The Dynamics of Stochastic Choice

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Abstract

The random utility model is the standard empirical framework for modelling stochastic choice behaviour in applied settings. Though the distribution of stochastic choice has important implications for both testing behavioural theories and predicting behaviour, the theoretical and empirical foundations of this distribution are not well understood. Moreover, the random utility framework has so far been agnostic about the dynamics of the decision process that are of considerable interest in psychology and neuroscience, captured by a class of bounded accumulation models which relate decision times to stochastic behaviour. This article demonstrates that a random utility model can be derived from a general class of bounded accumulation models, in which particular features of this dynamic process restrict the form of the relationship between observables and the distribution of stochastic choice.

1 Introduction

Stochastic choice behaviour is an established empirical phenomenon and the Random Utility Model (RUM; Becker, DeGroot, and Marschak, 1963) has become the standard framework for modelling it in applied settings (McFadden, 2001). The RUM has proven successful because it provides a highly-flexible empirical framework for relating observables to an individual’s choice behaviour (Train, 2009). However, it has been well-documented that the form of the random utility distribution has important implications for testing behavioural theories, often with starkly different predictions for the underlying theory (Loomes and Sugden, 1995). For instance, violations of the independence axiom found

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1ARUM is an implementation of stochastic revealed preference which seeks to identify whether distributions of observed choices within a population are consistent with utility maximization (McFadden, 2005). Here, I return to the original interpretation of a “population” as consisting of multiple choices by the same individual in the same choice situation (Becker, De-Groot, and Marschak, 1963; McFadden, 1981), as opposed to a single observation from multiple individuals (McFadden, 1974).
in sample moments of experimental data can instead be reconciled by the Expected Utility (EU) model with an additive stochastic specification (Wilcox, 2008). Relating the variance of the additive error to observables yields improved prediction in choice under risk (Hey, 1995) and cautions against rejecting EU in favour of more general theories (Buschena and Zilberman, 2000); however a theory for how the error distribution (particularly its variance and skew) depends on observables remains absent (Loomes, 2005). The crucial role played by the stochastic element has led to calls for an empirical foundation for stochastic choice models (Loomes, 2005; Hey, 2005), and speculation that neuroeconomics can offer a richer, data-driven, approach (Harrison, 2008).

I propose that recent advances in modelling the dynamics of decision-making offer a neurobiological foundation to the specification of discrete choice models. Significant progress has now been made in understanding the neural processes that underlie choice behaviour (for reviews, see Glimcher, 2011; Fehr and Rangel, 2011; Glimcher and Fehr, 2013), with particular emphasis placed on the timing of a decision. It is now established that decision time gives insight both into the neural computations involved in decision-making (Roitman and Shadlen, 2002; Wang, 2002; Gold and Shadlen, 2007) and the prediction of choice behaviour (Milosavljevic, Malmaud, Huth, Koch, and Rangel, 2010; Krajbich, Armel, and Rangel, 2010; Clithero and Rangel, 2014). To capture this relationship between time and choice, a general class of Bounded Accumulation Model (BAM) posits the existence of a decision variable for each choice alternative, and model the stochastic evolution of these variables towards a decision threshold (for reviews, see Ratcliff and Smith, 2004; Gold and Shadlen, 2007). Therefore a BAM predicts which decision variable hits a threshold first (the choice) and when it hits (the decision time). Of this large model class, the Drift Diffusion Model (DDM; Ratcliff, 1978) of binary choice is a well-known special case, with much recent focus shifted towards models with competing decision variables that race to a fixed threshold, broadly referred to as Race models (Usher and McClelland, 2001; Kiani, Corthell, and Shadlen, 2014).

However in economic applications, data on decision time has not typically been reported, leaving open the question of how such insights can be applied to economic models. The results presented below demonstrate that a BAM implies a form of RUM, and characterize how the resulting distribution of stochastic choice is related to the distribution of decision time — a subject on which the discrete choice literature has so far remained agnostic. After briefly reviewing the two model classes in Section 2, a novel result formally demonstrating that the general class of BAM implies a RUM is provided in Section 3. The intuition is straightforward: a BAM implements the maximization required by a RUM, but with a stochastic element. Crucially, the implied distribution of random utility incorporates the distribution of decision time, and therefore depends on the specification of the accumulation model. This yields falsifiable statements about the form of choice probabilities even if data on decision time is not observed. More accurately, standard models make distributional assumptions which ignore the role of dynamics explored in this article. Of course, a direct observation of decision time data can additionally identify the distribu-
To demonstrate this point, Section 4 presents results for examples of BAMs which vary in their degree of analytical tractability:

- For an accumulation model with a fixed decision time $\bar{t}$ (i.e. exogenously determined by the experimenter), the random utility distribution is Gaussian with a variance which decreases non-linearly in $\bar{t}$. Therefore choice probabilities are given by the Probit model with a heteroskedastic error term of known form.

- In the case of the DDM, it is well-known that closed-form solutions for the choice probabilities reduce to the familiar Logit formulation (Cox and Miller, 1965). I derive properties of the random utility distribution implied by the DDM, show that it is correlated over alternatives, and verify that it still yields the Logit. Therefore the binary Logit can be derived from a different class of random utility distributions than typically stated in the literature (i.e. the independent Gumbel distribution; McFadden, 1974).

- Beyond the DDM, I derive the joint density of the random variables which comprise random utility in a Race model. For the special case of indifference (in binary choice), this density is used to characterize how the variance of random utility increases with observables. This result lies in contrast to a model with constant variance (i.e. the Logit), and implies choice probabilities which depend on both the relative difference and the magnitude of observables. While results for the general case are elusive, the extension of this result is presented by direct calculation of the moments via the joint density.

- For the general class of BAM, neither the choice probabilities nor the moments of key random variables can be characterized in closed form. In such cases, a method for numerically approximating the choice probabilities is presented. How well this approximation captures the actual choice process – and choice data in particular – then becomes an empirical question.

Section 5 demonstrates the econometric implications of the above results in a well-known experimental condition for choice under uncertainty (‘Multiple Price Lists’, Holt and Laury, 2002). Because different BAMs have different implications for the distribution of random utility, the properties of structural parameter estimates depend crucially on the specification of the accumulation model. For instance, if choices in the Holt and Laury (2002) experiment are generated using a Race model, an estimate of the coefficient of relative risk aversion will be biased. This bias can be partially corrected using results established in Section 4, namely including observables in the specification of the variance. Including observations of decision time in the variance specification also reduces the bias, while a specification derived directly from the BAM eliminates it.

The growth of digital data collection methods, in both field and experimental conditions (e.g. Rubinstein, 2013), only strengthens the need for examining this relationship.
The incorporation of bounded accumulation models into the random utility framework therefore has important implications for discrete choice modelling. For choice modellers, it provides a means of constraining a stochastic choice model to the empirical features of the choice process – either through data on decision time or insight from models of the dynamic choice process – thereby guiding applied research. In the other direction, these results clarify that it is possible to distinguish between the models in the large class of BAM using choice data alone. Moreover, the RUM framework offers a number of econometric advantages that would be impractical through modelling at the level of neural dynamics (Webb, Glimcher, Levy, Lazzaro, and Rutledge, 2013). This interaction emphasizes the gains from modelling choice behaviour at different levels of abstraction.

2 Models of Stochastic Choice

2.1 The Random Utility Model

Consider a choice set comprised of \( n \) alternatives (indexed \( i = 1 \ldots n \)) and a vector of measured observables \( x_i \) for each alternative. Denote \( P_i \) the probability that alternative \( i \) is chosen from the choice set. The RUM posits a vector of random variables \( u \), with element \( u_i \), such that

\[
\Pr[u_i > u_j, \forall j \neq i] = P_i.
\]

Conditions placed on \( P_i \) determine whether observed behaviour is consistent with the principle of utility maximization (Block and Marschak, 1960; McFadden, 1974; Falmagne, 1978), with the equivalent choice criterion defining the chosen alternative \( i^* \),

\[
i^* = \arg \max_i \{u_i\}.
\]

The formulation of a RUM as an empirical tool for relating observables to choice probabilities was made possible by imposing an insight from psychophysics (McFadden, 1981, 2001). Let the function \( V(x_i; \beta) : x_i \rightarrow v_i \) denote a behavioural theory linking observables to valuations \( v = [v_1, \ldots, v_n] \). If we assume an additive form for utility \( u = v + \eta \), in which \( \eta \equiv [\eta_1, \ldots, \eta_n] \) is a random vector, the choice criterion is

\[
i^* = \arg \max_i \{v_i + \eta_i\} = \arg \max_i \{V(x_i, \beta) + \eta_i\}.
\]

This additivity assumption yields the “Fechner” model that has found application throughout the discrete choice literature (McFadden, 2001). Its usefulness arises from a convenient specification of the choice probabilities: the probability of choosing an alternative \( i \) depends on the comparison between the magnitude difference \( v_i - v_j \) and the random variable \( \tilde{\eta}_{ji} \equiv \eta_j - \eta_i \),

\[
P_i(v) = \Pr[v_i - v_j > \tilde{\eta}_{ji}, \forall j \neq i].
\]
However an assumption for the joint distribution of $\eta$ is required, carrying with it important implications for the form of the choice probabilities. For instance, if $\eta_i$ follows an independent Gumbel distribution, the choice probabilities are given by the Multinomial Logit model and obey the axiom of Independence of Irrelevant Alternatives (IIA; Luce and Suppes, 1965; McFadden, 1974). Departures from this assumption induce choice probabilities which violate IIA (Hausman and Wise, 1978; Train, 2009), however the class of such RUMs is large and can entail starkly different forms for the choice probabilities (McFadden and Train, 2000).

Since the additive formulation of (1) implies that a $u$ can be defined for any $v + \eta$, every Fechner model is a RUM (Becker, DeGroot, and Marschak, 1963; Batley, 2008). However, the reverse implication does not hold (Loomes and Sugden, 1995). An example of this which nicely demonstrates the importance of correctly specifying the distribution of $\eta$ can be found in the experimental literature on choice under uncertainty.

Suppose $v_i$ is given by the expected utility $EU(p_i; \alpha)$ of a lottery over some reward space $p_i$, parametrised with a risk preference $\alpha$. Under a Fechner RUM (2), choice behaviour which deviates from EU – such as a violation of first-order stochastic dominance – is captured by the covariance structure placed on $\eta$. Therefore for the purposes of testing theory, the Fechner RUM is a model of choice errors. However the pattern of EU violations typically observed in experiments can not be reconciled by $\eta_i$ which are assumed to be independent, have constant variance, and/or are symmetric (Loomes, 2005). Indeed, it has been demonstrated that the variance of $\tilde{\eta}_j$ depends on the time a decision is made (Hey, 1995) as well as other observable features of the set of lotteries (Buschena and Zilberman, 2000).

This one example typifies a larger issue in the discrete choice literature, namely that the stochastic specification has important implications for testing theory. In a thorough analysis of the interaction between stochastic choice and the structural theories of choice under risk and uncertainty, Wilcox (2008) concludes that the stochastic element of choice is “at least as important in determining sample properties as structures are.” Such renewed focus on the stochastic element presents an opportunity for insight from neuroeconomics, with structural error parameters and more flexible specifications that are driven by both neural and behavioural data (Harrison, 2008). The goal of this article is to do just this: to provide a foundation for the distribution of a Fechner RUM derived from the current neuroscientific approaches to modelling the dynamics of decision-making.

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4Webb, Glimcher, and Louie (2014) explores how neurobiological constraints on the processing of information, or equivalently, the bounding of the utility representation, implies observed choice probabilities which violate the IIA axiom.

5This issue extends to the domain of strategic choice. A stochastic best-response alters the equilibrium predictions away from Nash equilibrium (Palfrey, 1995), though a unique prediction depends crucially on a unique specification of the random utility distribution; otherwise, any observed outcome can be an equilibrium (Haile, Hortacsu, and Kosenok, 2008).
2.2 Bounded Accumulation Models

I now lay out the general formulation for a BAM, composed of two elements: a stochastic process which accumulates subjective value, and a stopping rule. The class of such models is large, and our formal statement is designed to capture this class in a manner that is tractable to relate to a RUM. Appendix 7.1 presents a review of the state of the BAM literature.

2.2.1 The Accumulation Process

Each alternative \( i \) in the choice set is associated with a decision variable \( Z_i(t) \) which accumulates \( v_i \) over time.\(^6\) In empirical practice, the decision variable is widely taken to be the activity level of a population of neurons associated with alternative \( i \). In formal modelling, the \( Z_i(t) \) form the vector-valued Markov process \( Z(t) \) in the continuous state space \( \mathbb{R}^n \), for a continuous time index \( t \in \mathbb{R}_+ \).\(^7\) The general accumulation process is given by the following set of stochastic differential equations:

\[
dZ(t) = [v + \nu(t) + \Gamma(t)Z(t)] \, dt + \sigma(t) \, dB(t),
\]

where \( B(t) \) is a vector-valued Brownian motion.\(^8\)

The diffusion of \( B(t) \) is governed by the \((n \times n)\) matrix-valued function \( \sigma(t) \), allowing for time-varying correlation between the stochastic process for each alternative. Similarly, the \((n \times n)\) matrix-valued function \( \Gamma(t) \) allows for both a time- and state-varying relationship in the accumulators. This general formulation also includes a time-dependent drift term \( \nu(t) > 0 \) (often termed an “urgency” signal) which does not depend on \( v \) or \( i \).

One additional assumption is imposed on the stochastic process, namely that the parameters that govern the statistics of the accumulation are symmetric across alternatives.

**Assumption 1.** The symmetric \( n \times n \) matrices \( \Gamma(t) \) and \( \sigma(t) \) are of the form \( a(t)I + (b(t) - a(t))I \), for some \( a(t) \) and \( b(t) \), where \( I \) is the \( n \times n \) identity matrix and \( 1 \) is a \( n \times n \) matrix of ones.

This assumption is necessary for our results and will be maintained throughout.\(^9\) Of special interest is the case of binary choice with two accumulators.

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\(^6\)Since \( v \) is interpreted to be a neural quantity, it is natural to assume \( v_i > 0 \). I also assume \( Z_i(0) = 0 \). A BAM concerned with neural implementation imposes the constraint that \( Z_i(t) \geq 0 \) (e.g. Usher and McClelland, 2001).

\(^7\)The continuous time process can be derived as the limit of a discrete time random-walk, and has the advantage that it includes the class of Gaussian processes for which solutions can be derived in closed form. Discrete time may, in fact, be the more appropriate approach to modelling neural activity (Shadlen, Hanks, Churchland, and Kiani, 2006). Our main result can be shown for a discrete state space and discrete time (see the working paper, Webb, 2013).

\(^8\)The technical details of constructing the differential equations, and their solutions, can be found in Cox and Miller (1965) and Smith (2000).

\(^9\)Note that the symmetry of the accumulation process between alternatives does not preclude that the underlying behavioural theory \( V(x, \beta) \) allow for ordering or context effects (e.g. Salant and Rubinstein, 2008).
governed by $\mathbf{\Gamma} = \left( \begin{array}{cc} \gamma & \psi \\ \psi & \gamma \end{array} \right)$, $\psi < 0$. Under this process, the multiple accumulators mutually inhibit each other if $\psi < 0$, implying that each $v_i$ influences the rate of accumulation of $Z_j(t)$, $\forall j \neq i$, therefore the distribution of decision time (e.g. Usher and McClelland, 2001).

To clarify the general parametrization given in (3), it is instructive to consider special cases defined in terms of restrictions of the stochastic differential equations. Unlike (3), the solution to each of these cases yields a Gaussian process.

**Definition 1.** An accumulation process $Z_i(t)$ is uncoupled if $\mathbf{\Gamma}(t)$ is a diagonal matrix with element $\gamma(t)$.

This restriction implies that each stochastic process no longer depends on the processes governing the other alternatives (i.e. $\psi = 0$),

$$dZ_i(t) = [v_i + \nu(t) + \gamma(t)Z_i(t)] dt + \sigma(t) dB_i(t), \quad \forall i, \forall t \geq 0. \quad (4)$$

However the statistics of the accumulation are still allowed to depend on both time and state. This is an important requirement. Woodford (2014) demonstrates that optimal information accumulation given a capacity constraint yields an uncoupled process in which the optimal drift depends on the state of the accumulation.

Additionally, much empirical focus has been placed on whether the accumulation perfectly integrates $v_i$ (e.g. Kiani, Hanks, and Shadlen, 2008). For instance, when $\gamma(t) < 0$, the accumulation is “leaky” in the sense that early realizations of the accumulation are discounted. Eliminating these possibilities brings us to a familiar process which is both time- and state-homogenous.

**Definition 2.** An accumulation process $Z_i(t)$ is a Brownian motion (with drift) if it is uncoupled, time-homogenous, and $\gamma = 0$:

$$dZ_i(t) = v_i dt + \sigma dB_i(t), \quad \forall i, \forall t \geq 0. \quad (5)$$

Brownian motion describes the continuous time properties of a discrete random walk. It is a Gaussian process with a closed-form solution for $Z(t)$, and its mathematical tractability greatly simplifies the derivation of choice probabilities and other features of the accumulation process. For this reason, it is often applied in practice.

### 2.2.2 Stopping Rules

The second element of a BAM is a stopping rule which determines when/if each accumulator (or a function of each accumulator) has reached a decision threshold. As with the accumulation process, many forms of stopping rules have been proposed in practice. Here, I present the stopping rule in a general form which allows the choice criterion to be generally stated, and examine three example rules which have received particular attention.
Let $t^*$ denote the time a decision is made, with associated density $g_*(t)$. The choice $i^*$ is determined by the largest accumulator at $t^*$,

$$i^* = \arg\max_i \{ Z_i(t^*) \}. \quad (6)$$

In the simplest form of a stopping rule, a fixed decision time $\bar{t}$ is imposed.

**Definition 3.** A “Fixed” stopping rule is given by $t^* = \bar{t}$ for constant $\bar{t}$.

While a fixed stopping rule has the feature of accumulating each $v_i$ over time via a stochastic process, thus introducing stochasticity in the choice, it does not yield the skewed distribution of decision times that is typically observed when decision time is not exogenously controlled (Luce, 1986). To address this fact, accumulation models implement a rule which terminates the decision once the accumulator reaches some threshold (or more generally, exits some region). Formally, the decision time is a random variable given by

$$t^* = \inf \{ t : Z_i(t) \notin \mathcal{R}(t) \}, \quad (7)$$

for some region $\mathcal{R}(t) \subset \mathbb{R}^n$.

The various accumulation models proposed in the literature differ in how the region $\mathcal{R}(t)$ is specified and yield different properties for the distribution of decision time, primarily how it depends on $v$.

**Definition 4.** A “Differenced” stopping rule is given by equation (7) and a region

$$\mathcal{R}(t) = \{ x : |x_i - x_j| < \theta(t), \quad \forall i \neq j \}. \quad (8)$$

The Differenced stopping rule terminates the decision when one accumulator is larger than all others by $\theta(t)$. This rule is particularly noteworthy for its application in binary choice when the accumulators are given by Brownian motion.

**Remark 1.** For $n=2$, if $Z_i(t)$ is a Brownian motion and the stopping rule is Differenced, then $Z_1(t) - Z_2(t)$ is a one-dimensional Brownian motion to dual thresholds given by $\theta(t)$ and $-\theta(t)$.

A formal statement can be found in Bogacz, Brown, Moehlis, and Holmes (2006). For intuition appeal to Figure 1; the differenced stopping rule projects $Z(t)$ onto the one-dimensional coordinate axis, $Z_1(t) - Z_2(t)$. If we further restrict $\theta(t) = \theta$ to be constant in time, this model is commonly referred to as the Drift Diffusion Model and has been largely successful in matching the properties of reaction time data in binary choice (Ratcliff, 1978; Ratcliff and Smith, 2004). In particular, as the drift in the stochastic process $v_i - v_j$ grows larger, the expected decision time shrinks.

By definition, the DDM requires that the difference between accumulators is constant at $t^*$, but puts no restriction on the magnitude of each accumulator.\footnote{Experimental conditions which implement this stopping rule are called “interrogation protocols” since the subject is not free to determine the decision time.}
Figure 1: Sample paths of a Brownian motion accumulator for the Difference (left) and Race (right) stopping rules. Paths of $Z_1(t) - Z_2(t)$ are also depicted as a function of time (below), with the associated density of decision times $g_*(t)$. 
This feature is at odds with much of the neurobiological data (see Appendix 7.1). The following stopping rule has the opposite feature: the threshold is fixed, but there is no constraint on the relative magnitudes.

**Definition 5.** A Race stopping rule is given by equation (7) and a region  
\[ R = \{ x : b_i < x < a_i \quad \forall i \}, \text{ where } -\infty \leq b_i \leq a_i < \infty. \]  

(9)

The Race stopping rule traces out a hypercube in \( \mathbb{R}^n \) that is fixed in time (Figure 1). The simplest interpretation of this rule occurs when \( b_i = -\infty \) and \( a_i = \theta \), resulting in a decision when an accumulator surpasses a fixed threshold \( \theta \). Recent models which pair the Race rule with various forms of accumulator are detailed in Appendix 7.1.

### 3 Derivation of Fechner Random Utility Model

In an accumulation model, the probability that an accumulator reaches a particular threshold (thereby implementing a choice) depends on the vector \( \mathbf{v} \). The open question is whether these choice probabilities can be represented by an additive form of the RUM, and how it relates to \( \mathbf{v} \).

For the special case of the DDM, the answer is straightforward because the choice probabilities can be derived in closed form; it is well-known that the DDM implies logistic choice probabilities (Cox and Miller, 1965). These choice probabilities are also implied by a RUM of the Logit form.

**Proposition 1.** For \( n = 2 \), if \( Z_i(t) \) is a Brownian motion (5) and the stopping rule is differenced (8) with constant threshold \( \theta(t) = \theta \), then the resulting choice probabilities are logistic. Furthermore, they can be represented by a Fechner RUM in which \( \eta_i \) are independently drawn from the Gumbel distribution.

**Proof.** \( Z_1(t) - Z_2(t) \) is a one-dimensional Brownian motion to dual threshold (see Remark 1). Cox and Miller (1965) give the probability that this process hits the boundary for alternative \( i \) before \( j \) as

\[
P_i(\mathbf{v}) = \left( 1 + e^{-2(\mathbf{v}_i - \mathbf{v}_j)^2 / \sigma^2} \right)^{-1}.
\]

(10)

The binary logistic choice probabilities are implied by a Fechner RUM where the \( \eta_i \) are independently drawn from the Gumbel distribution (Luce and Suppes, 1965; McFadden, 1974).

Proposition 1 states that the choice probabilities implied by a DDM can be represented by the familiar Logit model. However, the Gumbel distribution is not unique in yielding the Logit for binary choice. The actual distribution implied by the DDM will be taken up in Section 4.1.\(^{11}\)

\(^{11}\)Proposition 1 also highlights an identification issue that will be familiar to practitioners.
For the general class of accumulation models and stopping rules given by equations (3) and (7), closed form expressions for the choice probabilities are not known. However we can still demonstrate that a BAM yields a Fechner RUM. To provide some intuition, let us consider the simple case of a Brownian-motion with drift (5).

Begin by noting that the choice criterion (6) is preserved under a linear scaling, \( \lambda > 0 \), at time \( t^* > 0 \). Therefore,

\[
i^* = \arg\max_i \{Z_i(t^*)\} = \arg\max_i \{\lambda Z_i(t^*)\},
\]

and equivalently for the choice probabilities,

\[
P_{i^*} = \Pr\left[i^* = \arg\max_i \{\lambda Z_i(t^*)\}\right].
\]

The solution to the differential equation (5) at time \( t^* \) is:

\[
Z_i(t^*) = v_i t^* + B_i(t^*).
\]

This expression for \( Z_i(t^*) \) is fully characterized by separable terms consisting of the exogenous value \( v_i \) and the realizations of the stochastic process \( B_i(t) \) for \( t \in [0, t^*] \).

Substituting (12) into (11) yields the choice criterion

\[
i^* = \arg\max_i \{\lambda v_i t^* + \lambda B_i(t^*)\}.
\]

Now all that remains is to choose a suitable value for \( \lambda \). Define

\[
\lambda = \frac{1}{t^*} > 0,
\]

therefore the choice criterion becomes

\[
i^* = \arg\max_i \left\{v_i + \frac{\sigma B_i(t^*)}{t^*}\right\}.
\]

This expression has the form of a Fechner RUM (1) in which \( \eta_i \) is comprised of the location of the Brownian motion \( B_i(t^*) \), scaled by the stopping time \( t^* \),

\[
\eta_i \equiv \frac{\sigma}{t^*} B_i(t^*),
\]

If \( v_i - v_j \) is known, it is clear that only the term \( \beta \equiv \frac{2\sigma}{t^*} \) is identified by choice data alone. Therefore the variance of the stochastic process must be normalized before a Logit model can be applied. However there is still something to be gained from the accumulation model. Incorporating a second observable – the decision time \( t^* \) – identifies an additional parameter of the DDM. This results in a more efficient estimate of the relationship between \( v_i - v_j \) and the logistic choice probabilities (10) than can be achieved through choice data alone (Clithero and Rangel, 2014).

12 This is equivalent to observing that the scale of the random utility model is arbitrary, thus requires normalization.
with choice probabilities given by

\[ P_i = \Pr \left[ v_i - v_j > \tilde{\eta}_{ji}, \quad \forall j \neq i \right] \]
\[ = \Pr \left[ v_i - v_j > \frac{\sigma}{t^*} (B_j(t^*) - B_i(t^*)), \quad \forall j \neq i \right] \]
\[ = \Pr \left[ v_i - v_j > \frac{\sigma \tilde{B}_{ji}(t^*)}{t^*}, \quad \forall j \neq i \right]. \quad (15) \]

This derivation of \( \eta \) for the special case of Brownian motion clarifies the link between the stochastic process governing accumulation and the choice probabilities from the implied RUM. Propositions 2 and 3 state similar results for more general forms of accumulation within the class of BAM.

First, for the case of choice between \( n \) alternatives via uncoupled accumulators:

**Proposition 2.** For \( n \geq 2 \), the accumulator given by (4), any stopping rule given by (7), and choice criterion given by (6), the resulting choice probabilities can be represented by a Fechner RUM in which

\[ \eta_i \equiv \int_0^{t^*} \frac{U^{-1}(\tau) \left[ \nu(\tau) d\tau + \sigma(\tau) dB_i(\tau) \right]}{U^{-1}(\tau) d\tau}, \]

and

\[ U(t) = e^{\int_0^t \gamma(\tau) d\tau > 0, \quad \forall t > 0.} \]

\[ \text{Proof. In appendix.} \]

Second, for the case of binary choice and coupled accumulators:

**Proposition 3.** For \( n = 2 \), the accumulator given by (3) with \( \Gamma(t) = \psi t + (\gamma - \psi)I \), any stopping rule given by (7), and choice criterion given by (6), the resulting choice probabilities can be represented by a Fechner RUM.

\[ \text{Proof. In appendix.} \]

The derivation of a Fechner RUM from a BAM has two important implications for discrete choice modelling. First, the choice probabilities implied by a BAM can be captured within the familiar framework currently used in econometric applications. Since the entire class of RUMs can be approximated by a Generalized Extreme Value model (Dagsvik, 1995; McFadden and Train, 2000), this means that the choice probabilities resulting from a BAM can also be approximated by GEV with an appropriate covariance structure and fit to choice data alone. Proposition 1 is an (exact) special case of this fact. For the general class of BAMS, it therefore remains for the choice modeller to specify the correct covariance structure of \( \eta \).

\[ ^{13} \text{A derivation for coupled accumulators in discrete time can be found in a working paper (Webb, 2013).} \]
This is where the second implication provides guidance. The dynamics of the accumulation process (either the distribution of decision time implied by a BAM, or direct observation of decision times) can be used to inform the distribution of $\eta$. As the general class of BAMs is empirically constrained (whether via behavioural or neural data), this guides the choice of structure to impose on the behavioural model. Economic datasets which collect data on decision time will yield a more accurate characterization of this distribution, therefore also the distribution of $\tilde{\eta}_{ji}$. In some cases, the relationship between $t^*$ and $v$ arising from accumulation will allow the moments of $\tilde{\eta}_{ji}$ to be characterized in terms of $v$ alone. These relationships are now explored in greater detail.

4 The Distribution of Stochastic Choice

The distribution of stochastic choice that arises from a BAM depends on the form of accumulation (e.g. $\gamma, \psi$), the statistics of the stochastic process (e.g. $\sigma$), and the distribution of stopping time $t^*$ which results from the stopping rule. In particular, the presence of the the random variable $t^*$ requires some consideration.

Some intuition can be gleaned from the case of a Brownian motion derived in (14) and a Fixed stopping time. As previously noted, the random variable $\tilde{\eta}_{ji}$ can be expressed in terms of the stochastic processes $B(t^*)$, scaled by the stopping time $t^*$,

$$\tilde{\eta}_{ji} = \frac{\sigma(B_j(t^*) - B_i(t^*))}{t^*}. \tag{16}$$

Therefore under a fixed stopping rule (i.e. $t^* = \bar{t}$), we have the following result for the distribution of $\tilde{\eta}_{ji}$.

**Proposition 4.** For a Brownian motion accumulator (5) and a Fixed stopping time $\bar{t}$, $\tilde{\eta}_{ji}$ is distributed $\mathcal{N}(0, \frac{2\sigma^2}{t^*})$.

**Proof.** For some constant $\bar{t}$, $B_i(\bar{t}) \sim \mathcal{N}(0, \bar{t}), \forall i$, and

$$\tilde{\eta}_{ji} = \frac{\sigma(B_j(\bar{t}) - B_i(\bar{t}))}{\bar{t}}.$$

Therefore $\tilde{\eta}_{ji} \sim \mathcal{N}(0, \frac{2\sigma^2}{\bar{t}})$ with variance decreasing in $\bar{t}$. $\square$

A BAM in which the time allotted to a decision is exogenously varied yields a Probit model in which the variance decreases in time. This simple example demonstrates the role that time plays in determining the distribution of stochastic choice.

In fact, the re-expression of $\tilde{\eta}_{ji}$ as a ratio in which $t^*$ appears in the denominator also holds for more general processes and stopping rules (see the proofs of Propositions 2 and 3), with the caveat that $t^*$ is a random variable. Though the density of a ratio of random variables is not easily characterized, the fact that the denominator increases in $t^*$ will allow us to make statements about the
distribution and/or moments of $\tilde{\eta}_{ij}$ for various cases of the general formulation. In particular, we can investigate how the distribution of $t^*$ depends on $v$ and how it impacts the choice probabilities for various formulations of BAM.

In doing so, it is convenient to introduce the concept of a first passage time. For each accumulator $Z_i(t)$, the random variable $t_i$ is defined as the first time $Z_i(t)$ exits the region defined by the stopping rule. For example, under a Race stopping rule, $t_i = \inf\{t : Z_i(t) \geq \theta_i\}$. The CDF of the first passage time distribution is denoted by $G(t; v_i)$ where the dependence on the valuation $v_i$ is explicit. In the literature on stochastic processes, first passage times are, arguably, the primary objects of mathematical investigation. In the current application, they will prove useful in demonstrating the following results.

1. Proposition 1 stated that a Brownian motion accumulator combined with the Difference stopping rule (i.e. the DDM) yields logistic choice probabilities. These closed form choice probabilities are derived, in part, from the first passage time distributions (Cox and Miller, 1965). Section 4.1 verifies that the formulation of $\tilde{\eta}_{ij}$ resulting from the DDM recovers the Logit model from a different class of distributions for $\eta$ than the Gumbel distribution.

2. The first passage times provide a means to characterize $t^*$ by virtue of the stopping rule. A useful example is the Race stopping rule where $t^*$ is given by the first order statistic $t^* = \min\{t_1, \ldots, t_n\}$. Section 4.2 demonstrates how the density of $t^*$, and its dependence on $v$, can be characterized for the class of accumulators given by (4) and the Race stopping rule. By means of Proposition 2, this yields insight into how the distribution of $\tilde{\eta}$ depends on $v$.

3. For the general class of accumulators in (3), where no closed form expressions for the distributions of $Z(t)$, $t^*$, or even $t_i$ exist, numerical approximations of the first passage time distributions are still possible. Through sampling from these distributions, stopping rules given by equation (7) can be implemented via simulation, yielding simulated choice probabilities. These methods are described in Section 4.3 before moving to an example application in Section 5.

4.1 Difference stopping rule

Let us begin by providing some intuition for how Proposition 2 relates to Proposition 1. Under a Difference stopping rule (8), the magnitude of the difference between the accumulators at $t^*$ must be fixed at $\theta$. Therefore

$$Z_i(t^*) - Z_j(t^*) = R,$$

and

$$B_i(t^*) - B_j(t^*) = \frac{R - t^*(v_i - v_j)}{\sigma},$$

(17)
where $R$ is a random variable which takes on $\theta$ if the upper threshold is crossed (alternative $i$ is chosen), and $-\theta$ otherwise. For intuition, consider the case in which $v_i - v_j = 0$ and $\sigma = 1$; then $B_j(t^*) - B_i(t^*)$ must either be $\theta$ or $-\theta$ for the accumulation to terminate (Figure 1).

Substituting (17) into (16) (noting the subscript $i$ and $j$) then yields

$$\tilde{\eta}_{ji} = \frac{-R}{t^*} + (v_i - v_j),$$

with choice probabilities given by

$$P_i(v) = \Pr \left[ v_i - v_j > \frac{-R}{t^*} + (v_i - v_j), \quad \forall j \neq i \right]$$

$$= \Pr \left[ \frac{R}{t^*} > 0, \quad \forall j \neq i \right].$$

Since the probability that $R > 0$ is given by the logistic function in equation (10), the Logit model has been recovered, but from a different class of distributions for $\eta$ than previously noted in the literature (i.e. the independent Gumbel; Luce and Suppes, 1965; McFadden, 1974). There are two issues to note regarding this derivation.

First, the necessary and sufficient relationship between the Gumbel and the Logit requires independence (McFadden, 1974), and only holds for cases in which there are more than two choice alternatives (Yellott Jr., 1977). The necessary condition does not hold in the case of the DDM since $\eta_i$ and $\eta_j$ are not independent through their relation to $t^*$. Moreover, the existing closed form results for the DDM can only be applied in the binary case.

Second, the distribution of $\eta$ (or $\tilde{\eta}_{ji}$) derived from the DDM is particularly unusual. While $t^* > 0$ scales the random variable $R$, it has no bearing on whether the ratio $\frac{R}{t^*}$ is greater or less than 0. This is why the distribution of $t^*$ does not alter the choice probabilities from the logistic form. It turns out this property is unique to the Differenced stopping rule, as demonstrated in the following section.

Beyond the Brownian motion process, closed forms for the choice probabilities under the Differenced stopping rule are not available. Busemeyer and Townsend (1992) give an expression under a restricted version of (4) (where $\nu()$, $\gamma()$, and $\sigma()$ are constant) in terms of a ratio of definite integrals, and numerical simulations suggest these probabilities indeed are of the logistic form. However when this restriction is relaxed, a drift rate that varies in time leads to choice probabilities which deviate from the Logit (Srivastava, Feng, and Shen-hav, 2015). For the general accumulation process (3), no analytic statements for the choice probabilities are available under the Differenced stopping rule.

However for binary choice, the choice probabilities resulting from the Differenced rule and general accumulators can be approximated. Define $\hat{t}$ as the first stopping time at the upper boundary for the one-dimensional process $Z_1(t) - Z_2(t)$, conditional on not having crossed the lower. The first passage time density $g_{\hat{t}}(t)$ can be approximated by numerical methods given in Smith.
Since the process terminating at one boundary (vs. the other) is mutually exclusive, the choice probability for alternative 1 is given by integrating \(g(t)\),

\[
P_1(v) = \int_0^\infty g(t) \, dt.
\]

If a dataset also contains information on decision times, likelihood methods can be used to incorporate this added information to yield a more efficient estimate via,

\[
P_1(v, t^*) = \int_0^{t^*} g(t) \, dt.
\]

Clithero and Rangel (2014) give an empirical demonstration of this result in the case of the DDM. Conveniently, these methods also apply to common extensions of the differenced stopping rule, in particular a threshold \(\theta(t)\) which depends on time (Drugowitsch, Moreno-Bote, Churchland, Shadlen, and Pouget, 2012; Fudenberg, Strack, and Strzalecki, 2015). Numerical simulations suggest that these probabilities deviate from the Logit.

### 4.2 Race stopping rule

A Race stopping rule is an example of a stopping rule in which the levels, not just the differences, of each accumulator are relevant to the decision. Importantly, it yields a straightforward characterization of the stopping time as an order statistic of the first passage times for each accumulator, \(t^* = \min[t_1, \ldots, t_n]\), with associated density \(g_\ast(t; v)\). Given the expression for \(\eta_{ji}\) derived via Proposition 2, it is therefore possible to derive the joint density of \(\tilde{B}(t^*)\) and \(t^*\).

For special cases of the accumulation process, this density yields insight into how the moments of \(\tilde{\eta}_{ji}\) depend on \(v\), and importantly, extends to more general cases.

Consider two uncoupled accumulators, governed by a simple Brownian motion with drift and the Race stopping rule. Figure 2 depicts the variance of \(\tilde{\eta}\) (the \(ji\) subscript is dropped for the binary case) resulting from such a model, calculated via (16) and a joint density to be described shortly. In particular, note that \(\text{Var}(\tilde{\eta})\) depends on the magnitudes \(v_1, v_2, \text{ and } v_1 - v_2\). A model with constant variance (such as the Logit) would exhibit none of these relationships.

To understand why the Race stopping rule yields a relationship between the variance of \(\tilde{\eta}\) and \(v\), it is useful to examine the joint density of \(\tilde{B}(t^*)\) and \(t^*\). Let \(h_{\tilde{B}(t^*), t^*}(b, t)\) denote this density on the support \(b \in \mathbb{R}\) and \(t \in \mathbb{R}_+\). Given the linear form of the accumulation, it can be expressed as

\[
h_{\tilde{B}(t^*), t^*}(b, t) = h_{\tilde{Z}(t^*), t^*}(z, t), \quad \text{where } z = (v_j - v_i)t + b.
\]

\(^{14}\)Computational methods for implementing these approximations are available under a BSD license at https://github.com/jdrugo/dm

\(^{15}\)Note that the density of \(B(t^*)\), conditional on one of the processes having stopped at \(t^* = t\), is decidedly not Gaussian. As we will see, this is due to the threshold imposed by the stopping rule. Intuitively, the threshold truncates sample paths that would have crossed the threshold and then returned below the threshold.
Since the random variable \( Z_2(t^*) - Z_1(t^*) \) takes on either a value \( Z_2(t^*) - \theta \) or \( \theta - Z_1(t^*) \), depending on which accumulator has won, \( h(z, t) \) can be derived from the joint density that a Brownian motion accumulator has reached the bound at time \( t \), and the losing accumulator is at location \( z < \theta \). Moreno-Bote (2010) gives this density as the product of two functions:

1. the density of the stopping time for the winning accumulator (i.e. the Inverse Gaussian density with mean \( \frac{\theta}{v} \) and variance \( \frac{\theta^2}{v^2} \); Cox and Miller, 1965, p. 221),
   \[
   g(t; v) = \frac{\theta}{\sqrt{2\pi \sigma^2 t}} e^{-\frac{(\theta - vt)^2}{2\sigma^2 t}},
   \]
   \( (18) \)

2. the location of the accumulator which did not win at time \( t \),
   \[
   f(z, t; v) = \begin{cases} 
   \frac{1}{\sqrt{2\pi \sigma^2 t}} \left[ e^{-\frac{(z-vt)^2}{2\sigma^2 t}} - e^{-\frac{(z-2\theta-vt)^2}{2\sigma^2 t}} \right], & \text{if } z < \theta, \\
   0, & \text{otherwise.}
   \end{cases}
   \]
   \( (19) \)

Therefore \( Z_2(t^*) - Z_1(t^*) \) takes on either a value \( Z_2(t^*) - \theta \) or \( \theta - Z_1(t^*) \) with corresponding density proportional to \( f(z + \theta, t; v_2) \) or \( f(\theta - z, t; v_1) \). This yields the desired joint density

\[
\begin{align*}
  h(z, t; [v_1, v_2]) &= g(t; v_1)f(z + \theta, t; v_2) + g(t; v_2)f(\theta - z, t; v_1),
\end{align*}
\]

and the choice probability

\[
P_1 = \Pr[t_1 < t_2] = \int_0^\infty g(\tau; v_1) \left(1 - G(\tau; v_2)\right) d\tau,
\]

where \( G(t; v) \) is the CDF of the Inverse Gaussian distribution.
While closed-form expressions for the integral $P_1$ are not available (Section 4.3 details how it can be approximated numerically), Figure 3 depicts $h(z, t)$ for two exemplary cases of $v$: $v_1 = v_2$, and $v_1 > v_2$.

In the case of $v_1 = v_2$, the joint density is given by

$$h(z, t; [v, v]) = g(t; v)(f(z + \theta, t; v) + f(\theta - z, t; v)),$$

(21)

where $v$ is used to denote the value of $v_1$ and $v_2$. As $v$ increases, more density is placed on smaller $t$ (Figure 3, top and middle panels). The fact that shorter stopping times enter the denominator of $\tilde{\eta}$ (via Proposition 2) is of primary relevance for the increasing relation between $\text{Var}(\tilde{\eta})$ and $v$, stated formally in the following proposition:

**Proposition 5.** For a brownian motion $Z_i(t)$, $i \in \{1, 2\}$, with $v_1 = v_2 = v$, and a Race stopping rule:

a. $G_\ast(t; v)$ is increasing in $v$  

b. $E[\tilde{\eta}] = 0$,  

c. $\text{Var}(\tilde{\eta}) = \int_0^\infty g_\ast(t; v)\frac{1}{t^2} \text{Var}(\tilde{Z}_{ji}(t)|t^* = t) \, dt$,  

d. $\text{Var}(\tilde{Z}(t)|t^* = t)$ is decreasing in $v$.

**Proof.** In appendix \[ \square \]

In particular, Proposition 5.c states the variance of $\tilde{\eta}$ in terms of the stopping time density $g_\ast(t; v)$ and the conditional variance of $\tilde{Z}(t)$. As $v$ increases, the stopping time density $g_\ast(t; v)$ places greater weight on smaller values of $t$ and increases $\text{Var}(\tilde{\eta})$ via the factor $\frac{1}{t^2}$. This shrinking denominator appears to outweigh the (small) decrease in $\text{Var}(\tilde{Z}(t)|t^* = t)$ that arises from Proposition 5.d (Figure 3, top panels).\footnote{Note that $\text{Var}(\tilde{Z}(t)|t^* = t)$ is decreasing in $t$, except for small values of $t$. This fact is demonstrated in Lemma 2 in the appendix.}

When $v_1 > v_2$, a formal statement is complicated by the fact that $E[\tilde{\eta}_{ji}] \neq 0$, however the intuition underlying the effect of increasing $v_1$ is similar: as more density is placed on shorter stopping times (see Lemma 1 in appendix), the variance of $\tilde{\eta}$ increases.\footnote{Formally, $\text{Var}(\tilde{\eta}) = \int_0^\infty g_\ast(t; v)\frac{1}{t^2} \text{Var}(\tilde{Z}(t)|t^* = t) \, dt + \int_0^\infty g_\ast(t; v)\frac{1}{t^2} E^2[\tilde{Z}(t)|t^* = t] \, dt - \left[\int_0^\infty g_\ast(t; v)\frac{1}{t^2} E[\tilde{Z}(t)|t^* = t] \, dt\right]^2$. Note each term has a factor of $t$ in the denominator of the integrand.}

The relationship between $\text{Var}(\tilde{\eta})$ and $v$ comes with an interesting implication for choice sets in which the valuations are scaled, yet the relative differences are preserved. Consider two choice sets, one with valuations $v$ and another with valuations $v' = v + \alpha$ where $\alpha = [\alpha, \ldots, \alpha] > 0$. Denote $P_1$ as the probability of choosing the highest-valued alternative (i.e. $v_i - v_j > 0$, $i \neq j$), and define the set $D$ as the realizations of $\tilde{\eta}$ for which $i$ is chosen:

$$D_v = \{\tilde{\eta} : v_i - v_j > \tilde{\eta}_{ji}, \; \forall j \neq i\}.$$

\[ \square \]
Figure 3:
Corollary 1. Any scaling of \( v \) (i.e. \( \alpha > 0 \)) which transfers probability density towards realizations of \( \tilde{\eta} \notin D \) implies that \( P_i(v) > P_i(v + \alpha) \).

**Proof.** In appendix. \( \square \)

Figure 4 illustrates two such examples. The first is the Brownian motion process just considered. The second is the version of an uncoupled process (i.e. \( \psi < 0 \)) that is at the forefront of current practice in the neuroscience and psychology literature (the Leaky Competing Accumulator model; Usher and McClelland, 2001).\(^{18}\) In both cases, the choice probability of the highest-valued alternative (and logically, the other alternatives as well) depends on both the differences in subjective valuations and the magnitudes. A Fechner RUM with constant variance, such as the Logit, does not share this property.

It is important to note that the scaling of the variance with \( v \) does not imply that preferences (in a stochastic sense) are re-ordered. If \( P_i(v) > P_j(v) \), then \( P_i(v + \alpha) > P_j(v + \alpha) \). Moreover, the increase in variance of \( \tilde{\eta}_{ji} \) from increasing any \( v_i \) does not reduce the probability that \( i \) is chosen. This result can be stated explicitly for the case of a Brownian motion accumulator.

**Proposition 6.** For a Brownian motion accumulator (5) and a Race stopping rule (9), then \( P_i(v \alpha^\top) > P_i(v) \) for some \( \alpha = [\alpha_1, \ldots, \alpha_n] \) where \( \alpha_i > 1 \) and \( \alpha_j = 1, \forall j \neq i \).

**Proof.** In appendix. \( \square \)

Therefore any increase in the value of an alternative will increase the probability that its accumulator will hit the threshold first, regardless of the stochasticity added by the decrease in the expected decision time.

\(^{18}\)For reasons of neurobiological plausibility the LCA model restricts \( Z_i(t) > 0, \forall i. \)
The dependence between choice stochasticity and the magnitude of valuations carries with it an important implication. In the absence of some mechanism for re-scaling valuations, the Race stopping rule implies more stochasticity in choice behaviour for more valuable choice sets than would be predicted by a model with constant variance. From the perspective of optimal choice, this suggests that the brain might have some mechanism to re-scale, or normalize, valuations relative to a given threshold, reducing the number of errors. A recent article explores how such a mechanism can result from known neurobiological constraints (Webb, Glimcher, and Louie, 2014).

4.3 Numerical Methods for Approximating Choice Probabilities

Where closed form expressions for the distributions of $Z(t)$, $t^*$, or $t_i$ do not exist, as in the general accumulation (3), a numerical method for approximating the first passage time density is still possible. Since the Race stopping rule allows easy characterization of the stopping time $t^*$, it will provide a useful demonstration that will be relevant for the empirical example in Section 5. Note that this method generalizes to other stopping rules, including the Differeced rule as well as rules not pursued in this article. All that is required is a functional relation deriving the choice $i^*$ from the first passage times $\{t_1, \ldots, t_n\}$.

In the case of the Race stopping rule, this function is given by $i^* = \arg\min\{t_1, \ldots, t_n\}$. One immediate implication is that the random variables $t_i$ can be directly interpreted as a random utility,

$$i^* = \arg\max_i \{u_i\} = \arg\max_i \{Z_i(t^*)\} = \arg\min_i \{t_i\},$$

since the random vector $[t_1, \ldots, t_n]$ represents the choice probabilities (Marley and Colonius, 1992). Therefore, the result for the binary choice probability noted earlier extends to $n$ alternatives,

$$P_i = \Pr[t_i < t_j, \ \forall i \neq j].$$

(22)

Since the density $g_i(t; \nu)$ can be approximated numerically for a wide range of stochastic processes (Smith, 2000), repeated sampling from this density yields a simulated approximation of $P_i$ via (22), and the methods of Maximum Simulated Likelihood can be applied (Train, 2009). An example of this method, and the results from the earlier sections, is now pursued in the context of choice under uncertainty.

5 Example: Measuring Risk Aversion

The link between a BAM and a Fechner RUM carries with it a range of implications for economic modelling. As noted in the Section 4, these include

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19 Computational methods for implementing these approximations are available under a BSD license at https://github.com/jdrugo/dm
hypotheses for how the distribution of stochastic choice depends on $v$ (or not), and methods for incorporating additional data on decision time into an econometric specification. To give a simple example, consider the estimation of a structural parameter for risk aversion from a well-known experiment on choice over uncertainty (Holt and Laury, 2002).

Table 1: Lottery Choices in the Holt-Laury Risk Aversion Experiment (reproduced from Harrison and Rutstrom, 2008)

<table>
<thead>
<tr>
<th>Lottery 1 (Safe)</th>
<th>Lottery 2 (Risky)</th>
<th>$EV_1$</th>
<th>$EV_2$</th>
<th>$EV_1-EV_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{S1}$</td>
<td>$P_{S2}$</td>
<td>$P_{S1.60}$</td>
<td>$P_{S1.85}$</td>
<td>$P_{S0.10}$</td>
</tr>
<tr>
<td>$P_{S0}$</td>
<td>$P_{S1}$</td>
<td>$P_{S1.2}$</td>
<td>$P_{S1.3}$</td>
<td>$P_{S1.4}$</td>
</tr>
<tr>
<td>0.1</td>
<td>$2$</td>
<td>0.9</td>
<td>$1.60$</td>
<td>0.1</td>
</tr>
<tr>
<td>0.2</td>
<td>$2$</td>
<td>0.8</td>
<td>$1.60$</td>
<td>0.2</td>
</tr>
<tr>
<td>0.3</td>
<td>$2$</td>
<td>0.7</td>
<td>$1.60$</td>
<td>0.3</td>
</tr>
<tr>
<td>0.4</td>
<td>$2$</td>
<td>0.6</td>
<td>$1.60$</td>
<td>0.4</td>
</tr>
<tr>
<td>0.5</td>
<td>$2$</td>
<td>0.5</td>
<td>$1.60$</td>
<td>0.5</td>
</tr>
<tr>
<td>0.6</td>
<td>$2$</td>
<td>0.4</td>
<td>$1.60$</td>
<td>0.6</td>
</tr>
<tr>
<td>0.7</td>
<td>$2$</td>
<td>0.3</td>
<td>$1.60$</td>
<td>0.7</td>
</tr>
<tr>
<td>0.8</td>
<td>$2$</td>
<td>0.2</td>
<td>$1.60$</td>
<td>0.8</td>
</tr>
<tr>
<td>0.9</td>
<td>$2$</td>
<td>0.1</td>
<td>$1.60$</td>
<td>0.9</td>
</tr>
<tr>
<td>1</td>
<td>$2$</td>
<td>0</td>
<td>$1.60$</td>
<td>1</td>
</tr>
</tbody>
</table>

In the Holt and Laury experiment, each subject was presented with a list of lottery pairs (Table 1), ordered by the difference in their expected value. This method is commonly referred to as a ‘Multiple Price List’ experiment. As each subject picks a lottery from each of the pairs, their willingness to choose a lottery with negative expected value is taken to reflect the degree of risk aversion in their preferences.

From choice data of this type, it is common to estimate a structural model of risk preferences using standard likelihood methods (Camerer and Ho, 1994; Hey and Orme, 1994; Holt and Laury, 2002; Harrison and Rutstrom, 2008), provided one places an assumption on the distribution of stochastic choice. However note that while the lotteries in the Price List are symmetric (around zero) in their expected values, the magnitude of the expected values increases as you move down the list. If there is a relationship between the magnitude of the expected utilities and the degree of stochasticity in choice, this relationship should be accounted for in the specification of the model.

For instance, Section 4.2 describes a relationship between the magnitude of valuations and the variance of $\tilde{\eta}$ under a Race stopping rule. As noted in Proposition 5, the variance of $\tilde{\eta}$ depends on the magnitude of $v$, specifically on the expected utility of the safe and risky lottery. This implies that the variance of $\tilde{\eta}$ will depend on the lottery pair in the Multiple Price List experiment. Econometrically, this amounts to a Fechner RUM with heteroskedasticity of a form that, if ignored in estimation, will lead to misspecification of the choice
probabilities and biased estimates of model parameters.

To demonstrate this fact, lottery choices were simulated for twenty risk-neutral subjects, with choice stochasticity introduced using a BAM with a Race stopping rule. The accumulator was parameterized as a Brownian motion (equation 5), with the valuation of lottery $i$ given by $v_i = EU_i$, and the decision terminated via the Race stopping rule with $\theta = 10$. The variance of the accumulation, $\sigma$, was normalized to 1. Risk preferences were then estimated via utility curvature parameter $\alpha$ from a CRRA utility function $u(x) = x^\alpha$.

Table 2 reports the average estimate, $\hat{\alpha}$, from 1000 simulated datasets, estimated under the common assumption of logistic choice probabilities with a constant standard deviation $s$. In the notation of (10), denote this specification

$$H_0 : \frac{\sigma^2}{2\theta} = s,$$

with $P_i = \left(1 + e^{\frac{EU_1 - EU_2}{s}}\right)^{-1}$. Under the specification of constant variance, the average estimate $\hat{\alpha}$ is 0.96, with a Type I error rate of 10.1% given a 5% significance-level test. Since $\alpha$ enters the utility function non-linearly, this degree of bias varies depending on the lottery, but it is consequential. For the 50/50 version of Lottery 2, it amounts to a risk premium of $\sim 5\%$ of the expected value. For a 50/50 lottery between $0$ and $100$, the risk premium would amount to $\sim 17\%$.

The bias in the estimate of $\hat{\alpha}$ arises because the variance of $\tilde{y}_{ji}$ depends, in part, on the magnitude of the expected utilities of each of the lotteries. In the Holt and Laury experiment, the lotteries with the largest expected utilities are found in pairs where $EU_1 - EU_2 < 0$, increasing the choice stochasticity for these lotteries. As a result, the choice probabilities, as a function of $EU_1 - EU_2$, are not symmetric around zero (as implied by a logistic function) and the estimator attempts to compensate by biasing the estimate of $\alpha$ downwards (Figure 5).

Table 2: Average estimates of the CRRA coefficient from 1000 simulations of a risk-neutral chooser ($\alpha = 1$), for various specifications of the stochastic choice distribution.

<table>
<thead>
<tr>
<th></th>
<th>$\hat{\alpha}$</th>
<th>Type I Error Rate</th>
<th>$\hat{\sigma}^2$</th>
<th>$\hat{k}$</th>
<th>$\hat{\lambda}_1$</th>
<th>$\hat{\lambda}_2$</th>
<th>$\hat{\lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0$</td>
<td>0.960</td>
<td>0.101</td>
<td>0.101</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_1$</td>
<td>0.973</td>
<td>0.076</td>
<td>0.032</td>
<td>-0.368</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_2$</td>
<td>0.981</td>
<td>0.071</td>
<td>0.039</td>
<td>-0.216</td>
<td>-6.784</td>
<td>-0.066</td>
<td></td>
</tr>
<tr>
<td>$H_3$</td>
<td>0.978</td>
<td>0.085</td>
<td>0.184</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_4$</td>
<td>0.997</td>
<td>0.053</td>
<td>0.0414</td>
<td></td>
<td></td>
<td></td>
<td>-0.014</td>
</tr>
</tbody>
</table>

Fortunately, a range of methods are available to achieve less biased (or even unbiased) results, depending on the specification of choice stochasticity and the
availability of decision time data. Since the variance of $\hat{\eta}_{ji}$ depends on $v$, one possibility is to include the difference in expected utility in the specification, 

$$H_1: \frac{\sigma^2}{2\theta} = s \sqrt{e^{k(EU_1 - EU_2)}},$$

as well as the magnitudes,

$$H_2: \frac{\sigma^2}{2\theta} = s \sqrt{e^{k(EU_1 - EU_2)} + e^{h_1 EU_1} + e^{h_2 EU_2}}.$$

As can be observed in Table 2, including functions of $v$ reduce the bias of $\hat{\alpha}$ by nearly half and reduce Type I errors to 7.1%. However, these linear specifications do not fully capture the dependence of the choice probabilities on $v$, since the bias is not completely eliminated.

Another possibility is to use data on decision time directly. Since $v$ directly impacts the distribution of stopping times, the effect on the variance of the random utility model can be equivalently stated in terms of time: the variance of $\hat{\eta}_{ji}$ increases as expected decision times becomes shorter. This leads to the following specification,

$$H_3: \frac{\sigma^2}{2\theta} = \sqrt{s^2 + gt^*}.$$

Including stopping time data has a similar effect to including a linear function
of \( v \), reducing the bias in \( \hat{\alpha} \) by half and the Type I error rate to 8.5%\(^{20}\).

Finally, if the choice probabilities are correctly specified, the methods in Section 4.3 can be used to arrive at estimates for \( \alpha \) and \( \theta \), even if data on decision times are not available. As noted in Section 4.2, the Race stopping rule implies that the stopping times are distributed Inverse Gaussian with mean \( \frac{\alpha}{EU_1} \) and variance \( \theta^2 \). Drawing a large number of samples from these distributions (for each lottery) yields an approximation of \( P_i \) via equation (22) for a given \( \alpha \) and \( \theta \). Denote this specification \( H_4 \), with \( \theta \) reported in Table 2 as \( \frac{1}{2s^2} \) to conform with the other specifications. As expected, a Maximum Simulated Likelihood estimation of \( \alpha \) appears unbiased, with a Type I error rate of 5.3%.

6 Conclusion

Significant progress has been made in understanding the neural processes underlying choice, with particular focus on the dynamics of a decision. From this line of theoretical and empirical research, a class of models, termed Bounded Accumulation Models, have been developed to link both behavioural and neural data. This work demonstrates that a “Fechner” Random Utility Model – the benchmark framework for discrete choice in economic applications – can be derived from a general class of BAMs, providing a neurobiological foundation for the choice process represented by random utility maximization.

The relationship between these two approaches might not be surprising given the origins of the RUM in psychophysical research (McFadden, 2001). However this relationship has important consequences for testing economic theory and predicting choice behaviour. The specific parameterization of a BAM – particularly its dynamics, stopping rule, and implied distribution of decision time – influence the resulting distribution of stochastic choice.

In the special case of a Brownian motion to dual thresholds (better known as a Drift Diffusion model for binary choice), the resulting choice probabilities are logistic, but arise from a different class of error distributions than traditionally reported in the literature. More neurobiologically-plausible models imply that the variance of the random utility distribution will depend on observables. An example of the Race stopping rule, for which the distribution of decision times can be derived, is pursued. Under this rule, the variance of the choice distribution will increase in the magnitude of the valuation of each of the alternatives, or equivalently, decrease in the observed decision times. This relation alters the choice probabilities, attenuating the choice probabilities for larger valued choice sets compared to a model with constant variance.

These result carry important implications for econometric applications. Estimates of structural choice parameters can be biased if common (mis)-specifications of the stochastic choice distribution are employed. Specifications of the variance which depend on either observables characteristics or decision time data

\(^{20}\)To ensure this linear specification of variance is greater than zero, a cutoff rule was used with the likelihood heavily penalized. Notably, a non-linear specification using the exponential function did not reduce bias.
can partially correct for this bias. A full correction is possible since the true choice probabilities can be approximated by simulating the stopping time distributions. Conveniently, these methods apply for the general formulation of a BAM.

Finally, this article also points to a methodological goal. Knowledge of the choice process in the brain will help constrain behavioural models (Bernheim, 2009; Webb, 2011). Clearly, much more work is required before a sharper specification of the stochastic choice distribution is possible, including further empirical study of the class of bounded accumulation models at both the neural and behavioural level. However, the formal relationship between these two levels of analysis implies that advances in neuroscientific modelling can reduce the degrees of freedom that applied economics researchers must consider, and vice-versa. In essence, modelling at the level of behaviour and at the level of dynamic neural processes is a symbiotic exercise.

7 Appendix

7.1 State of BAM Literature

The dynamic process by which the brain reaches a decision is a topic of intense research interest in psychology and neuroscience. Originally, this work focused on the dynamics of perceptual decision-making in which subjects are required to determine the state of an objectively known stimulus (e.g. the direction of motion of erratically moving dots; Britten, Shadlen, Newsome, and Movshon, 1992). The class of Bounded Accumulation Model (BAM) has come to address how such sensory evidence is accumulated over time by neural activity, integrated into a decision variable, and how a decision is implemented once this variable reaches some boundary (for reviews see Ratcliff and Smith, 2004; Smith and Ratcliff, 2004; Gold and Shadlen, 2007).

The pairing of an accumulator with a stopping rule defines a particular BAM. The definition of a general accumulation process (3) and a general stopping rule (7) ensures that our results will apply as widely as possible to the models explored in the neuroscience and psychology literature. However, it will be useful to describe the state of this literature in more detail.

An important feature of all bounded accumulation models is the prediction of a distribution for decision times which matches empirical observations (Ratcliff, 1978; Luce, 1986; Roitman and Shadlen, 2002). Typically, these distributions are positively skewed (e.g. Figure 1). In the case of binary choice, the DDM (with the added feature of a stochastic initial condition) is considered the benchmark model (for a review, see Smith and Ratcliff, 2004) and is particularly appealing if one assumes no opportunity cost of time. Under this assumption, the DDM implements a decision in the least amount of time for a given error rate, achieving a normative solution (Gold and Shadlen, 2002).21

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21Consider a decision-maker who receives a temporal sequence of independent, stochastic, signals about a binary state of the world, and must select an appropriate policy. The decision-
However when a cost of time is introduced – such that the decision-maker must tradeoff between improving accuracy of the current decision (by accumulating more evidence) versus beginning a new decision – the optimal stopping rule collapses the boundaries so that $\theta(t)$ is no longer constant in time (Drugowitsch, Moreno-Bote, Churchland, Shadlen, and Pouget, 2012; Fudenberg, Strack, and Strzalecki, 2015). Intuitively, the decision maker is willing to decrease accuracy in the current decision to arrive at a new decision problem earlier.

For the case of three (or more) alternatives, the optimal stopping rule(s) are not known. Various extensions of the differenced stopping rule have been proposed in the literature (McMillen and Holmes, 2006; Krajbich and Rangel, 2011; Niwa and Ditterich, 2008; Ditterich and Churchland, 2012), however these approximate an optimal solution only asymptotically (as the degree of choice error goes to zero).

This, and other, issues have led researchers to explore alternative accumulation processes and stopping rules. In particular, there is recognition that modelling binary choice as resulting from a single accumulator to dual thresholds arises from mathematical convenience, not from neurobiological plausibility (Kiani, Corthell, and Shadlen, 2014). The neurobiological evidence suggests multiple, competing accumulators $Z(t)$ are the primitive objects underlying the neural processes of decision (Mazurek, Roitman, Ditterich, and Shadlen, 2003; Bogacz, Usher, Zhang, and McClelland, 2007; Beck, Ma, Kiani, Hanks, Churchland, Roitman, Latham, and Pouget, 2008; Churchland, Kiani, and Shadlen, 2008; Furman and Wang, 2008; Thevarajah, Webb, Ferrall, and Dorris, 2010). In particular, the evidence is incompatible with a stopping rule which depends on the difference between accumulators, and instead points to a fixed threshold that is constant over trials (Roitman and Shadlen, 2002; Churchland, Kiani, and Shadlen, 2008; Niwa and Ditterich, 2008, though see Heitz and Schall, 2012). The Race rule has this feature, and moreover, is easily generalizable to $n$ choice alternatives.

A Race stopping rule also has the feature that the “confidence” of a decision (the posterior belief that $v_1 > v_2$) can be inferred from the magnitude of $Z_1(t^*) - Z_2(t^*)$ at the time of decision (Drugowitsch and Pouget, 2012). Intriguingly, neural measurements of this quantity correlate with the decision to opt out of an uncertain choice decision in favour of a certain option (Kiani and Shadlen, 2009; Kiani, Corthell, and Shadlen, 2014). This observation is not compatible with a simple differenced stopping rule since $Z_1(t^*) - Z_2(t^*) = \theta$ is constant by definition.

However for a simple Brownian motion accumulator, the Race rule carries

22Moreover each of them requires determining which accumulator is greatest at every $t$ then comparing it to other alternatives. From a computational standpoint, if the role of a BAM is to implement a “max” operator on subjective value, it would be (infinitely) recursive to embed a second “max” operation inside the stopping rule.
with it the odd prediction that decision times do not depend on the magnitude
difference between choice alternatives, an empirical regularity that has been ob-
served in a wide range of experimental datasets (Luce, 1986). For this reason,
models which aim for both neurobiological plausibility and realistic character-
ization of decision times use the more general stochastic process given in (3)
in tandem with the Race rule (Usher and McClelland, 2001; Roxin and Led-
berg, 2008; Tsetsos, Gao, McClelland, and Usher, 2012). The coupling of
the accumulators is grounded in principles of neural computation (i.e. mutual inhi-
bition; Wang, 2002; Wong and Wang, 2006) and ensures that the accumulation
is influenced by all of the choice alternatives, thus matching the same features
of decision time distribution as the DDM (Ratcliff and Smith, 2004).

There is still much ongoing debate over the particulars of the dynamic pro-
cess, the form of the boundary, and how a dynamic decision process can be im-
plemented in neural architecture (e.g. Cisek, 2006; Kiani, Hanks, and Shadlen,
2008; Tsetsos, Usher, and Chater, 2010; Ditterich, 2010; Hunt, Kolling, Soltani,
Woolrich, Rushworth, and Behrens, 2012; Tsetsos, Gao, McClelland, and Usher,
2012; Ditterich and Churchland, 2012; Heitz and Schall, 2012; Liston and Stone,
2013; Standage, You, Wang, and Dorris, 2013; Kiani, Corthell, and Shadlen,
2014).

It is important to emphasize that in all of these implementations, the equa-
tion determining the choice is given by (6) and the difference between BAMs
lies in the distribution of stopping times implied by the choice of accumulator
and stopping rule. As I will demonstrate below, the fact that all BAMs can be
written with this choice criterion and an appropriate distribution of stopping
times allows a derivation of equivalence between bounded accumulation models
and the Fechner RUM.

7.2 Proofs from Section 3

PROPOSITION 2 For $n \geq 2$, the accumulator given by (4), any stopping rule
given by (7), and choice criterion given by (6), the resulting choice probabilities
can be represented by a Fechner RUM.

Proof. Let us begin by noting that the choice criterion (equation 6) is preserved
under a scaling, $\Lambda(t^*) > 0$, for $t^* > 0$. Therefore,
\[
i^* = \arg\max_i \{Z_i(t^*)\} = \arg\max_i \{\Lambda(t^*)Z_i(t^*)\},
\]

The Ito calculus can be used to derive a solution to (4) (Smith, 2000). Solving
this equation at $t^*$ yields
\[
Z_i(t^*) = \int_0^{t^*} \frac{U(t^*)}{U(\tau)} \left[ v_i + \nu(\tau) \right] d\tau + \sigma(\tau) dB_i(\tau),
\]
\[
= v_i \int_0^{t^*} \frac{U(t^*)}{U(\tau)} d\tau + \left. \int_0^{t^*} \frac{U(t^*)}{U(\tau)} [\nu(\tau) d\tau + \sigma(\tau) dB_i(\tau)] \right|_{\tau=0}^{\tau=t^*}.
\]
where
\[ U(t) = e^{\int_0^t \gamma(\tau) \, d\tau} > 0, \quad \forall t > 0. \]

Following the intuition from our previous example, we multiply through equation (23) by \( \Lambda(t^*) = \left[ U(t^*) \int_0^{t^*} U^{-1}(\tau) \, d\tau \right]^{-1} > 0 \), yielding
\[
\Lambda(t^*) Z_i(t^*) = v_i + \Lambda(t^*) U(t^*) \int_0^{t^*} U^{-1}(\tau) [\nu(\tau) \, d\tau + \sigma(\tau) \, dB_i(\tau)].
\]
Substituting this term into the choice criterion,
\[ i^* = \operatorname{argmax}_i \{ \Lambda(t^*) Z_i(t^*) \}, \]
\[ = \operatorname{argmax}_i \{ v_i + \eta_i \}, \]
yields our result with
\[ \eta_i = \frac{\int_0^{t^*} U^{-1}(\tau) [\nu(\tau) \, d\tau + \sigma(\tau) \, dB_i(\tau)]}{\int_0^{t^*} U^{-1}(\tau) \, d\tau}. \]

Note that the random variable relevant for choice behaviour, \( \tilde{\eta}_{ij} \), is thus given by
\[ \tilde{\eta}_{ij} = \frac{\int_0^{t^*} U^{-1}(\tau) \sigma(\tau) \, d(B_j(\tau) - B_i(\tau))}{\int_0^{t^*} U^{-1}(\tau) \, d\tau}, \]
since the the urgency term \( \nu(\tau) \) is not specific to alternatives \( i \) and \( j \), therefore cancels.

**Proposition 3** For \( n = 2 \), the accumulator given by (3) with \( \Gamma(t) = \psi I + (\gamma - \psi) I \), any stopping rule given by (7), and choice criterion given by (6), the resulting choice probabilities can be represented by a Fechner RUM.

**Proof.** Begin with the solution to the differential equation (3), for \( \Gamma(t) = \Gamma = \begin{pmatrix} \gamma & \psi \\ \psi & \gamma \end{pmatrix} \), as given in Smith (2000). Let the \( n \times n \) matrix function \( \Delta(t) \) be the fundamental solution to the first-order linear differential system
\[ \Delta'(t) = \Gamma \Delta(t), \quad \Delta(0) = I. \]
Then the solution to equation (3) at \( t^* \) is
\[
Z(t^*) = \Delta(t^*) \int_0^{t^*} \Delta^{-1}(\tau) [\nu + \nu(\tau)] \, d\tau + \Delta(t^*) \int_0^{t^*} \Delta^{-1}(\tau) \sigma(\tau) \, dB(\tau)
= \int_0^{t^*} \Delta(t^*) \Delta^{-1}(\tau) \nu \, d\tau + \int_0^{t^*} \Delta(t^*) \Delta^{-1}(\tau) [\nu(\tau) \, d\tau + \sigma(\tau) \, dB(\tau)]
= \int_0^{t^*} \Delta(t^*) \Delta^{-1}(\tau) \nu \, d\tau + C(t^*),
\]
where for exposition we define the portion of the integral that does not depend on \( v \) as \( C(t^*) \).

To proceed, observe that the solution \( \Delta(t) \) takes the form
\[
\Delta(t) = e^{\gamma t} \begin{pmatrix} \cosh(\psi t) & \sinh(\psi t) \\ \sinh(\psi t) & \cosh(\psi t) \end{pmatrix}.
\]
Therefore
\[
\Delta(t^*) \Delta^{-1}(\tau) = e^{\gamma(t^* - \tau)} \begin{pmatrix} \cosh(\psi(t^* - \tau)) & \sinh(\psi(t^* - \tau)) \\ \sinh(\psi(t^* - \tau)) & \cosh(\psi(t^* - \tau)) \end{pmatrix},
\]
where \( \cosh(\psi(t^* - \tau)) > 0 \) and \( \sinh(\psi(t^* - \tau)) > 0 \), for all \( \tau < t \).

Substituting this term in to the solution to the differential equation yields
\[
Z(t^*) = \int_0^{t^*} e^{\gamma(t^* - \tau)} \left[ \cosh(\psi(t^* - \tau)) v + \sinh(\psi(t^* - \tau)) \begin{pmatrix} 1 \\ v_1 \end{pmatrix} \right] d\tau + C(t^*)
\]
\[
= v \int_0^{t^*} e^{\gamma(t^* - \tau)} \cosh(\psi(t^* - \tau)) d\tau + \int_0^{t^*} e^{\gamma(t^* - \tau)} \sinh(\psi(t^* - \tau)) \begin{pmatrix} 1 \\ v_1 \end{pmatrix} d\tau + C(t^*),
\]
where we now explicitly separate the portion of the accumulation of \( Z_i \) which depends on \( v_i \) from the portion which depends on the other alternative.

From this point, proceed as in the earlier proof to Proposition 2. Define
\[
\Lambda(t^*) = \left[ \int_0^{t^*} e^{\gamma(t^* - \tau)} \cosh(\psi(t^* - \tau)) d\tau \right]^{-1} > 0,
\]
and multiply through, yielding
\[
\Lambda(t^*)Z(t^*) = v + \Lambda(t^*) \int_0^{t^*} e^{\gamma(t^* - \tau)} \sinh(\psi(t^* - \tau)) \begin{pmatrix} 1 \\ v_1 \end{pmatrix} d\tau + \Lambda(t^*) C(t^*).
\]
Finally, substituting into the choice criterion yields
\[
i^* = \arg\max \{ \Lambda(t^*)Z(t^*) \} = \arg\max \{ v + \eta \},
\]
where
\[
\eta = \Lambda(t^*) \int_0^{t^*} e^{\gamma(t^* - \tau)} \cosh(\psi(t^* - \tau)) \begin{pmatrix} 1 \\ v_1 \end{pmatrix} d\tau + \Lambda(t^*) C(t^*)
\]
\[
= \int_0^{t^*} e^{-\gamma \tau} \sinh(\psi(t^* - \tau)) d\tau \begin{pmatrix} 1 \\ v_1 \end{pmatrix} + \Lambda(t^*) C(t^*).
\]
The variance of \( \eta \) is therefore given by
\[
\text{Var}(\eta) = \text{Var} \left( \int_0^{t^*} e^{-\gamma \tau} \begin{pmatrix} \cosh(\psi(t^* - \tau)) & \sinh(\psi(t^* - \tau)) \\ \sinh(\psi(t^* - \tau)) & \cosh(\psi(t^* - \tau)) \end{pmatrix} \sigma(\tau) dB(\tau) \right).
\]
\[
\Box
\]
7.3 Proofs from Section 4

We begin by establishing that an increase in the valuation of any alternative will result in a distribution of stopping times that is first order stochastically dominated.

**Lemma 1.** For a Brownian motion accumulator (5) and a Race stopping rule (9), for any $v'$ and $v$ where $v'_j \geq v_j, \forall j$, and $\exists i$ where $v'_i > v_i$, then $G_*(t;v') > G_*(t;v)$.

**Proof.** The stopping time $t^*(v) = \min\{t_1, \ldots, t_n\}$ is an order statistic with the following cumulative distribution function,

$$G_*(t;v) = 1 - \prod_{i=1}^{n} (1 - G(t;v_i)). \quad (24)$$

For $v'_i > v_i$, the first passage time density $g(t;v)$ satisfies the monotone likelihood ratio property for increasing $v$. Therefore $G(t;v'_i) > G(t;v_i)$. By means of (24), this implies $G_*(t;v') > G_*(t;v)$.

Before we proceed with a proof of Proposition 5, we must characterize how the density of the losing accumulator depends on $v$ and $t$. Let $F(x; t, v)$ denote the conditional distribution at $x > 0$ for a given $t$ and $v$, derived from integrating (19),

$$F(z; t, v) = \frac{1}{\tilde{F}} \int_{0}^{z} f(\theta - x, t; v)dx$$

where the normalization constant $\tilde{F}$ is given by

$$\tilde{F} = \int_{0}^{\infty} f(\theta - z, t; v)dz$$

$$= \frac{1}{2} \left[ \text{Erfc} \left( \frac{tv - \theta}{\sqrt{2}\sigma^2} \right) - e^{\frac{tv^2}{2\sigma^2}} \text{Erfc} \left( \frac{tv + \theta}{\sqrt{2}\sigma^2} \right) \right]$$

$$= 1 - G(t;v).$$

We now establish the following two Lemmas regarding the conditional density $f_{(\theta-z,t,v)}$.

**Lemma 2.** $F(z'; t', v) < F(z; t, v)$ for some $t' > t$. 

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Proof. As $t \to 0$,
\[
\frac{f(\theta - z, t; v)}{1 - G(t; v)} \to \delta(z), \quad \text{where } \delta(z) = \begin{cases} \infty, & z = \theta \\ 0, & z \neq \theta \end{cases}
\]
is the dirac delta function centered on $\theta$. Therefore,
\[
\lim_{t \to 0} F(z; t, v) = 1, \quad \forall z > \theta.
\]
However, for some small $t' > 0$ and all $z > \theta$, it can be shown that $\frac{f(\theta - z, t'; v)}{1 - G(t'; v)} > 0$. Therefore $\exists t' > t$ for which $F(z; t', v) < F(z; t, v)$. \hfill \square

Lemma 3. $F(z; t, v') > F(z; t, v)$ for all $v' > v$.

Proof. The ratio of the conditional densities is
\[
\frac{f(\theta - z, t, v)}{1 - G(t; v)} \equiv \left[ e^{-(z-\theta)^2/2\sigma^2} - e^{2\theta(z-\theta) + 2(z-\theta)^2/2\sigma^2} \right] \frac{1 - G(t; v')}{1 - G(t; v)}.
\]
Since this ratio is increasing in $z$,
\[
\frac{\delta \left[ f(\theta - z, t, v)/(1 - G(t; v)) \right]}{\delta z} = \left[ \frac{(v' - v)}{\sigma^2} e^{2\theta(z-\theta) + 2(z-\theta)^2/2\sigma^2} \right] \frac{1 - G(t; v')}{1 - G(t; v)} > 0,
\]
the densities satisfy the monotone likelihood ratio property. This implies $F(z; t, v') > F(z; t, v)$. \hfill \square

PROPOSITION 5. For a brownian motion $Z_i(t)$, $i \in \{1, 2\}$, with $v_i = v_j = v$, and a Race stopping rule:

a. $G_*(t, v)$ is increasing in $v$

b. $E[\tilde{\eta}_{ji}] = 0$

c. $\text{Var}(\tilde{\eta}_{ji}) = \int_0^\infty g_*(t; v) \frac{1}{2\pi} \text{Var}(\tilde{Z}_{ji}(t) \ | t^* = t) \, dt$

d. $\text{Var}(\tilde{Z}_{ji}(t) \ | t^* = t)$ is decreasing in $v$.

Proof. Result a. is a special case of Lemma 1 where $n=2$.

From the derivation of $\tilde{\eta}_{ji}$ (via Proposition 2) and definition of $Z_i(t)$:
\[
E[\tilde{\eta}_{ji}] = E \left[ \frac{\tilde{B}_{ji}(t^*)}{t^*} \right] = E \left[ \frac{\tilde{Z}_{ji}(t^*)}{t^*} \right] - (v_i - v_j)
\]
\[
= E_t^* \left[ \frac{1}{t} E \left[ \tilde{Z}_{ji}(t^*) \mid t^* = t \right] \right].
\]
where the last equality follows from the Law of Iterated Expectations. Moreover,

\[
E \left[ \tilde{Z}_{ji}(t^*) | t^* = t \right] = \frac{1}{4(1 - G(t; v))} \int_{-\infty}^{\infty} z (f(z + \theta) + f(\theta - z)) \, dz
\]

\[= 0,
\]

since the function given by \(f(z + \theta) + f(\theta - z)\) is even. This yields result b.

Similarly, from Proposition 2 and the definition of \(Z_i(t)\),

\[
\text{Var} (\tilde{\varphi}_{ji}) = \text{Var} \left( \tilde{B}_{ji}(t^*) \right) = \text{Var} \left( \frac{\tilde{Z}_{ji}(t^*)}{t^*} \right).
\]

From the definition of conditional variance,

\[
\text{Var} \left( \frac{\tilde{Z}_{ji}(t^*)}{t^*} \right) = E_{t^*} \left[ \text{Var} \left( \frac{\tilde{Z}_{ji}(t)}{t} \bigg| t^* = t \right) \right] + \text{Var}_{t^*} \left( E \left[ \frac{\tilde{Z}_{ji}(t)}{t} \bigg| t^* = t \right] \right)
\]

\[= E_{t^*} \left[ \frac{1}{t^2} \text{Var} \left( \tilde{Z}_{ji}(t) \bigg| t^* = t \right) \right] + 0
\]

\[= \int_{0}^{\infty} g_*(t) \frac{1}{t^2} \text{Var} (\tilde{Z}_{ji}(t) \bigg| t^* = t) \, dt.
\]

This yields result c.

Regarding result d., a similar argument to b. holds,

\[
\text{Var}(\tilde{Z}_{ji}(t^*) | t^* = t) = \int_{-\infty}^{\infty} z^2 h(z | t^* = t) \, dz
\]

\[= \frac{1}{2(1 - G(t; v))} \left[ \frac{1}{2} \int_{-\infty}^{0} z^2 f(z + \theta, t) \, dz + \frac{1}{2} \int_{0}^{\infty} z^2 f(\theta - z, t) \, dz \right]
\]

\[= \frac{1}{2(1 - G(t; v))} \int_{0}^{\infty} z^2 f(\theta - z, t) \, dz,
\]

where the final equality arises since \(f(z + \theta) = f(\theta - (z)), \forall z\) (i.e. \(h(z, t)\) is symmetric about 0).

Since the density \(\frac{f(\theta - z, t)}{1 - G(t; v)}\) places less weight in its right tail for larger \(v\) (Lemma 3), this yields result d. via (26).

\[
\square
\]

**COROLLARY 1** Any scaling of \(v\) (i.e. \(\alpha > 0\)) which transfers probability density towards realizations of \(\tilde{\eta} \notin D\) implies that \(P_i(v) > P_i(v + \alpha)\).

**Proof.** The choice probabilities are given by equation (2), restated here as an integral over the set of realizations of \(\tilde{\eta}\) for which \(\tilde{i}\) is chosen (Train, 2009).

\[
P_i(v) = \int_{\tilde{\eta} \in D_v} p_v(\tilde{\eta}) \, d\tilde{\eta}.
\]
and $p_v(\tilde{\eta})$ is the density of $\tilde{\eta}$ implied by the BAM with valuation $v$.

Since $v + \alpha$ preserves the relative differences between alternatives, the elements of the set $D$ are identical under $v$ and $v + \alpha$:

$$D_{v + \alpha} = \{ \tilde{\eta}: v_i + \alpha - v_j - \alpha > \tilde{\eta}_{ji}, \quad \forall j \neq i \} = D_v.$$

Therefore any scaling of $v$ (i.e. $\alpha > 0$) which transfers probability density towards realizations of $\tilde{\eta} \not\in D$ will imply that $P_i(v) > P_i(v + \alpha)$. \hfill \Box

**Proposition 6** For a Brownian motion accumulator (5) and a Race stopping rule (9), then $P_i(v\alpha^T) > P_i(v)$ for some $\alpha = [\alpha_1, \ldots, \alpha_n]$ where $\alpha_i > 1$ and $\alpha_j = 1, \forall j \neq i$.

**Proof.** Recalling the binary choice probability given in (20), the probabilities for the general case of $n$ accumulators are given by

$$P_i(v) = \int_0^\infty g(t; v_i) \prod_{j \neq i} (1 - G(t; v_j)) dt,$$

and

$$P_i(v\alpha^T) = \int_0^\infty g(t; \alpha v_i) \prod_{j \neq i} (1 - G(t; v_j)) dt.$$

Since $G(t; v_i)$ is increasing in $v_i$, $g(t; \alpha v_i)$ places more density on smaller values of $t$ than does $g(t; v_i)$. Since $1 - G(t; v_j)$ is decreasing in $t$, $P_i(v\alpha^T) > P_i(v)$. \hfill \Box

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References


The neural basis of decision making,” Annual Review of Neuroscience, 30, 535–574.


