions on utility functions, consumption sets and initial endowments, this price adjustment process converges to a Walrasian equilibrium price system for almost all economies and almost all starting price systems.

Apparently, several processes in distinct areas of research have been shown to be universally convergent. The reason for these strong convergence properties has remained mysterious up to now, and the convergence proofs were rather ad hoc as a consequence. The aim of the current paper is to point out that convergence of each one of these processes can be understood from fixed point theory and is not even related to differentiability. This makes our proofs very different from the original convergence proofs. It also increases our understanding as to why these distinct adjustment processes converge. This understanding is useful to develop other universally convergent mechanisms that may incorporate features that are lacking in current processes.

Some alternatives and extensions have already been suggested. The procedure described in Yamamoto (1993) may serve as an alternative to the tracing procedure, and Joosten and Talman (1997) describe an alternative price adjustment process. Extensions have been made to economies with linear or constant returns to scale production (see van den Elzen, 1993, 1997; van den Elzen et al., 1994), and to economies with short-run price rigidities (see Herings, 1996; Herings et al., 1997, 1998, 1999). All these extensions can be understood as well from the unifying treatment that is given in this paper.

The organization of the paper is as follows. In Section 2, we outline the general structure behind the approach that is used in our proofs and we present the most important tool required, Browder's fixed point theorem. In Section 3, we apply this methodology to the tracing procedure of Harsanyi and Selten (1988) and in Section 4 to the equilibrium selection procedure proposed by McKelvey and Palfrey (1995). Next we turn to price adjustment processes. We treat Kamiya's process in Section 5, the process proposed in van der Laan and Talman (1987) in Section 6, and Smale's global Newton method in Section 7. Section 8 discusses and illustrates how the approach suggested can be used to derive new adjustment processes. Section 9 concludes.

2. A unifying approach

Before turning to the specific adjustment processes, it is helpful to highlight the approach that can be used to give a unifying treatment of convergence. Usually, dynamic processes are defined by a system of first-order differential equations

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = g(x(t)),$$

where $x(t) \in \mathbb{R}^m$ denotes the state vector reached at time $t \in \mathbb{R}_+$ and g is a function from some subset of the state space $\mathbb{R}^m$ into $\mathbb{R}^m$. The vector x typically corresponds to a mixed strategy combination in case of a strategy adjustment process, and to a price system for a price adjustment process. The function g specifies the way in which players adjust their strategies, or prices adjust in general equilibrium. The initial state $x(0)$ is assumed to be given.

Conditions for which the system of differential equations has a solution are well-known, see for instance Hirsch and Smale (1974). The orbit $\gamma(x(0))$ is the set of state vectors that is generated by the system of first-order differential equations when the initial state is $x(0)$,

$$\gamma(x(0)) = \{x \in \mathbb{R}^m | \exists t \geq 0, \quad x = x(t)\}.$$

We denote the closure of $\gamma(x(0))$ by $\bar{\gamma}(x(0))$, and call $\bar{\gamma}(x(0))$ an orbit as well.

Although all adjustment processes we consider can be formulated as a system of differential equations, they can alternatively be described by the orbit that they generate. In fact, all adjustment processes considered share the property that the easiest way to formulate them is in terms of the orbit that they generate. For each adjustment process, we define a system of equations whose solutions correspond to the orbit of the adjustment process. We study the properties of the set of solutions to the system of equations by means of fixed point theory and not by the theory of dynamic systems. In this paper we argue that the convergence of various adjustment processes is best understood from a single fixed point theorem that is introduced in Browder (1960).

Theorem 2.1 (Browder's fixed point theorem). *Let S be a non-empty, compact, convex subset of $\mathbb{R}^m$ and let $\varphi : [0, 1] \times S \to S$ be a continuous function. Then the set of fixed points, $F_\varphi = \{(\lambda, s) \in [0, 1] \times S | s = \varphi(\lambda, s)\}$ contains a connected set, F_φ^{c}, such that $(\{0\} \times S) \cap F_\varphi^{\mathrm{c}} \neq \emptyset$ and $(\{1\} \times S) \cap F_\varphi^{\mathrm{c}} \neq \emptyset$.*

Theorem 2.1 implies that for all $\lambda \in [0, 1]$, $(\{\lambda\} \times S) \cap F_\varphi \neq \emptyset$. That property would also follow from a repeated application of the well-known fixed point theorem of Brouwer (1912). The surprising part of the theorem is that there exists a connected set F_φ^{c} with those properties. Notice that along the connected set of fixed points, it is not necessarily the case that λ increases monotonically from 0 to 1. The value of λ increases initially, may decrease later on, and will eventually increase until it reaches the value 1.

In all sections, the strategy of proof is the same. A function φ satisfying Browder's fixed point theorem is constructed such that the fixed points in the connected set F_φ^{c} correspond in a one-one way to the orbit generated by the adjustment process. The value of λ

indicates the amount of progress of the adjustment process. At $\lambda = 0$, a fixed point corresponds to the initial state vector $x(0)$. At $\lambda = 1$, a fixed point yields an equilibrium state vector.

Our assumptions on primitives are so weak, that orbits are not necessarily nicely behaved sets, that is differentiable paths or loops. In exceptional cases it is for instance possible that pitchforks may arise, or even higher dimensional solution sets. Such exceptional cases are usually excluded by making differentiability assumptions and employing a transversality argument.

Let Φ be the set of twice continuously differentiable functions $\varphi : [0, 1] \times S \to S$ endowed with the topology of uniform convergence of the values of the function and its first partial derivatives. Theorem 2 in Mas-Colell (1974) asserts that there is an open and dense set $\Phi' \subset \Phi$, such that for every $\varphi \in \Phi'$ the set F^{c}_{φ} is a closed segment, that is a set diffeomorphic to the unit interval $[0, 1]$, and only the end points of the segment intersect $\{0, 1\} \times S$, so one end point intersects $\{0\} \times S$ and the other $\{1\} \times S$. This result is not completely surprising as the set F^{c}_{φ} is defined by a number of equations equal to the dimension of S in a number of unknowns equal to the dimension of $S+1$, leaving one degree of freedom for the solution set. Mas-Colell's result does not require $\{0\} \times S$ to be unique. This, however, is important for algorithmic purposes, as it avoids the problem which point in $\{0\} \times S$ is connected to $\{1\} \times S$.

Since an orbit $\bar{\gamma}(x(0))$ of an adjustment process corresponds in a one-one way to the fixed points of the mapping φ, it follows as a consequence of Theorem 2 in Mas-Colell (1974) that generically an orbit is a nicely behaved set that does not allow for pitchforks or higher dimensional solution sets. In particular, an orbit $\bar{\gamma}(x(0))$ connects $x(0)$ to a unique equilibrium. It is for this reason that we concentrate, in this paper, on the property that $\bar{\gamma}(x(0))$ connects $x(0)$ to some equilibrium. The stronger property that $\bar{\gamma}(x(0))$ is connected by a segment to a unique equilibrium follows by suitable differentiability assumptions and transversality arguments.

In the strategy adjustment processes of Sections 3 and 4, the parameter λ is an explicit part of the adjustment process and is inversely related to the weight given to the prior and the level by which players make errors, respectively. The function $\varphi : [0, 1] \times S \to S$ is such that $(\lambda, s) \in \bar{\gamma}(x(0))$ if and only if $(\lambda, s) \in F^{c}_{\varphi}$.

In the price adjustment processes of Sections 5–7, there is no explicit parameter λ. For $\lambda \in [0, 1]$, a subset $T(\lambda)$ of the state space is defined, which is strictly increasing in λ. The set $T(0)$ corresponds to the starting price system p^0, whereas $T(1)$ contains all price systems of interest. The number $\lambda(x) \in [0, 1]$ is defined to be such that $\lambda(x) = \lambda$ if x belongs to the boundary of $T(\lambda)$. The number $\lambda(x)$, therefore, measures the distance of x to $x(0)$. The function $\varphi : [0, 1] \times S \to S$ is constructed such that $x \in \bar{\gamma}(x(0))$ if and only if $(\lambda(x), x) \in F^{c}_{\varphi}$.

Our construction also suggests alternative functions φ that lead to orbits $\bar{\gamma}(x(0))$ corresponding to novel adjustment processes, for instance by specifying alternative subsets $T(\lambda)$. Section 8 illustrates how such a new adjustment process can be derived. Section 8 also discusses the reverse question. How to specify a system of differential equations that leads to $\bar{\gamma}(x(0))$ as an orbit? Section 8 shows that this can be achieved by applying the so-called Davidenko equations (Davidenko, 1953) to the system of equations characterizing the orbit $\bar{\gamma}(x(0))$.

3. The tracing procedure of Harsanyi and Selten

The tracing procedure is used repeatedly in the equilibrium selection theory of Harsanyi and Selten (1988) to find a unique solution of the so-called basic games. It is also used to define risk-dominance relationships between Nash equilibria. It models a process of convergent expectations that rational players can use to find a particular Nash equilibrium as the solution for a given game. Before applying the tracing procedure, players are assumed to have a common probability distribution expressing their expectations about the strategy choices of the other players. This common probability distribution is called a prior. In the linear tracing procedure the information on the best replies to the prior is gradually fed back into the expectations of the players. As the linear tracing procedure proceeds, both the prior and the best responses will gradually change until both converge to some Nash equilibrium of the game.

Consider a non-cooperative N-person normal form game $\Gamma = (\Phi_1, \ldots, \Phi_N, R_1, \ldots, R_N)$. Each player $i = 1, \ldots, N$, has M_i pure strategies. The k-th pure strategy of player i is denoted by (i, k). The set of pure strategies of player i is denoted by Φ_i. The total number of strategies is given by $M = \sum_{i=1}^{N} M_i$. The set of all pure strategy combinations is given by $\Phi = \prod_{i=1}^{N} \Phi_i$. The function $R_i : \Phi \to \mathbb{R}$ denotes the payoff function of a player i and it is extended in the standard way to the set of all mixed strategy combinations $S = \prod_{i=1}^{N} S^{M_i}$. Here we identify all probability distributions on Φ_i with $S^{M_i} = \{s_i \in \mathbb{R}_{+}^{M_i} \mid \sum_{j=1}^{M_i} s_{ij} = 1\}$. Given a mixed strategy combination $s \in S$ and a mixed strategy $\bar{s}_i \in S_i$, we denote by $s \setminus \bar{s}_i$ the mixed strategy combination that results from replacing s_i by $\bar{s}_i$. The set of Nash equilibria of Γ is denoted $\mathrm{NE}(\Gamma)$.

A probability distribution $s^0 \in S$, called the prior, is given for the remainder of this paper. The prior describes the initial beliefs of all players about the strategies played by the other players. The prior is assumed to be the same for all players, and the determination of the prior is part of the equilibrium selection theory of Harsanyi and Selten (1988). For every $\lambda \in [0, 1]$, the linear tracing procedure generates a Nash equilibrium of the game $\Gamma^{\lambda} = (\Phi_1, \ldots, \Phi_N, H_1^{\lambda}, \ldots, H_N^{\lambda})$, where the payoff function $H_i^{\lambda} : \Phi \to \mathbb{R}$ of player i is defined by

$$H_i^{\lambda}(\phi) = \lambda R_i(\phi) + (1 - \lambda) R_i(s^0 \setminus \phi_i).$$

It is straightforward to extend H_i^{λ} to the set of all mixed strategy combinations $S = \prod_{i=1}^{N} S^{M_i}$. The game Γ^0 corresponds to a trivial game, where all players believe that all their opponents play with probability 1 according to the prior beliefs. The game Γ^1 coincides with the game Γ. The linear tracing procedure links a Nash equilibrium of the game Γ^0 to a Nash equilibrium of Γ^1. Let $\mathcal{L}$ denote the set of all Nash equilibria related to the games Γ^{λ}, $\lambda \in [0, 1]$, so

$$\mathcal{L} = \{(\lambda, s) \in [0, 1] \times S | s \in \mathrm{NE}(\Gamma^{\lambda})\}.$$

The linear tracing procedure is said to be feasible if there exists a continuous function $\gamma : [0, 1] \to \mathcal{L}$, i.e. a path, such that $\gamma(0) \in \mathcal{L} \cap (\{0\} \times S)$ and $\gamma(1) \in \mathcal{L} \cap (\{1\} \times S)$. The linear tracing procedure is said to be well-defined if each path $\gamma : [0, 1] \to \mathcal{L}$ such that $\gamma(0) \in \mathcal{L} \cap (\{0\} \times S)$ and $\gamma(1) \in \mathcal{L} \cap (\{1\} \times S)$ has the same image. We consider feasibility as the more difficult property to establish. Indeed, as it has been argued in Section 2, it is possible to go from feasibility to well-definedness by invoking certain regularity properties of $\mathcal{L}$, for a rigorous proof in the context of the tracing procedure, see Herings and Peeters (2001).

Since $\mathcal{L}$ is a set that can be described by a finite union of sets described by a finite number of polynomial inequalities, it is a semi-algebraic set. All the components of $\mathcal{L}$, that is all maximally connected subsets of $\mathcal{L}$, are also path-connected. Therefore, any two points in a component of $\mathcal{L}$ can be joined by a path, see for instance Schanuel et al. (1991) for a nice introduction into the properties of semi-algebraic sets. To show that the linear tracing procedure is feasible, it is sufficient to show that $\mathcal{L}$ has a component that intersects both the sets $\{0\} \times S$ and $\{1\} \times S$.

The proof of feasibility of the linear tracing procedure presented here is not new. It coincides with one of the proofs proposed in Herings (2000). It is repeated here for illustrational purposes, as the connection between Browder's fixed point theorem and the tracing procedure is the closest of all the adjustment processes that we will consider.

Let the function $\sigma_i : [0, 1] \times S \to S^{M_i}$ be defined by

$$\sigma_i(\lambda, s) = \arg \max_{\bar{s}_i \in S^{M_i}} \lambda R_i(s \setminus \bar{s}_i) + (1 - \lambda) R_i(s^0 \setminus \bar{s}_i) - \|\bar{s}_i - s_i\|_2^2.$$

The function σ_i is well-defined and continuous because it is the argmax of a function that is strictly concave, because its first two terms are linear in $\bar{s}_i$, while the third is strictly concave.

We define the function $f : [0, 1] \times S \to S$ by

$$f(\lambda, s) = (\sigma_1(\lambda, s), \ldots, \sigma_N(\lambda, s)).$$

The fixed points of f are closely related to the strategies in the set $\mathcal{L}$.

Theorem 3.1. *For any non-cooperative N-person game Γ, for any prior s^0, it holds that $(\lambda, s) \in \mathcal{L}$ if and only if $f(\lambda, s) = s$.*

Proof. It is obvious that $(\lambda, s) \in \mathcal{L}$ implies $f(\lambda, s) = s$.

Suppose there is $(\bar{\lambda}, \bar{s}) \in [0, 1] \times S$ such that $f(\bar{\lambda}, \bar{s}) = \bar{s}$, but $(\bar{\lambda}, \bar{s}) \notin \mathcal{L}$. Then, for some $s_i \in S^{M_i}$, $H_i^{\bar{\lambda}}(\bar{s} \setminus s_i) - H_i^{\bar{\lambda}}(\bar{s}) = h > 0$. Since $H_i^{\bar{\lambda}}(\bar{s} \setminus s_i) = \sum_{(i,k) \in \Phi_i} s_{ik} H_i^{\bar{\lambda}}(\bar{s} \setminus (i, k))$, it holds that, for $0 < \varepsilon < 1$, $H_i^{\bar{\lambda}}(\bar{s} \setminus \varepsilon s_i + (1 - \varepsilon)\bar{s}_i) - H_i^{\bar{\lambda}}(\bar{s}) = \varepsilon h > 0$. Now, $\|(\varepsilon s_i + (1 - \varepsilon)\bar{s}_i) - \bar{s}_i\|^2 = \varepsilon^2 \|s_i - \bar{s}_i\|^2 < \varepsilon h$, for small enough ε, contradicting that $\bar{s}_i$ is the argument maximizing the expression in the definition of $\sigma_i(\bar{\lambda}, \bar{s})$. □

The argument given in the proof of Theorem 3.1 is the same as the one used in Geanakoplos (1996), where Brouwer's fixed point theorem, as opposed to Kakutani's fixed point theorem, is used to show the existence of a Nash equilibrium in a finite non-cooperative N-person game.

Theorem 3.2. *For any non-cooperative N-person game Γ, for any prior s^0, the tracing procedure is feasible.*

Proof. It is immediate that f satisfies the conditions of Browder's fixed point theorem and so there is a component F^c of $F = \{(\lambda, s) \in [0, 1] \times S | s = f(\lambda, s)\}$ such that

$(\{0\} \times S) \cap F^{\mathrm{c}} \neq \emptyset$ and $(\{1\} \times S) \cap F^{\mathrm{c}} \neq \emptyset$. By Theorem 3.1 it follows that $F = \mathcal{L}$, so F^{c} is a subset of $\mathcal{L}$ that connects a best response to the prior s^0 to a Nash equilibrium s^*. □

Theorem 3.2 demonstrates that feasibility of the tracing procedure is a corollary to Browder's fixed point theorem.

4. The quantal response equilibria of McKelvey and Palfrey

Quantal response equilibria as introduced in McKelvey and Palfrey (1995), are statistical versions of Nash equilibria, where each player's payoff is subject to random error. One possible interpretation is that players make errors according to some random process when calculating their expected payoffs. An alternative interpretation is that players calculate expected payoffs correctly, but have an additive payoff disturbance associated with each available pure strategy. For a given specification of the error structure, a quantal response equilibrium is a mixed strategy combination that is consistent with optimizing behavior subject to the error structure.

Consider a non-cooperative N-person normal form game $\Gamma = (\Phi_1, \dots, \Phi_N, R_1, \dots, R_N)$. Player i's payoff when playing pure strategy (i, k) against a mixed strategy combination s is subject to error and is given by

$$\hat{R}_i(s \setminus (i, k)) = R_i(s \setminus (i, k)) + \varepsilon_{ik}.$$

Player i's error vector $\varepsilon_i = (\varepsilon_{i1}, \dots, \varepsilon_{iM_i})$ is distributed according to a joint distribution with density function ϕ_i. Given the vector of payoffs that player i receives when playing his pure strategies and when errors are absent, $\bar{R}_i = (R_i(s \setminus (i, 1)), \dots, R_i(s \setminus (i, M_i)))$ for some $s \in S$, the ik-response set $E_{ik}(\bar{R}_i)$ is defined as the set of error vectors that make pure strategy (i, k) the best response, so

$$E_{ik}(\bar{R}_i) = \{\varepsilon_i \in \mathbb{R}^{M_i} | \bar{R}_{ik} + \varepsilon_{ik} \geq \bar{R}_{ij} + \varepsilon_{ij}, \quad j = 1, \dots, M_i\}.$$

The probability of choosing pure strategy (i, k) is then given by

$$\sigma_{ik}(\bar{R}_i) = \int_{\varepsilon_i \in E_{ik}(\bar{R}_i)} \phi_i(\varepsilon_i)\, \mathrm{d}\varepsilon_i.$$

A quantal response equilibrium is a mixed strategy combination $s^* \in S$ that is consistent with the error structure, thus

$$s^*_{ik} = \sigma_{ik}(R_i(s^* \setminus (i, 1)), \dots, R_i(s^* \setminus (i, M_i))), \quad i = 1, \dots, N, \quad k = 1, \dots, M_i.$$

The following specification of the error structure is quite common in the theory of individual choice behavior (see Luce, 1959) and leads to the logistic quantal response equilibria. For any parameter $\theta \geq 0$, the logistic quantal response function is defined by

$$\sigma_{ik}(\bar{R}_i) = \frac{\exp(\theta \bar{R}_{ik})}{\sum_{j=1}^{M_i} \exp(\theta \bar{R}_{ij})}, \quad \bar{R}_i \in \mathbb{R}^{M_i},$$

and is obtained when ϕ_i corresponds to the extreme value distribution. The parameter θ is inversely related to the error level. When $\theta = 0$, the choice of all players is completely determined by the errors, and corresponds to playing all pure strategies with equal probability. When θ approaches infinity, the influence of the errors disappears. This suggests a way of selecting Nash equilibria analogously to the tracing procedure. Start from the quantal response equilibrium at $\theta = 0$ and let the influence of errors go to zero. McKelvey and Palfrey (1995) show that for generic games, this approach selects a unique Nash equilibrium. We show that for all games the quantal response equilibrium at $\theta = 0$ is connected by a set of quantal response equilibria to at least one Nash equilibrium. As has been argued in Section 2, the stronger property that a unique Nash equilibrium is selected follows from differentiability assumptions and transversality arguments.

Given an error level corresponding to θ, the set of quantal response equilibria of Γ is denoted by $\mathrm{QRE}^\theta(\Gamma)$. Let $\mathcal{Q}$ denote the set of all quantal response equilibria for varying values of $\theta \in \mathbb{R}_+$, so

$$\mathcal{Q} = \{(\theta, s) \in \mathbb{R}_+ \times S | s \in \mathrm{QRE}^\theta(\Gamma)\},$$

or alternatively

$$\mathcal{Q} = \left\{(\theta, s) \in \mathbb{R}_+ \times S | s_{ik} = \frac{\exp(\theta R_i(s \setminus (i,k)))}{\sum_{j=1}^{M_i} \exp(\theta R_i(s \setminus (i,j)))}, \right.$$
$$\left. i = 1, \dots, N, \quad k = 1, \dots, M_i \right\}.$$

To investigate whether the quantal response equilibrium at $\theta = 0$ is connected to a Nash equilibrium, it is useful to make the transformation $\theta = \lambda/(1-\lambda)$ and to define

$$\tilde{\mathcal{Q}} = \left\{(\lambda, s) \in [0,1) \times S | s_{ik} = \frac{\exp((\lambda/(1-\lambda)) R_i(s \setminus (i,k)))}{\sum_{j=1}^{M_i} \exp((\lambda/(1-\lambda)) R_i(s \setminus (i,j)))}, \right.$$
$$\left. i = 1, \dots, N, \quad k = 1, \dots, M_i \right\}.$$

We define the function $f : [0,1) \times S \to S$ by

$$f_{ik}(\lambda, s) = \frac{\exp((\lambda/(1-\lambda)) R_i(s \setminus (i,k)))}{\sum_{j=1}^{M_i} \exp((\lambda/(1-\lambda)) R_i(s \setminus (i,j)))}, \quad i = 1, \dots, N, \quad k=1, \dots, M_i.$$

The fixed points of f are closely related to the strategies in the set $\tilde{\mathcal{Q}}$.

Theorem 4.1. *For any non-cooperative N-person game Γ, it holds that $(\lambda, s) \in \tilde{\mathcal{Q}}$ if and only if $f(\lambda, s) = s$.*

Proof. Obvious. □

The following result follows immediately from Browder's fixed point theorem, so a proof is omitted.

Theorem 4.2. *For any non-cooperative N-person game* Γ, *for any* $\bar{\lambda} \in (0, 1)$, *there is a component* $\tilde{\mathcal{Q}}^{\mathrm{c}}$ *of* $\tilde{\mathcal{Q}}$ *such that* $(\{0\} \times S) \cap \tilde{\mathcal{Q}}^{\mathrm{c}} \neq \emptyset$ *and* $(\{\bar{\lambda}\} \times S) \cap \tilde{\mathcal{Q}}^{\mathrm{c}} \neq \emptyset$.

The theorem makes clear that the unique quantal response equilibrium at $\theta = 0$ is connected by quantal response equilibria to a quantal response equilibrium for an arbitrarily high value of θ.

The next step is to extend Theorem 4.2 and to consider what happens in the limit. In particular, we want to show that the quantal response equilibrium at $\theta = 0$ is connected by quantal response equilibria to a Nash equilibrium. To this end, we define

$$\bar{\mathcal{Q}} = \tilde{\mathcal{Q}} \cup (\{1\} \times \mathrm{NE}(\Gamma))$$

and we show the following result.

Theorem 4.3. *For any non-cooperative N-person game* Γ, *there is a component* $\bar{\mathcal{Q}}^{\mathrm{c}}$ *of* $\bar{\mathcal{Q}}$ *such that* $(\{0\} \times S) \cap \bar{\mathcal{Q}}^{\mathrm{c}} \neq \emptyset$ *and* $(\{1\} \times S) \cap \bar{\mathcal{Q}}^{\mathrm{c}} \neq \emptyset$.

Proof. For $n \in \mathbb{N}$, denote the component $\tilde{\mathcal{Q}}^{\mathrm{c}}$ of $\tilde{\mathcal{Q}}$ such that $(\{0\} \times S) \cap \tilde{\mathcal{Q}}^{\mathrm{c}} \neq \emptyset$ and $(\{1 - (1/n)\} \times S) \cap \tilde{\mathcal{Q}}^{\mathrm{c}} \neq \emptyset$ by $\tilde{Q}^n$. Note that, for $n \in \mathbb{N}$, $\tilde{Q}^n \subset \tilde{Q}^{n+1}$. By Mas-Colell (1985) (Theorem A.5.1.(ii), page 10), the closed limit of the sequence $\tilde{Q}^n$, denoted $\bar{Q}^{\mathrm{c}}$, is connected. We show that $\bar{Q}^{\mathrm{c}} \subset \bar{Q}$.

Let $(\bar{\lambda}, \bar{s})$ be an element of $\bar{\mathcal{Q}}^{\mathrm{c}}$. Then there exists a sequence of points $\{(\lambda^n, s^n)\}_{n \in \mathbb{N}}$ such that $\lambda^n < 1$, $f(\lambda^n, s^n) = s^n$, and $(\lambda^n, s^n) \to (\bar{\lambda}, \bar{s})$. If $\bar{\lambda} < 1$, then the continuity of f implies $(\bar{\lambda}, \bar{s}) \in \tilde{\mathcal{Q}} \subset \bar{\mathcal{Q}}$. Suppose $\bar{\lambda} = 1$, and suppose $\bar{s}$ is not a Nash equilibrium. Then there is a player i, a pair of pure strategies (i, k) and (i, l), and $\varepsilon > 0$ such that $\bar{s}_{ik} > 0$, but $R_i(\bar{s} \setminus (i, k)) + \varepsilon < R_i(\bar{s} \setminus (i, l))$. Since $s^n \to \bar{s}$, there is $\bar{n}$ such that $R_i(s^n \setminus (i, k)) + \varepsilon < R_i(s^n \setminus (i, l))$ for all $n \geq \bar{n}$. However,

$$\begin{aligned} \lim_{n \to \infty} f_{ik}(\lambda^n, s^n) &= \lim_{n \to \infty} \frac{\exp(\lambda^n/(1 - \lambda^n) R_i(s^n \setminus (i, k)))}{\sum_{j=1}^{M_i} \exp(\lambda^n/(1 - \lambda^n) R_i(s^n \setminus (i, j)))} \\ &\leq \lim_{n \to \infty} \frac{\exp(\lambda^n/(1 - \lambda^n) R_i(s^n \setminus (i, k)))}{\exp(\lambda^n/(1 - \lambda^n) R_i(s^n \setminus (i, l)))} = 0. \end{aligned}$$

Therefore,

$$0 < \bar{s}_{ik} = \lim_{n \to \infty} s_{ik}^n = \lim_{n \to \infty} f_{ik}(\lambda^n, s^n) = 0,$$

a contradiction. We have shown that $\bar{\mathcal{Q}}^{\mathrm{c}} \subset \bar{\mathcal{Q}}$.

The property that $(\{0\} \times S) \cap \bar{\mathcal{Q}}^{\mathrm{c}} \neq \emptyset$ and $(\{1\} \times S) \cap \bar{\mathcal{Q}}^{\mathrm{c}} \neq \emptyset$ is immediate. □

As was the case for the tracing procedure, Browder's fixed point theorem provides an elegant way to show the connectedness of the quantal response equilibrium at $\theta = 0$ to a Nash equilibrium.

5. The price adjustment process of Kamiya

In Kamiya (1990), the prices of commodities are normalized by assuming that $\sum_{l=1}^{L}(p_l)^2 = 1$. An adjustment process is defined for a total excess demand function $z : \mathbb{R}^L_+ \setminus \{0\} \to \mathbb{R}^L$ and a starting price system $p^0 \in \mathbb{R}^L_{++}$ with $\sum_{l=1}^{L}(p_l^0)^2 = 1$. The following assumption is made throughout this section.

Assumption 1. The function $z : \mathbb{R}^L_+ \setminus \{0\} \to \mathbb{R}^L$ satisfies continuity, homogeneity, Walras' law and the following boundary behavior:

For $p \in \mathbb{R}^L_+ \setminus \{0\}$, for $l = 1, \dots, L$, $p_l = 0$ implies $z_l(p) > 0$.

Assumption 1 is a weak version of the assumptions in Kamiya (1990), where also twice continuous differentiability of z on $\mathbb{R}^L_{++}$ is assumed. Since prices are normalized such that $\sum_{l=1}^{L}(p_l)^2 = 1$, Walras' law implies that we may replace the excess demand function z by the excess demand function $\hat{z} : \dot{B}^{L-1}_+ \to \mathbb{R}^{L-1}$, where

$$\dot{B}^{L-1}_+ = \left\{ \hat{p} \in \mathbb{R}^{L-1}_+ \,\middle|\, \sum_{l=1}^{L-1} (\hat{p}_l)^2 < 1 \right\}$$

and $\hat{z}_l(\hat{p}) = z_l(\hat{p}_1, \dots, \hat{p}_{L-1}, \sqrt{1 - \sum_{l=1}^{L-1}(\hat{p}_l)^2})$, $l = 1, \dots, L-1$. The function $\hat{z}$ is obtained by omitting the last component of z and making use of the price normalization.

Kamiya's process is a weighted average of Smale's global Newton method, $\partial\hat{z}(\hat{p})(\mathrm{d}\hat{p}/\mathrm{d}t) = -\lambda(\hat{p})\hat{z}(\hat{p})$, and Walrasian tatonnement, $(\mathrm{d}\hat{p}/\mathrm{d}t) = \hat{z}(\hat{p})$. The weights depend on the norm of the excess demand and the distance between $\hat{p}$ and the initial price system $\hat{p}^0$, where $\hat{p}^0$ denotes the initial price system with component L left out. When formulated as a differential equation, Kamiya's process is given by

$$\left(\frac{\partial \hat{z}(\hat{p})}{\|\hat{z}(\hat{p})\|_2} - \frac{I}{\|\hat{p} - \hat{p}^0\|_2} \right) \frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = -\lambda(\hat{p})\hat{z}(\hat{p}),$$

where I is the $(L-1) \times (L-1)$ identity matrix and λ is an arbitrary scalar function of $\hat{p}$ such that

$$\mathrm{sign}(\lambda(\hat{p})) = \mathrm{sign}\det\left(\frac{I}{\|\hat{p} - \hat{p}^0\|_2} - \frac{\partial \hat{z}(\hat{p})}{\|\hat{z}(\hat{p})\|_2} \right).$$

Although $\mathrm{d}\hat{p}/\mathrm{d}t$ is not directly defined at $\hat{p} = \hat{p}^0$ or for a competitive equilibrium price system $\hat{p}$, it can be appropriately defined by taking a limit. The process corresponds to Walrasian tatonnement at $\hat{p}^0$, and it becomes Smale's global Newton method as it approaches an equilibrium.

As Kamiya (1990) shows, prices generated by the differential equation belongs to the set

$$P = \left\{ \hat{p} \in \mathbb{R}^{L-1}_+ \,\middle|\, \sum_{l=1}^{L-1} (\hat{p}_l)^2 < 1 \quad \exists \theta \in [0,1], \quad \text{for } l = 1, \dots, L-1,\ \theta\hat{z}_l(\hat{p}) = (1-\theta)(\hat{p}_l - p_l^0) \right\}.$$

It is easily verified that $\theta = 0$ yields $p = p^0$ as the unique solution, so $p^0 \in P$. By considering $\theta = 1$ it follows that if p^* is a Walrasian equilibrium price system with $\sum_{l=1}^{L}(p_l^*)^2 = 1$, then $(p_1^*, \dots, p_{L-1}^*) \in P$. From the definition of the set P it follows that the differential equation adjusts prices in such a way that the excess demand at a price system is proportional to the difference between this price system and the initial price system.

Kamiya (1990) shows that under suitable differentiability assumptions, for a generic economy, the component of P containing p^0 is a path that connects p^0 to a Walrasian equilibrium price system. Since we do not make any differentiability assumptions, nor do we restrict ourselves to generic economies, we want to show that the component of P containing p^0 connects p^0 to a Walrasian equilibrium price system. Kamiya's adjustment process is said to be convergent if this latter property holds.

First, we give a different characterization of the set P. It follows from the boundary behavior and the continuity of z that there exists $\bar{\varepsilon} > 0, \bar{\varepsilon} \leq p_L^0$, such that $z_L(p) > 0$ whenever $p_L \leq \bar{\varepsilon}$ and $\sum_{l=1}^{L}(p_l)^2 = 1$. We introduce the set

$$B_+^{L-1}(\bar{\varepsilon}) = \left\{ \hat{p} \in \mathbb{R}_+^{L-1} \,\middle|\, \sum_{l=1}^{L-1}(\hat{p}_l)^2 \leq 1 - \bar{\varepsilon}^2 \right\}.$$

For any non-empty, closed, convex subset X of $\mathbb{R}^m$, the continuous function $\pi_X : \mathbb{R}^m \to X$ is the orthogonal projection on X, so $\pi_X(y) = x$ if $x \in X$ and $\|y - x\|_2 \leq \|y - \bar{x}\|_2$, for all $\bar{x} \in X$, i.e. $\pi_X(y)$ is the closest point in X to y. We extend the excess demand function $\hat{z}$ to a function $\tilde{z}$ defined on $\mathbb{R}^{L-1}$ by setting

$$\tilde{z}(\hat{p}) = \hat{z}(\pi_{B_+^{L-1}(\bar{\varepsilon})}(\hat{p})), \quad \hat{p} \in \mathbb{R}^{L-1}.$$

We define the set $\tilde{P}$ by omitting the non-negativity constraints on prices in P and replacing $\hat{z}(\hat{p})$ by $\tilde{z}(\hat{p})$, so

$$\tilde{P} = \{\hat{p} \in \mathbb{R}^{L-1} | \exists \theta \in [0, 1], \quad \text{for } l = 1, \dots, L-1, \quad \theta \tilde{z}_l(\hat{p}) = (1-\theta)(\hat{p}_l - p_l^0)\}.$$

Lemma 5.1. *For any excess demand function z satisfying Assumption 1, for any $p^0 \in \mathbb{R}_{++}^L$ with $\sum_{l=1}^{L}(p_l^0)^2 = 1$, it holds that $P = \tilde{P}$.*

Proof. Consider some $\hat{p} \in P$. First, it is shown that $\hat{p} \in B_+^{L-1}(\bar{\varepsilon})$. Suppose not, then $0 < \sqrt{1 - \sum_{l=1}^{L-1}(\hat{p}_l)^2} < \bar{\varepsilon}$, so $z_L(\hat{p}_1, \dots, \hat{p}_{L-1}, \sqrt{1 - \sum_{l=1}^{L-1}(\hat{p}_l)^2}) > 0$. By Walras's law it follows that

$$0 > \sum_{l=1}^{L-1} \hat{p}_l \hat{z}_l(\hat{p}).$$

Since $\hat{p} \in P$ there is $\theta \in [0, 1]$ such that $\theta \hat{z}_l(\hat{p}) = (1-\theta)(\hat{p}_l - p_l^0)$, $l = 1, \dots, L-1$. If $\theta = 0$, then $\hat{p} = \hat{p}^0$, which implies $\sqrt{1 - \sum_{l=1}^{L-1}(\hat{p}_l)^2} = p_L^0 \geq \bar{\varepsilon}$, contradicting our supposition. If $\theta = 1$, then $\hat{z}(\hat{p}) = 0$, which contradicts $0 > \sum_{l=1}^{L-1} \hat{p}_l \hat{z}_l(\hat{p})$. It follows that $\theta \in (0, 1)$. But then, using the definition of the set P,

$$\sum_{l=1}^{L-1} \hat{p}_l \hat{z}_l(\hat{p}) = \frac{1-\theta}{\theta} \sum_{l=1}^{L-1} \hat{p}_l(\hat{p}_l - p_l^0) > 0,$$

where the inequality comes from $\hat{p}^0 \neq \hat{p}$, $\sum_{l=1}^{L-1} p_l^0 \leq \sum_{l=1}^{L-1} \hat{p}_l$, and $\theta < 1$. This contradicts $0 > \sum_{l=1}^{L-1} \hat{p}_l \hat{z}_l(\hat{p})$. Consequently, $\hat{p} \in B_+^{L-1}(\bar{\varepsilon})$, from which it is obtained that $\tilde{z}(\hat{p}) = \hat{z}(\hat{p})$ and $\hat{p} \in \tilde{P}$.

Now consider some $\hat{p} \in \tilde{P}$. Suppose $\hat{p} \notin B_+^{L-1}(\bar{\varepsilon})$. Denote the projection $\pi_{B_+^{L-1}(\bar{\varepsilon})}(\hat{p})$ by $\hat{\pi}$. Then there is l' such that $\hat{\pi}_{l'} = 0$ or $\sum_{l=1}^{L-1} \hat{\pi}_l = 1 - \bar{\varepsilon}$. In the former case, it holds that $\hat{p}_{l'} \leq 0$ and $\tilde{z}_{l'}(\hat{p}) > 0$, and for some $\theta \in (0, 1)$,[1]

$$0 < \theta \tilde{z}_{l'}(\hat{p}) = (1-\theta)(\hat{p}_{l'} - p_{l'}^0) < 0,$$

a contradiction. In the latter case it holds that $0 > \sum_{l=1}^{L-1} \hat{\pi}_l \tilde{z}_l(\hat{p})$ and, for some $\theta \in (0, 1)$, $\theta \tilde{z}_l(\hat{p}) = (1-\theta)(\hat{p}_l - p_l^0)$, $l = 1, \ldots, L-1$. Because of the contradiction obtained to $\hat{\pi}_{l'} = 0$ in the former case, we may assume without loss of generality that $\hat{\pi} \gg 0$, so $\hat{p}$ is projected onto the strictly positive part of $B_+^{L-1}(\bar{\varepsilon})$, so $\hat{p} = \lambda \hat{\pi}$ for some $\lambda > 1$. Therefore,

$$0 > \sum_{l=1}^{L-1} \hat{\pi}_l \tilde{z}_l(\hat{p}) = \lambda \sum_{l=1}^{L-1} \hat{p}_l \tilde{z}_l(\hat{p}) = \lambda \frac{1-\theta}{\theta} \sum_{l=1}^{L-1} \hat{p}_l(\hat{p}_l - p_l^0) > 0,$$

a contradiction. Consequently, $\hat{p} \in B_+^{L-1}(\bar{\varepsilon})$, from which it is obtained that $\tilde{z}(\hat{p}) = \hat{z}(\hat{p})$ and $\hat{p} \in P$. □

The lemma makes clear that we may either study the set P or the set $\tilde{P}$ in order to study the adjustment process.

For $\lambda \geq 0$, we define the set

$$T^{L-1}(\lambda) = \{\hat{p} \in \mathbb{R}^{L-1} | \|\hat{p} - \hat{p}^0\|_2 \leq \lambda\}.$$

In Fig. 1 the set $T^{L-1}(\lambda)$ is shown for various values of λ for the case $L = 3$. The set $T^{L-1}(0)$ contains only the point p^0. The set $T^{L-1}(\lambda)$ expands when λ increases, and $T^{L-1}(1)$ contains the set $\dot{B}_+^{L-1}$. For $\hat{p} \in \mathbb{R}^{L-1}$, we define $\lambda(\hat{p})$ as the distance to $\hat{p}^0$, $\lambda(\hat{p}) = \|\hat{p} - \hat{p}^0\|_2$. It is immediate that $\hat{p} \in T^{L-1}(\lambda)$ for all $\lambda \geq \lambda(\hat{p})$.

We define the function $f : [0, 1] \times T^{L-1}(1) \to T^{L-1}(1)$ by

$$f(\lambda, p) = \pi_{T^{L-1}(\lambda)}(p + \tilde{z}(p)).$$

The fixed points of f coincide with the prices in the set P.

Theorem 5.2. *For any excess demand function z satisfying Assumption 1, for any p^0 with $\sum_{l=1}^{L}(p_l^0)^2 = 1$, it holds that $\hat{p} \in P$ if and only if there is $\lambda \in [0, 1]$, such that $f(\lambda, \hat{p}) = \hat{p}$.*

[1] The argument that $\theta \in (0, 1)$ is similar to the one in the first part of this proof.

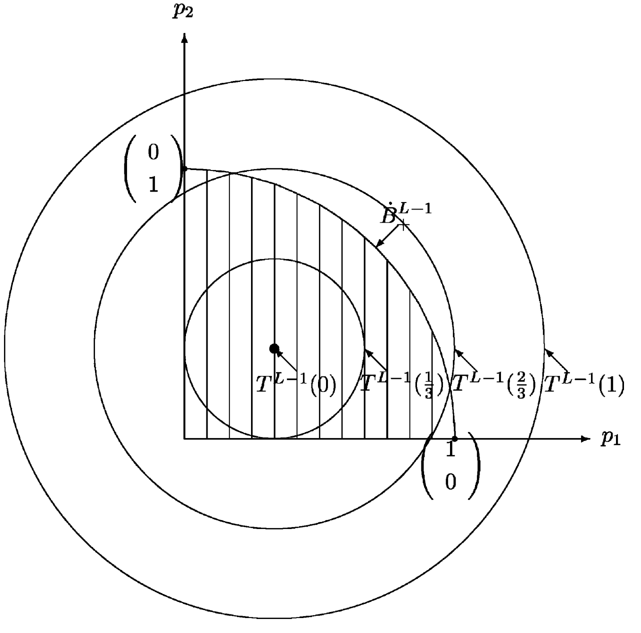


Fig. 1. The sets $\dot{B}_+^{L-1}$, $T^{L-1}(0)$, $T^{L-1}(1/3)$, $T^{L-1}(2/3)$ and $T^{L-1}(1)$, for $\hat{p}^0 = (1/3, 1/3)^\top$.

Moreover, either $\hat{z}(\hat{p}) \neq 0$ and $\lambda = \lambda(\hat{p})$ *or* $z(\hat{p}) = 0$ *and* $f(\lambda, \hat{p}) = \hat{p}$ *for all* $\lambda \geq \lambda(\hat{p})$.

Proof. Consider a fixed point $\hat{p}$ of $f(\lambda, \cdot)$, so $\hat{p} = f(\lambda, \hat{p}) = \pi_{T^{L-1}(\lambda)}(\hat{p} + \tilde{z}(\hat{p}))$. We show that $\hat{p} \in \tilde{P}$, from which it follows that $\hat{p} \in P$ by Lemma 5.1.

Since $f(0, \hat{p}) = \hat{p}^0$, it is obvious that $f(0, \hat{p}) = \hat{p}$ implies $\hat{p} = \hat{p}^0$, so $\hat{p} \in \tilde{P}$.

Consider the case $\lambda > 0$. The projection of an arbitrary vector x on the set $T^{L-1}(\lambda)$ is determined by the following optimization problem.

$$\min_{y \in \mathbb{R}^{L-1}} \sum_{l=1}^{L-1} \frac{1}{2}(y_l - x_l)^2 \quad \text{s.t.} \quad \lambda^2 - \sum_{l=1}^{L-1}(y_l - p_l^0)^2 \geq 0.$$

The necessary and sufficient Kuhn–Tucker conditions for an optimum are given by

$$y_l - x_l + 2\mu(y_l - p_l^0) = 0, \quad l = 1, \dots, L-1,$$

$$\mu\left(\lambda^2 - \sum_{l=1}^{L-1}(y_l - p_l^0)^2\right) = 0,$$

$$\lambda^2 - \sum_{l=1}^{L-1}(y_l - p_l^0)^2 \geq 0,$$

$$\mu \geq 0,$$

where y equals the projection $\pi_{T^{L-1}(\lambda)}(x)$ and μ denotes the shadow price of the constraint $\lambda^2 - \sum_{l=1}^{L-1}(y_l - p_l^0)^2 \geq 0$.

It follows that there exists $\mu \geq 0$ such that

$$\tilde{z}_l(\hat{p}) = 2\mu(\hat{p}_l - p_l^0), \quad l = 1, \ldots, L-1.$$

Since $\mu \geq 0$, it follows that $\hat{p} \in \tilde{P}$.

Consider some $\hat{p} \in \tilde{P}$. If $\hat{z}(\hat{p}) = 0$, then it is trivially the case that $f(\lambda, \hat{p}) = \hat{p}$ for all $\lambda \geq \lambda(\hat{p})$. Suppose $\hat{z}(\hat{p}) \neq 0$. If $\lambda(\hat{p}) = 0$, then $\hat{p} = \hat{p}^0$ and trivially $f(0, \hat{p}^0) = \hat{p}^0$. Suppose $\hat{z}(\hat{p}) \neq 0$ and $\lambda(\hat{p}) > 0$. We need to show that $f(\lambda(\hat{p}), \hat{p}) = \hat{p}$, which is equivalent to the statement that the projection of $\hat{p} + \tilde{z}(\hat{p})$ on $T^{L-1}(\lambda(\hat{p}))$ equals $\hat{p}$.

Since $\lambda(\hat{p}) > 0$ there exists $\theta \in (0, 1)$ such that $\theta \tilde{z}_l(p) = (1-\theta)(p_l - p_l^0)$, $l = 1, \ldots, L-1$. Substitute in the Kuhn–Tucker conditions, $y_l = \hat{p}_l$, $x_l = \hat{p}_l + \tilde{z}_l(\hat{p})$, $\mu = \theta/(2(1-\theta))$ and observe that all equalities and inequalities in the Kuhn–Tucker conditions are satisfied. □

When p^* is a competitive equilibrium, then $(p_1^*, \ldots, p_{L-1}^*)$ is a fixed point of f for any value of λ exceeding $\lambda(p_1^*, \ldots, p_{L-1}^*)$. When $\hat{p}$ is a price system generated by the adjustment process, but does not correspond to a competitive equilibrium, then $\hat{p}$ is a fixed point of $f(\lambda(\hat{p}), \cdot)$.

At $\hat{p}^0$ the value of $\lambda(\cdot)$ is zero. Along the path of the adjustment process, the value of $\lambda(\cdot)$ increases initially, but it may decrease later on. Eventually, it will increase until it reaches the value 1, and a competitive equilibrium has been found.

Theorem 5.3. *The price adjustment process converges for any excess demand function satisfying Assumption 1, for any* $p^0 \in \mathbb{R}_{++}^L$ *with* $\sum_{l=1}^{L}(p_l^0)^2 = 1$.

Proof. It is immediate that f satisfies the conditions of Browder's fixed point theorem and so there is a component F^c of $F = \{(\lambda, \hat{p}) \in [0,1] \times T^{L-1}(1) | \hat{p} = f(\lambda, \hat{p})\}$ such that $(\{0\} \times T^{L-1}(1)) \cap F^c \neq \emptyset$ and $(\{1\} \times T^{L-1}(1)) \cap F^c \neq \emptyset$. Let the projection function $g : [0,1] \times T^{L-1}(1) \to T^{L-1}(1)$ be defined by $g(\lambda, \hat{p}) = \hat{p}$. By Theorem5.2 it follows that $g(F) = P$. Since g is continuous, $g(F^c)$ is a connected subset of P that connects the starting price system $\hat{p}^0$ to some competitive equilibrium price system $\hat{p}^*$. □

Convergence of the price adjustment process is a corollary to Browder's fixed point theorem.

6. The price adjustment process of van der Laan and Talman

van der Laan and Talman (1987) introduce a price adjustment process for an exchange economy. The prices of the commodities are normalized by $\sum_{l=1}^{L} p_l = 1$. Given a total excess demand function $z : \mathbb{R}_{++}^L \to \mathbb{R}^L$ and a starting price system $p^0 \in \mathbb{R}_{++}^L$ with $\sum_{l=1}^{L} p_l^0 = 1$, the adjustment process generates price systems in the set

$$P = \left\{ p \in \mathbb{R}^L_{++} \left| \sum_{l=1}^{L} p_l = 1, \quad \text{for } l' = 1, \dots, L, \quad z_{l'}(p) < 0 \right. \right.$$
$$\Rightarrow \frac{p_{l'}}{p^0_{l'}} = \min_{l=1,\dots,L} \frac{p_l}{p^0_l}, \quad \text{for } l' = 1, \dots, L, \quad z_{l'}(p) > 0$$
$$\left. \Rightarrow \frac{p_{l'}}{p^0_{l'}} = \max_{l=1,\dots,L} \frac{p_l}{p^0_l} \right\}.$$

Two types of restrictions are made on prices in the set P. The first is an innocuous price normalization, $\sum_{l=1}^{L} p_l = 1$. The second concerns the requirement that the relative price of a commodity, i.e. the ratio of the price of a commodity and its initial price, be minimal when the commodity is in positive excess supply, and maximal when the commodity is in positive excess demand. This is closely related to the ideas behind Walrasian tatonnement, where prices of commodities in positive excess supply are decreased and those of commodities in positive excess demand are increased. It is obvious that the starting price system p^0 belongs to P. It can also be verified that whenever p^* is a Walrasian equilibrium price system with $\sum_{l=1}^{L} p^*_l = 1$, then $p^* \in P$.

In Herings (1997), it is shown that under suitable differentiability assumptions, for a generic economy, the component of P containing p^0 is a path that connects p^0 to a Walrasian equilibrium price system. Since in this section we do not make any differentiability assumptions, nor do we restrict ourselves to generic economies, we want to show that the component of P containing p^0 connects p^0 to a Walrasian equilibrium price system. The adjustment process is said to be convergent if this latter property holds.

To simplify the exposition, we renormalize the units of measurement of quantities of commodities to make sure that $p^0 = (1/L, \dots, 1/L)$. It follows that the set P is given by

$$P = \left\{ p \in \mathbb{R}^L_{++} \left| \sum_{l=1}^{L} p_l = 1, \quad \text{for } l' = 1, \dots, L, \quad z_{l'}(p) < 0 \right. \right.$$
$$\Rightarrow p_{l'} = \min_{l=1,\dots,L} p_l, \quad \text{for } l' = 1, \dots, L, \quad z_{l'}(p) > 0$$
$$\left. \Rightarrow p_{l'} = \max_{l=1,\dots,L} p_l \right\}.$$

We may also take the value of excess demand $v(p)$, defined by a function $v : \mathbb{R}^L_{++} \to \mathbb{R}^L$, where

$$v_l(p) = p_l z_l(p), \quad l = 1, \dots, L,$$

instead of the excess demand $z(p)$, to define the set P. Since $v(p)$ is positive (negative) if and only if $z(p)$ is positive (negative), it follows that replacing $z(p)$ by $v(p)$ leaves the set P unchanged.

We assume that $z : \mathbb{R}^L_{++} \to \mathbb{R}^L$ is an excess demand function, so it satisfies Assumption 2.

Assumption 2. The function $z : \mathbb{R}^L_{++} \to \mathbb{R}^L$ satisfies continuity, homogeneity, Walras' law and the following boundary behavior:

If $(p^n)_{n\in\mathbb{N}}$ is a sequence converging to $\bar{p} \in \mathbb{R}^L_+ \setminus \{0\}$, then $\lim_{n\to\infty}\|z(p^n)\|_\infty = +\infty$.

Contrary to the boundary behavior in Assumption 1, Assumption 2 follows from standard assumptions on primitives, that is from standard assumptions on consumption sets, utility functions, and initial endowments.

The continuity and the boundary behavior of z imply that we can choose $\bar{\varepsilon} > 0$ such that for any $\bar{p} \notin S^L(\bar{\varepsilon}) = \{p \in S^L | p_l \geq \bar{\varepsilon}, \quad l = 1, \dots, L\}$, it holds that $z_l(\bar{p}) > 0$ for some l with $0 < \bar{p}_l \leq \bar{\varepsilon}$.

We modify the value function v near the boundary of the unit simplex S^L and extend it to a function $\tilde{v}$ defined on $T^L = \{p \in \mathbb{R}^L | \sum_{l=1}^L p_l = 1\}$ by setting

$$\tilde{v}(p) = v(\pi_{S^L(\bar{\varepsilon})}(p)), \quad p \in T^L.$$

We define the set $\tilde{P}$ by omitting non-negativity constraints and replacing $z(p)$ by $\tilde{v}(p)$, so

$$\tilde{P} = \{p \in T^L | \quad \text{for } l' = 1, \dots, L, \quad \tilde{v}_{l'}(p) < 0 \Rightarrow p_{l'} = \min_{l=1,\dots,L} p_l, \\ \text{for } l' = 1, \dots, L, \quad \tilde{v}_{l'}(p) > 0 \Rightarrow p_{l'} = \max_{l=1,\dots,L} p_l\}.$$

Lemma 6.1. *For any excess demand function z satisfying Assumption 2, it holds that* $P = \tilde{P}$.

Proof. Consider some $\bar{p} \in P$. It is immediate that $\bar{p} \in S^L(\bar{\varepsilon})$ and $\tilde{v}(\bar{p}) = v(\bar{p})$. Therefore, $\tilde{v}_l(p) > 0$ if and only if $z_l(p) > 0$ and $\tilde{v}_l(p) < 0$ if and only if $z_l(p) < 0$. So, $\bar{p} \in \tilde{P}$.

Consider some $\bar{p} \in \tilde{P}$. Suppose $\bar{p} \notin S^L(\bar{\varepsilon})$. There is l' such that $\pi_{S^L(\bar{\varepsilon})_{l'}}(\bar{p}) = \bar{\varepsilon}$ and $z_{l'}(\pi_{S^L(\bar{\varepsilon})}(\bar{p})) > 0$. But then $\tilde{v}_{l'}(\bar{p}) > 0$ and $\bar{p}_{l'} < \max_{l=1,\dots,L} \bar{p}_l$, a contradiction to $\bar{p} \in \tilde{P}$. Consequently, $\bar{p} \in S^L(\bar{\varepsilon})$. Therefore, $\tilde{v}_l(p) > 0$ if and only if $z_l(p) > 0$ and $\tilde{v}_l(p) < 0$ if and only if $z_l(p) < 0$. So, $\bar{p} \in P$. □

For $\lambda \geq 0$, we define the set

$$T^L(\lambda) = \{p \in T^L | p_k - p_l \leq \lambda, \quad k, l = 1, \dots, L, \quad k \neq l\}.$$

In Fig. 2 the set $T^L(\lambda)$ is shown for various values of λ. The set $T^L(0)$ contains only the point $(1/3, 1/3, 1/3)$. The set $T^L(\lambda)$ expands when λ increases. The set $T^L(1)$ contains the set S^L.

For $p \in \mathbb{R}^L$, we define $\lambda(p) = \max_{k\neq l} p_k - p_l$. It is immediate that $p \in T^L(\lambda)$ for all $\lambda \geq \lambda(p)$. We define the function $f : [0, 1] \times T^L(1) \to T^L(1)$ by

$$f(\lambda, p) = \pi_{T^L(\lambda)}(p + \tilde{v}(p)).$$

The fixed points of f coincide with the prices in the set P.

Theorem 6.2. *For any excess demand function z satisfying Assumption 2, it holds that $p \in P$ if and only if there is $\lambda \in [0, 1]$ such that $f(\lambda, p) = p$. Moreover, either $z(p) \neq 0$ and $\lambda = \lambda(p)$, or $z(p) = 0$ and $f(\lambda, p) = p$ for all $\lambda \geq \lambda(p)$.*

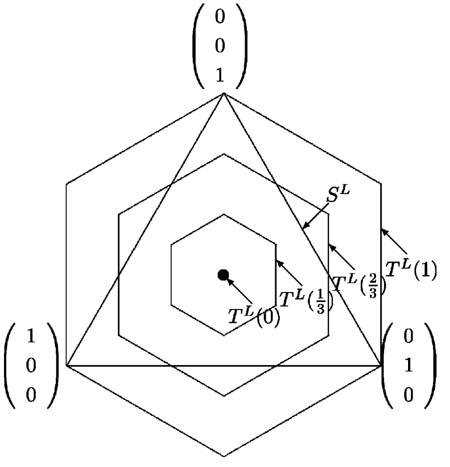


Fig. 2. The sets S^L, $T^L(0)$, $T^L(1/3)$, $T^L(2/3)$ and $T^L(1)$.

Proof. Consider a fixed point $\bar{p}$ of $f(\bar{\lambda}, \cdot)$, so $\bar{p} = f(\bar{\lambda}, \bar{p}) = \pi_{T^L(\bar{\lambda})}(\bar{p} + \tilde{v}(\bar{p}))$. We show that $\bar{p} \in \tilde{P}$, from which it follows that $\bar{p} \in P$ by Lemma 6.1.

The projection of an arbitrary vector x on the set $T^L(\lambda)$ is determined by the following optimization problem

$$\min_{y \in \mathbb{R}^L} \sum_{l=1}^{L} \frac{1}{2}(y_l - x_l)^2 \quad \text{s.t.} \quad \sum_{l=1}^{L} y_l - 1 = 0, \quad y_k - y_l - \lambda \geq 0, \quad k \neq l.$$

The necessary and sufficient Kuhn–Tucker conditions for an optimum are given by

$$y_l = x_l + \mu - \sum_{k \neq l} \mu_{l,k} + \sum_{k \neq l} \mu_{k,l}, \quad l = 1, \dots, L,$$

$$\sum_{l=1}^{L} y_l - 1 = 0,$$

$$\mu_{k,l}(y_k - y_l - \lambda) = 0, \quad k \neq l,$$

$$y_k - y_l - \lambda \geq 0, \quad k \neq l,$$

$$\mu_{k,l} \geq 0, \quad k \neq l,$$

where y equals the projection $\pi_{T^L(\lambda)}(x)$, μ denotes the shadow price of the constraint $\sum_{l=1}^{L} y_l - 1 = 0$, and $\mu_{k,l}$, $k \neq l$, denotes the shadow price of the constraint $y_k - y_l - \lambda \geq 0$.

Since $\bar{p} = \pi_{T^L(\bar{\lambda})}(\bar{p} + \tilde{v}(\bar{p}))$, it follows that there exists $\mu \in \mathbb{R}$ and $\mu_{k,l} \geq 0$, $k \neq l$, such that

$$\tilde{v}_l(\bar{p}) = -\mu + \sum_{k \neq l} \mu_{l,k} - \sum_{k \neq l} \mu_{k,l}, \quad l = 1, \dots, L.$$

Moreover,

$$1 = \sum_{l=1}^{L} \bar{p}_l = \sum_{l=1}^{L} \left(\bar{p}_l + \tilde{v}_l(\bar{p}) + \mu - \sum_{k \neq l} \mu_{l,k} + \sum_{k \neq l} \mu_{k,l} \right)$$
$$= 1 + \sum_{l=1}^{L} \tilde{v}_l(\bar{p}) + L\mu + \sum_{l=1}^{L} \left(- \sum_{k \neq l} \mu_{l,k} + \sum_{k \neq l} \mu_{k,l} \right) = 1 + L\mu,$$

so $\mu = 0$ and

$$\tilde{v}_l(\bar{p}) = \sum_{k \neq l} \mu_{l,k} - \sum_{k \neq l} \mu_{k,l}, \quad l = 1, \dots, L.$$

It also holds that

$$\mu_{k,l}(\bar{p}_k - \bar{p}_l - \bar{\lambda}) = 0, \quad k \neq l.$$

Suppose $\tilde{v}_{l'}(\bar{p}) < 0$ for some l'. Then $\mu_{k,l'} > 0$ for some k, so $\bar{p}_{l'} = \bar{p}_k - \bar{\lambda}$. Since for all l, $\bar{p}_l \geq \bar{p}_k - \bar{\lambda}$, it holds that $\bar{p}_{l'} = \min_{l=1,\dots,L} \bar{p}_l$. Similarly it can be shown that $\tilde{v}_{l'}(\bar{p}) > 0$ implies $\bar{p}_{l'} = \max_{l=1,\dots,L} \bar{p}_l$. Consequently, it holds that $\bar{p} \in \tilde{P}$.

Consider some $\bar{p} \in \tilde{P}$. If $z(\bar{p}) = 0$, then it is trivially the case that $f(\lambda, \bar{p}) = \bar{p}$ for all $\lambda \geq \lambda(\bar{p})$. Suppose $z(\bar{p}) \neq 0$. We need to show that $f(\lambda(\bar{p}), \bar{p}) = \bar{p}$, which implies that the projection of $\bar{p} + \tilde{v}(\bar{p})$ on $T^L(\bar{\lambda})$ equals $\bar{p}$. This is achieved by substituting in the Kuhn–Tucker conditions $y_l = \bar{p}_l$, $x_l = \bar{p}_l + \tilde{v}_l(\bar{p})$, $\mu = 0$, $\lambda = \lambda(\bar{p})$, $\mu_{k,l} = \tilde{v}_k(\bar{p})\tilde{v}_l(\bar{p})/v$ if $\tilde{v}_k(\bar{p}) > 0$ and $\tilde{v}_l(\bar{p}) < 0$, and $\mu_{k,l} = 0$, otherwise, where $v = \sum_{\{l | \tilde{v}_l(\bar{p}) < 0\}} \tilde{v}_l(\bar{p})$. Observe that all equalities and inequalities in the Kuhn–Tucker conditions are satisfied. □

When p^* is a competitive equilibrium, then p^* is a fixed point of f for any value of λ exceeding $\lambda(p^*)$. When p is a price system generated by the adjustment process, but not a competitive equilibrium, then p is a fixed point of $f(\lambda(p), \cdot)$.

At p^0 the value of $\lambda(\cdot)$ is zero. Along the path of the adjustment process, the value of $\lambda(\cdot)$ increases initially, but it may decrease later on. Eventually, it will increase until it reaches the value 1, and a competitive equilibrium has been found.

Theorem 6.3. *The price adjustment process converges for any excess demand function satisfying* Assumption 2.

Proof. It is immediate that f satisfies the conditions of Browder's fixed point theorem and so there is a component F^c of $F = \{(\lambda, p) \in [0, 1] \times T^L(1) | p = f(\lambda, p)\}$ such that $(\{0\} \times T^L(1)) \cap F^c \neq \emptyset$ and $(\{1\} \times T^L(1)) \cap F^c \neq \emptyset$. Let the projection function $g : [0, 1] \times T^L(1) \to T^L(1)$ be defined by $g(\lambda, p) = p$. By Theorem 6.2 it follows that $g(F) = P$. Since g is continuous, $g(F^c)$ is a connected subset of P that connects the starting price system p^0 to some competitive equilibrium price system p^*. □

Once again, the convergence of a price adjustment process is intimately connected to Browder's fixed point theorem.

7. The global Newton method of Smale

The following assumption on the excess demand function z is made throughout this section.

Assumption 3. The function $z : \mathbb{R}^L_+ \setminus \{0\} \to \mathbb{R}^L$ satisfies continuity, homogeneity, Walras' law and the following boundary behavior:

For every $p \in \mathbb{R}^L_+ \setminus (\mathbb{R}^L_{++} \cup \{0\})$, $z(p) - \bar{z}(p)\mathbb{1}$ is not radially outward pointing, i.e. there is no $\mu > 0$ such that $z(p) - \bar{z}(p)\mathbb{1} = \mu(p - (1/L)\mathbb{1})$, where $\bar{z}(p) = \sum_{l=1}^{L} z_l(p)/L$ is the mean excess demand at p, and $\mathbb{1}$ is a vector of ones of appropriate dimension.

Assumption 3 is a weak version of the assumption in Smale (1976), where twice continuous differentiability of z is assumed and a rather complicated and strong boundary condition is stated. An illustration of a radially outward pointing vector $z(p) - \bar{z}(p)\mathbb{1}$ for various values of p can be found in Fig. 3. It holds that $z(p) - \bar{z}(p)\mathbb{1}$ is radially outward pointing if it lies on the ray starting at 0 and passing through $p - (1/L)\mathbb{1}$. We normalize prices, such as to belong to the unit simplex S^L.

The assumed boundary behavior is weaker than the requirement that $z_l(p) > 0$ for some $l \in L$ for which $p_l = 0$, a requirement that is natural for a function defined on $\mathbb{R}^L_+ \setminus \{0\}$. Indeed, if l is such that $p_l = 0$ and $z_l(p) > 0$, then Walras' law implies that there is l' such that $p_{l'} > 0$ and $z_l(p) > z_{l'}(p)$. So, $z_l(p) - \bar{z}(p) > z_{l'}(p) - \bar{z}(p)$, whereas $-1/L = p_l - 1/L < p_{l'} - 1/L$, which implies that $z(p) - \bar{z}(p)$ is not radially outward pointing.

There are several versions of Smale's global Newton method. Here we combine the approaches suggested in Smale (1976) on page 117, and Varian (1977)[2] to apply Smale's method to the function $\tilde{z} : D^L \to \mathbb{R}^L$ defined by

$$\tilde{z}(p) = \tilde{\pi}(p) - p + (1 - \|p - (1/L)\mathbb{1}\|_2)\,(z(\tilde{\pi}(p)) - \bar{z}(\tilde{\pi}(p))\mathbb{1})\,, \qquad p \in D^L,$$

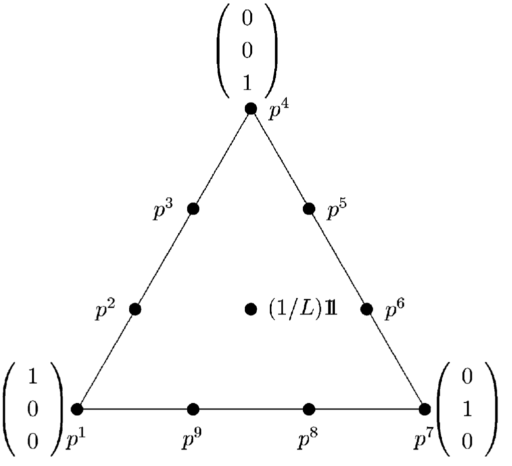
where $D^L = \{p \in \mathbb{R}^L \,|\, \sum_{l=1}^{L}(p_l - 1/L)^2 \leq 1 \text{ and } \sum_{l=1}^{L} p_l = 1\}$, a disk containing S^L in its interior, and $\tilde{\pi}$ denotes the radial projection on S^L. For p with $\sum_{l=1}^{L} p_l = 1$ not in S^L, the radial projection of p on S^L is given by the price system where the line between p and $(1/L)\mathbb{1}$ hits the boundary of S^L. If p with $\sum_{l=1}^{L} p_l = 1$ does not belong to S^L, then, for $l' \in \arg\min_{l=1,\dots,L} p_l$,

$$\tilde{\pi}(p) = \frac{1/L}{1/L - p_{l'}}\,p + \frac{-p_{l'}}{1/L - p_{l'}}(1/L)\mathbb{1}.$$

If $p \in S^L$, then $\tilde{\pi}(p) = p$.

The function $\tilde{z}$ is simply an extension of the function $z(\cdot) - \bar{z}(\cdot)\mathbb{1}$ multiplied by a positive number to a disk containing the unit simplex in its interior. The vector $\tilde{z}(p)$ is the sum of the terms $\tilde{\pi}(p) - p$ and $z(\tilde{\pi}(p)) - \bar{z}(\tilde{\pi}(p))\mathbb{1}$, where the latter term is multiplied by the non-negative number $(1 - \|p - (1/L)\mathbb{1}\|_2)$. It is a positive multiple of $z(p) - \bar{z}(p)\mathbb{1}$ on S^L,

[2] The construction of Varian (1977) applies to a more abstract problem on the unit disk, but our variation of his construction to the set S^L is rather straightforward.



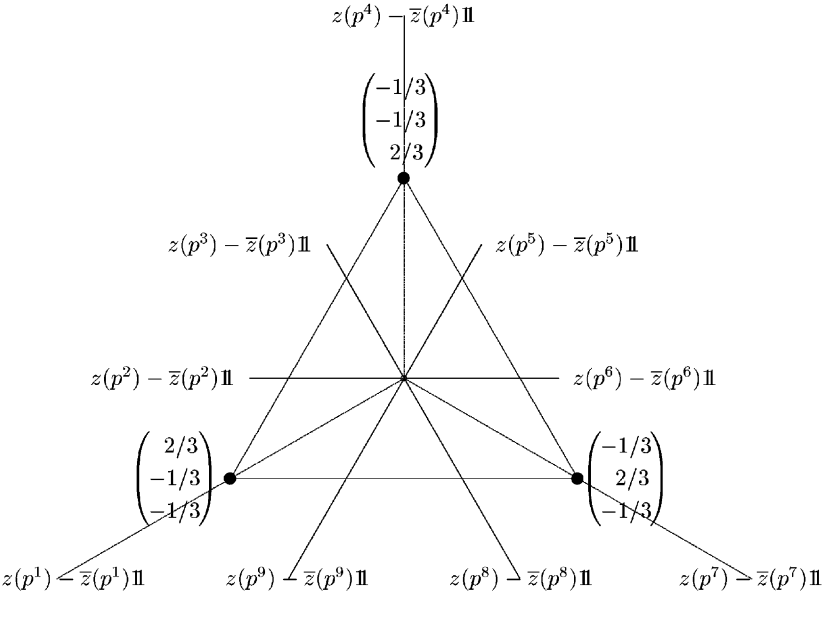


Fig. 3. A vector $z(p) - \bar{z}(\bar{p})\mathbb{1}$ that is radially outward pointing for $p = p^1, p^2, \ldots, p^9$.

since $\tilde{\pi}(p) = p$ for $p \in S^L$. The contribution of the term $z(\tilde{\pi}(p)) - \bar{z}(\tilde{\pi}(p))\mathbb{1}$ vanishes on the relative boundary of D^L, where it holds that $\|p - (1/L)\mathbb{1}\|_2 = 1$. This makes the function $\tilde{z}$ radially inward pointing on the relative boundary of D^L.

The zero points of z and $\tilde{z}$ coincide. Indeed, there are no equilibria of $\tilde{z}$ on the relative boundary of D^L as the term $1 - \|p - (1/L)\mathbb{1}\|_2$ vanishes there and the remaining term is

$\tilde{\pi}(p) - p$. On S^L, the function $\tilde{z}$ is a positive multiple of $z(\cdot) - \bar{z}(\cdot)\mathbb{1}$. Obviously the zero points of $z(\cdot) - \bar{z}(\cdot)\mathbb{1}$ and z coincide. Consider a point p not on the relative boundary of D^L and outside S^L. Then $\tilde{z}(p) = 0$ if and only if

$$\tilde{\pi}(p) - p + (1 - \|p - (1/L)\mathbb{1}\|_2)(z(\tilde{\pi}(p)) - \bar{z}(\tilde{\pi}(p))\mathbb{1}) = 0.$$

Then it holds that

$$\begin{aligned} z(\tilde{\pi}(p)) - \bar{z}(\tilde{\pi}(p))\mathbb{1} &= \frac{1}{1 - \|p - (1/L)\mathbb{1}\|_2}(p - \tilde{\pi}(p)) \\ &= \frac{-Lp_{l'}}{1 - \|p - (1/L)\mathbb{1}\|_2}(\tilde{\pi}(p) - (1/L)\mathbb{1}), \end{aligned}$$

where $l' \in \arg\min_{l=1,\dots,L} p_l$. This implies that z is radially outward pointing at $\tilde{\pi}(p)$, a contradiction to Assumption 3.4.

Let $\hat{z}$ be the function $\tilde{z}$ with the last component omitted. The differential equation of Smale's global Newton method is given by

$$\partial\hat{z}(p)\frac{\mathrm{d}p}{\mathrm{d}t} = -\lambda(p)\hat{z}(p),$$

$$\mathbb{1}^\top\frac{\mathrm{d}p}{\mathrm{d}t} = 0,$$

where λ is an arbitrary scalar function of p such that

$$\mathrm{sign}(\lambda(p)) = \mathrm{sign}\det\begin{pmatrix} -\partial\hat{z}(p) \\ -\mathbb{1}^\top \end{pmatrix}.$$

Since the sum of the components of $\tilde{z}(p)$ equals zero, it holds that $\mathbb{1}^\top\partial\tilde{z}(p) = 0$. Then $\partial\hat{z}(p)(\mathrm{d}p/\mathrm{d}t) = -\lambda(p)\hat{z}(p)$ implies $\partial\tilde{z}_L(p)(\mathrm{d}p/\mathrm{d}t) = -\lambda(p)\tilde{z}_L(p)$, so the adjustment of the price of commodity L is similar to the adjustment of the prices of the other commodities. Since $\mathbb{1}^\top(\mathrm{d}p/\mathrm{d}t) = 0$, the sum of the prices is kept equal to one.

The starting price system p^0 has to be chosen in the relative boundary of D^L to guarantee convergence to a competitive equilibrium price system. In Keenan (1981) it has been shown that Smale's process may not converge for starting price systems in the relative interior of D^L.

As Smale (1976) shows, his process generates price systems in the set

$$P = \{p \in D^L | \exists\theta \geq 0, \quad \tilde{z}(p) = \theta\tilde{z}(p^0)\}.$$

It is easily verified, by taking $\theta = 1$, that $p^0 \in P$, and, by taking $\theta = 0$, that $p^* \in P$ if $p^* \in S^L$ is an equilibrium price system. By the arguments given before there are no solutions for $\theta = 0$ with $p^* \in D^L \setminus S^L$. From the definition of the set P it follows that the differential equation adjusts prices in such a way that the excess demand remains proportional to the excess demand at the starting price system.

Under suitable differentiability assumptions, for a generic economy, Smale (1976) shows that the component of P containing p^0 is a path that connects p^0 to a Walrasian equilibrium

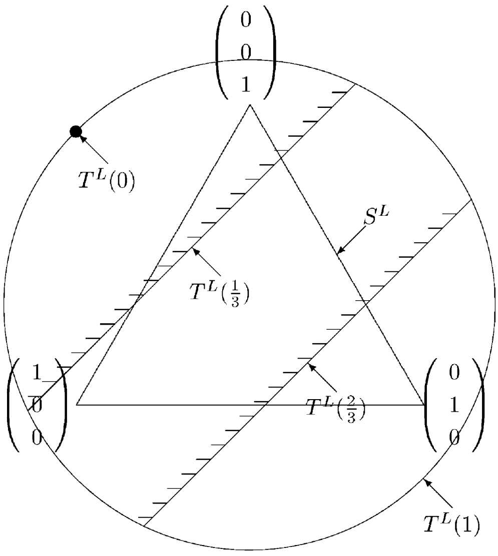


Fig. 4. The sets S^L, $T^L(0)$, $T^L(1/3)$, $T^L(2/3)$ and $T^L(1)$, for $p^0 = (0.545, -0.455, 0.91)^\top$.

price system. We show that even without such differentiability assumptions, and without restricting attention to generic economies, the component of P containing p^0 connects p^0 to a Walrasian equilibrium price system. Smale's global Newton method is said to be convergent if this latter property holds.

For $\lambda \geq 0$, we define the set

$$T^L(\lambda) = \{p \in D^L | p \cdot \tilde{z}(p^0) \leq (1 - 2\lambda)p^0 \cdot \tilde{z}(p^0)\}.$$

In Fig. 4 the set $T^L(\lambda)$ is shown for various values of λ. The set $T^L(0)$ contains only the point p^0. The set $T^L(\lambda)$ expands when λ increases. The set $T^L(1)$ equals D^L. For $p \in D^L$, we define $\lambda(p) = (p - p^0) \cdot \tilde{z}(p^0)/ - 2p^0 \cdot \tilde{z}(p^0)$. It holds that $p \in T^L(\lambda)$ if and only if $\lambda \geq \lambda(p)$.

We define the function $f : [0, 1] \times T^L(1) \to T^L(1)$ by

$$f(\lambda, p) = \pi_{T^L(\lambda)}(p + \tilde{z}(p)).$$

Theorem 7.1. *For any excess demand function z satisfying Assumption 3, for any p^0 in the relative boundary of D^L, it holds that $p \in P$ if and only if there is $\lambda \in [0, 1]$ such that $f(\lambda, p) = p$. Moreover, either $\tilde{z}(p) \neq 0$ and $\lambda = \lambda(p)$ or $\tilde{z}(p) = 0$ and $f(\lambda, p) = p$ for all $\lambda \geq \lambda(p)$.*

Proof. Consider a fixed point $\bar{p}$ of $f(\bar{\lambda}, \cdot)$, so $\bar{p} = f(\bar{\lambda}, \bar{p}) = \pi_{T^L(\bar{\lambda})}(\bar{p} + \tilde{z}(\bar{p}))$. We show that $\bar{p} \in P$.

Since $f(0, \bar{p}) = p^0$, it is obvious that $f(0, \bar{p}) = \bar{p}$ implies $\bar{p} = p^0$, so $\bar{p} \in P$. Next consider the case $\bar{\lambda} > 0$. For $\bar{p}$ in the relative boundary of D^L, $\tilde{z}(\bar{p})$ is radially inward pointing, so obviously $\pi_{T^L(\bar{\lambda})}(\bar{p} + \tilde{z}(\bar{p})) \neq \bar{p}$. Consider $\bar{p}$ in the relative interior of D^L.

Then $\bar{p} = \pi_{T^L(\bar{\lambda})}(\bar{p} + \tilde{z}(\bar{p}))$ if and only if the projection of $\bar{p} + \tilde{z}(\bar{p})$ on $\tilde{T}^L(\lambda) = \{p \in \mathbb{R}^L | p \cdot \tilde{z}(p^0) \leq (1 - 2\bar{\lambda}) p^0 \cdot \tilde{z}(p^0)\}$ equals $\bar{p}$.

The projection of an arbitrary vector x on the set $\tilde{T}^L(\lambda)$ is determined by the following optimization problem.

$$\min_{y \in \mathbb{R}^L} \sum_{l=1}^{L} \frac{1}{2}(y_l - x_l)^2 \quad \text{s.t.} \quad (1 - 2\lambda) p^0 \cdot \tilde{z}(p^0) - y \cdot \tilde{z}(p^0) \geq 0.$$

The necessary and sufficient Kuhn–Tucker conditions for an optimum are given by

$$\begin{aligned}
&y - x + \mu \tilde{z}(p^0) = 0, \\
&\mu((1 - 2\lambda) p^0 \cdot \tilde{z}(p^0) - y \cdot \tilde{z}(p^0)) = 0, \\
&(1 - 2\lambda) p^0 \cdot \tilde{z}(p^0) - y \cdot \tilde{z}(p^0) \geq 0, \\
&\mu \geq 0,
\end{aligned}$$

where y equals the projection $\pi_{\tilde{T}^L(\lambda)}(x)$ and μ denotes the shadow price of the constraint $(1 - 2\lambda) p^0 \cdot \tilde{z}(p^0) - y \cdot \tilde{z}(p^0) \geq 0$.

Since $\bar{p} = \pi_{T^L(\bar{\lambda})}(\bar{p} + \tilde{z}(\bar{p}))$, it follows that there exists $\mu \geq 0$ such that

$$\bar{p} - \bar{p} - \tilde{z}(\bar{p}) + \mu \tilde{z}(p^0) = 0,$$

so $\tilde{z}(\bar{p}) = \mu \tilde{z}(p^0)$, and $\bar{p} \in P$. This completes the first part of the proof.

Consider some $\bar{p} \in P$. If $\tilde{z}(\bar{p}) = 0$, then it is trivially the case that $f(\lambda, \bar{p}) = \bar{p}$ whenever $\bar{p} \in T^L(\lambda)$, i.e. when $\lambda \geq \lambda(\bar{p})$. Suppose $\tilde{z}(\bar{p}) \neq 0$. It is obvious that $f(\lambda, \bar{p}) \neq \bar{p}$ when $\lambda \neq \lambda(\bar{p})$. It remains to be shown that $f(\lambda(\bar{p}), \bar{p}) = \bar{p}$. If $\lambda(\bar{p}) = 0$, then $\bar{p} = p^0$ and trivially $f(0, p^0) = p^0$. Suppose $\tilde{z}(\bar{p}) \neq 0$ and $\lambda(\bar{p}) > 0$. There exists $\theta > 0$ such that $\tilde{z}(\bar{p}) = \theta \tilde{z}(p^0)$. From the necessary and sufficient Kuhn–Tucker conditions it follows that $\pi_{\tilde{T}^L(\lambda(\bar{p}))}(\bar{p} + \tilde{z}(\bar{p})) = \bar{p}$. Since $\bar{p} \in T^L(\lambda(\bar{p})) \subset \tilde{T}^L(\lambda(\bar{p}))$, it holds as well that $\pi_{T^L(\lambda(\bar{p}))}(\bar{p} + \tilde{z}(\bar{p})) = \bar{p}$. □

Theorem 7.1 establishes that the fixed points of f coincide with the prices in the set P. When p^* is a competitive equilibrium price system, then p^* is a fixed point of f for any value of λ exceeding $\lambda(p^*)$. When p is a price system generated by the adjustment process, but does not correspond to a competitive equilibrium, then p is a fixed point of $f(\lambda(p), \cdot)$, where $\lambda(p) < 1$.

At p^0 the value of $\lambda(\cdot)$ is zero. Along the path of the adjustment process, the value of $\lambda(\cdot)$ increases initially, but it may decrease later on. Eventually, it will increase until it reaches the value 1, and a competitive equilibrium has been found.

Theorem 7.2. *The price adjustment process converges for any excess demand function satisfying* Assumption 3, *for any p^0 in the relative boundary of D^L.*

Proof. It is immediate that f satisfies the conditions of Browder's fixed point theorem and so there is a component F^c of $F = \{(\lambda, p) \in [0, 1] \times T^L(1) | p = f(\lambda, p)\}$ such that $(\{0\} \times T^L(1)) \cap F^c \neq \emptyset$ and $(\{1\} \times T^L(1)) \cap F^c \neq \emptyset$. Let the projection function $g : [0,1] \times T^L(1) \to T^L(1)$ be defined by $g(\lambda, p) = p$. By Theorem 5.2 it follows that $g(F) = P$. Since g is continuous, $g(F^c)$ is a connected subset of P that connects the starting price system p^0 to some competitive equilibrium price system p^*. □

The proof of Theorem 7.2 show that convergence of the price adjustment process is a corollary to Browder's fixed point theorem.

8. From orbits to differential equations

In the previous five sections, the orbit of a number of adjustment processes is specified as being the set of fixed points of a continuous function $f : [0,1] \times S \to S$. In the strategy adjustment processes of Sections 3 and 4, the specification of f is straightforward. In the price adjustment processes in Sections 5–7, $f(\lambda, p) = \pi_{T^L(\lambda)}(p + \tilde{z}(p))$, with $\tilde{z}$ corresponding to some normalization of the excess demand function and $T^L(\lambda)$ a set that expands in λ and being such that $T^L(1) = S$. By choosing different normalizations for $\tilde{z}$ and different expanding sets $T^L(\lambda)$, it is possible to generate new price adjustment processes, as is illustrated at the end of this section.

It has been argued in Section 2 that, under suitable differentiability and transversality conditions, the orbits generated by the adjustment processes are well-behaved sets. If so, the reverse of the question treated so far in the exposition arises, i.e. whether it is possible to find a system of differential equations that generates a given orbit.

Consider first the case where the differentiable orbit $(\lambda(t), s(t))$ corresponding to the fixed points of f can be parameterized by arc length t. Notice that $\lambda(0) = 0$ and that $s(0)$ is the starting point of the adjustment process.

Let the function g be defined by $g(\lambda, s) = f(\lambda, s) - s$. Suppose that zero is a regular value of both g and of the restriction of g to $\{0,1\} \times S$. Let $J(\lambda, s)$ denote the Jacobian of g evaluated at (λ, s). The matrix $J(\lambda, s)$ is $L \times (L+1)$, with L the dimension of S, and has rank L because of the regularity assumptions made with respect to g.

These regularity assumptions also imply that there is a unique vector $T(J(0, s(0)))$ in the kernel of $J(0, s(0))$ satisfying $\|T(J(0, s(0)))\|_2 = 1$ and with the first component of $T(J(0, s(0)))$ positive. Let $T(J(\lambda, s))$ denote the unique vector in the kernel of $J(\lambda, s)$ satisfying $\|T(J(\lambda, s))\|_2 = 1$ and being such that

$$\det \begin{pmatrix} J(\lambda, s) \\ T(J(\lambda, s))^\top \end{pmatrix} = \det \begin{pmatrix} J(0, s(0)) \\ T(J(0, s(0)))^\top \end{pmatrix}.$$

It can be shown that the orbit of zero points induced by g is generated by the autonomous system of differential equations

$$(\dot{\lambda}, \dot{s}) = T(J(\lambda, s)),$$

where $(\dot{\lambda}, \dot{s})$ denotes differentiation with respect to arc length, see for instance Allgower and Georg (1983). The system of autonomous differential equations was first proposed by Davidenko (1953), and is also referred to as the Davidenko equations.

Using the Davidenko equations immediately leads to Smale's process. Indeed, the orbit of Smale's process is characterized by the system of equations

$$\tilde{z}(p) = (1-\lambda)\tilde{z}(p^0),$$

$$\sum_{l=1}^{L} p_l = 1.$$

It follows that

$$J(\lambda, p) = \begin{bmatrix} \tilde{z}(p^0) & \partial\tilde{z}(p) \\ 0 & \mathbb{1}^\top \end{bmatrix}.$$

The Davidenko equations specify that

$$(\dot{\lambda}, \dot{p}) = T(J(\lambda, p)),$$

from which it follows that

$$\partial\tilde{z}(p)\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\mathrm{d}\lambda}{\mathrm{d}t}\tilde{z}(p^0),$$

$$\mathbb{1}^\top\frac{\mathrm{d}p}{\mathrm{d}t} = 0.$$

Since $\tilde{z}(p) = (1-\lambda)\tilde{z}(p^0)$, the specification of Smale's process as in Section 7 follows.

By varying the set $T^L(\lambda)$ it is possible to obtain new adjustment processes. Consider for instance the case where the endogenous variable p belongs to a cube $[0, 1]^L$ and the excess demand function $\tilde{z} : [0, 1]^L \to \mathbb{R}^L$ satisfies the boundary condition $\tilde{z}_l(p) \geq 0$ if $p_l = 0$ and $\tilde{z}_l(p) \leq 0$ if $p_l = 1$.

The cube $[0, 1]^L$ could represent prices belonging to the set of extended real vectors of dimension L, and $\tilde{z}$ the excess demands of the first L commodities out of $L+1$. The price of commodity $L+1$ is normalized to be equal to some constant. By Walras' law it follows that the market for commodity $L+1$ clears when the excess demands for the first L commodities are zero.

Consider the case where $T^L(\lambda)$ is an expanding cube,

$$T^L(\lambda) = \{p \in C^L | (1-\lambda)p_l^0 \leq p_l \leq p_l^0 + \lambda(1-p_l^0), \quad l = 1, \ldots, L\}.$$

It follows that $p \in T^L(\lambda)$ if and only if $\lambda \geq \lambda(p)$, where

$$\lambda(p) = \max_{l=1,\ldots,L}\left\{\frac{p_l^0 - p_l}{p_l^0}, \frac{p_l - p_l^0}{1-p_l^0}\right\}.$$

The function $f : [0, 1] \times T^L(1) \to T^L(1)$ is defined by

$$f(\lambda, p) = \pi_{T^L(\lambda)}(p + \tilde{z}(p)).$$

To analyze the properties of f, consider first the projection of a vector $x \in \mathbb{R}^L$ on $T^L(\lambda)$. This projection is determined by the following minimization problem.

$$\min_{y \in \mathbb{R}^L} \sum_{l=1}^{L} \frac{1}{2}(y_l - x_l)^2 \quad \text{s.t.} \quad p_l^0 - \lambda p_l^0 \leq y_l \leq p_l^0 + \lambda(1 - p_l^0), \quad l = 1, \ldots, L.$$

The necessary and sufficient Kuhn–Tucker conditions, with μ_l^- and μ_l^+ the Lagrange multipliers corresponding to the two inequality constraints related to commodity l, lead to:

$$\begin{aligned}
&y_l - x_l - \mu_l^- + \mu_l^+ = 0,\\
&\mu_l^-(y_l - p_l^0 + \lambda p_l^0) = 0,\\
&\mu_l^+(-y_l + p_l^0 + \lambda(1 - p_l^0)) = 0,\\
&y_l - p_l^0 + \lambda p_l^0 \geq 0,\\
&-y_l + p_l^0 + \lambda(1 - p_l^0) \geq 0,\\
&\mu_l^- \geq 0,\\
&\mu_l^+ \geq 0.
\end{aligned}$$

It is easily verified that $x_l > p_l^0 + \lambda(1 - p_l^0)$ implies $\mu_l^+ > 0$, so $y_l = p_l^0 + \lambda(1 - p_l^0)$. Also, $x_l < p_l^0 - \lambda p_l^0$ implies $\mu_l^- > 0$, so $y_l = p_l^0 - \lambda p_l^0$. Otherwise, $y_l = x_l$ and $p_l^0 + \lambda(1 - p_l^0) \geq x_l \geq p_l^0 - \lambda p_l^0$.

Fixed points of f have nice properties. Consider $(\bar{\lambda}, \bar{p})$ such that $\bar{p} = \pi_{T^L(\bar{\lambda})}(\bar{p} + \tilde{z}(\bar{p}))$. If $\bar{p}_l + \tilde{z}(\bar{p}) > p_l^0 + \bar{\lambda}(1 - p_l^0)$, then $\bar{p}_l = p_l^0 + \bar{\lambda}(1 - p_l^0)$. If $\bar{p}_l + \tilde{z}(\bar{p}) < p_l^0 - \bar{\lambda} p_l^0$, then $\bar{p}_l = p_l^0 - \bar{\lambda} p_l^0$. Otherwise it holds that $\bar{p}_l = \bar{p}_l + \tilde{z}_l(\bar{p})$, so $\tilde{z}_l(\bar{p}) = 0$. It follows that the adjustment process generates price systems in the set

$$\begin{aligned}
P = \{&p \in [0, 1]^L | \exists \bar{\lambda} \in [0, 1], \quad \text{for } l = 1, \ldots, L,\\
&\tilde{z}_l(p) > 0 \Rightarrow \bar{p}_l = p_l^0 + \bar{\lambda}(1 - p_l^0), \quad \text{for } l = 1, \ldots, L,\\
&\tilde{z}_l(p) < 0 \Rightarrow \bar{p}_l = p_l^0 - \bar{\lambda} p_l^0, \quad \text{for } l = 1, \ldots, L,\\
&\tilde{z}_l(p) = 0 \Rightarrow p_l^0 - \bar{\lambda} p_l^0 \leq \bar{p}_l \leq p_l^0 + \bar{\lambda}(1 - p_l^0)\}.
\end{aligned}$$

It can be shown as before that the price adjustment process is convergent. Notice that the price adjustment process has a very nice intuitive interpretation. Prices of commodities l in excess demand are increased at a rate $(1 - p_l^0)$ with respect to the initial price p_l^0 and prices of commodities l in excess supply are decreased at a rate p_l^0 with respect to the initial price. Prices of commodities whose markets are in equilibrium are adjusted such that markets remain in equilibrium, as long as it is possible to do so for prices p_l satisfying $p_l^0 - \bar{\lambda} p_l^0 \leq \bar{p}_l \leq p_l^0 + \bar{\lambda}(1 - p_l^0)$.

Is it possible to formulate a system of differential equations that generates the orbit of the adjustment process above. Compared to the situation in the beginning of the section, an additional difficulty is that orbits can only be expected to be piecewise differentiable.

This problem can be solved by applying a so-called α-transformation as proposed in Garcia and Zangwill (1981). If we make the following substitutions into the system of first-order conditions that characterizes the fixed points of f,

$$\mu_l^- = [\max\{0, \alpha_l\}]^2,$$

$$p_l - p_l^0 + \lambda p_l^0 = [\min\{0, \alpha_l\}]^2,$$

$$\mu_l^+ = [\max\{0, \beta_l\}]^2,$$

$$-p_l + p_l^0 + \lambda(1 - p_l^0) = [\min\{0, \beta_l\}]^2,$$

and rearrange terms, we find that the fixed points of f are characterized by the solutions to

$$\tilde{z}_l([\min\{0, \alpha\}]^2 + p^0 - \lambda p^0) + [\max\{0, \alpha_l\}]^2 - [\max\{0, \beta_l\}]^2 = 0,$$

$$[\min\{0, \alpha_l\}]^2 + [\min\{0, \beta_l\}]^2 = \lambda.$$

Since the system above is differentiable in α, β and λ, it is possible to apply Davidenko's equations, and get a system of differential equations that generates the orbit of the adjustment process.

9. Conclusion

We have studied the convergence of a number of distinct adjustment processes in game theory and in general equilibrium theory. Convergence of the processes has been shown before in the literature by rather ad hoc arguments, and only for generic games and generic economies, under suitable differentiability assumptions. We have argued that the driving force behind convergence is to be found in Browder's fixed point theorem, which applies under very general conditions and does not involve any assumptions on differentiability. It is remarkable that not only existence of equilibrium, but also universal stability, is fundamentally based on fixed point theory. The use of Browder's result provides a uniform and simple way to show convergence of all the adjustment processes considered. It also enables us to design a sheer unlimited number of new adjustment processes, that are universally convergent.

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References

Allgower, E.L., Georg, K., 1983. Predictor–corrector and simplicial methods for approximating fixed points and zero points of nonlinear mappings. In: Bachem, A., Grötchel, M., Korte, B. (Eds.), Mathematical Programming; The State of the Art; Bonn 1982. Springer, Berlin, pp. 15–56.

Brouwer, L.E.J., 1912. Über abbildung von mannigfaltigkeiten. Mathematische Annalen 71, 97–115.

Browder, F.E., 1960. On continuity of fixed points under deformations of continuous mappings. Summa Brasiliensis Mathematicae 4, 183–191.

Davidenko, D., 1953. On a new method of numerical solution of systems of nonlinear equations. Doklady Akad. Nauk USSR 88, 601–602 (in Russian).

Debreu, G., 1974. Excess demand functions. Journal of Mathematical Economics 1, 15–21.

Eaves, B.C., Schmedders, K., 1999. General equilibrium models and homotopy methods. Journal of Economic Dynamics and Control 23, 1249–1279.

Garcia, C.B., Zangwill, W.I., 1981. Pathways to Solutions, Fixed Points, and Equilibria, Prentice-Hall Series in Computational Mathematics. Prentice-Hall, Englewood Cliffs.

Geanakoplos, J.D., 1996. Nash and Walras Equilibrium Via Brouwer, Cowles Foundation Discussion Paper No. 1131. New Haven, CT, pp. 1–18.

Harsanyi, J.C., 1975. The tracing procedure: A Bayesian approach to defining a solution for n-person noncooperative games. International Journal of Game Theory 4, 61–94.

Harsanyi, J.C., Selten, R., 1988. A General Theory of Equilibrium Selection in Games. MIT Press, Cambridge.

Herings, P.J.J., 1996. Static and Dynamic Aspects of General Disequilibrium Theory, Theory and Decision Library, Series C: Game Theory, Mathematical Programming and Operations Research. Kluwer Academic Publishers, Norwell, MA.

Herings, P.J.J., 1997. A globally and universally stable price adjustment process. Journal of Mathemats 27, 163–193.

Herings, P.J.J., 2000. Two simple proofs of the feasibility of the linear tracing procedure. Economic Theory 15, 485–490.

Herings, P.J.J., Peeters, R., 2001. A differentiable homotopy to compute Nash equilibria of n-person games. Economic Theory 18, 159–186.

Herings, P.J.J., van der Laan, G., Talman, A.J.J., Venniker, R., 1997. Equilibrium adjustment of disequilibrium prices. Journal of Mathematical Economics 27, 53–77.

Herings, P.J.J., van der Laan, G., Venniker, R.J.G., 1998. The transition from a Drèze equilibrium to a Walrasian equilibrium. Journal of Mathematical Economics 29, 303–330.

Herings, P.J.J., van der Laan, G., Talman, A.J.J., 1999. Price–quantity adjustment in a Keynesian economy. In: Herings, P.J.J., van der Laan, G., Talman, A.J.J. (Eds.), The Theory of Markets. North-Holland, Amsterdam, pp. 27–57.

Hirsch, M.W., Smale, S., 1974. Differential Equations, Dynamical Systems, and Linear Algebra. Academic Press, New York.

Joosten, R.A.M.G., Talman, A.J.J., 1997. A globally convergent price adjustment process for exchange economies. Journal of Mathematical Economics 29, 15–26.

Judd, K.L., 1997. Computational economics and economic theory: subsitutes or complements? Journal of Economic Dynamics and Control 21, 907–942.

Kamiya, K., 1990. A globally stable price adjustment process. Econometrica 58, 1481–1485.

Keenan, D.C., 1981. Further remarks on the global Newton method. Journal of Mathematical Economics 8, 159–165.

Luce, R.D., 1959. Individual Choice Behaviour: a Theoretical Analysis. Wiley, New York.

Mantel, R.R., 1974. On the characterization of aggregate excess demand. Journal of Economic Theory 7, 348–353.

McKelvey, R.D., Palfrey, T.R., 1995. Quantal response equilibria for normal form games. Games and Economic Behavior 10, 6–38.

Saari, D.G., 1985. Iterative price mechanisms. Econometrica 53, 1117–1131.

Saari, D.G., Simon, C.P., 1978. Effective price mechanisms. Econometrica 46, 1097–1125.

Scarf, H., 1960. Some examples of global instability of the competitive equilibrium. International Economic Review 1, 157–172.

Schanuel, S.H., Simon, L.K., Zame, W.R., 1991. The algebraic geometry of games and the tracing procedure. In: Selten, R. (Ed.), Game Equilibrium Models. II. Methods, Morals and Markets. Springer, Berlin, pp. 9–43.

Smale, S., 1976. A convergent process of price adjustment and global Newton methods. Journal of Mathematical Economics 3, 107–120.

Sonnenschein, H., 1972. Market excess demand functions. Econometrica 40, 549–563.

Sonnenschein, H., 1973. Do Walras' identity and continuity characterize the class of community excess demand functions? Journal of Economic Theory 6, 345–354.

Yamamoto, Y., 1993. A path-following procedure to find a proper equilibrium of finite games. International Journal of Game Theory 22, 249–259.

van den Elzen, A.H., 1993. Adjustment processes for exchange economies and noncooperative games, Lecture Notes in Economics and Mathematical Systems, vol. 402. Springer, Berlin.

van den Elzen, A.H., 1997. An adjustment process for the standard arrow/Debreu model with production. Journal of Mathematical Economics 27, 315–324.

van den Elzen, A.H., van der Laan, G., Talman, A.J.J., 1994. An adjustment process for an economy with linear production technologies. Mathematics of Operations Research 19, 341–351.

van der Laan, G., Talman, A.J.J., 1987. A convergent price adjustment process. Economics Letters 23, 119–123.

Varian, H.R., 1977. A remark on boundary restrictions in the global Newton method. Journal of Mathematical Economics 4, 127–130.