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e-companion

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Electronic Companion—"The Worst-Case Efficiency of Cost Sharing Methods in Resource Allocation Games" by Tobias Harks and Konstantin Miller, *Operations Research*, http://dx.doi.org/10.1287/opre.1110.0979.

Proofs of Statements

EC.1. Proof of Proposition 3.1.

PROPOSITION 3.1 *The only cost sharing method that is exactly budget balanced and fulfills Assumption* (4) *(single price per unit) in Definition 3.2 is average cost sharing.*

Proof. Observe that $C_r\left(\ell_r(\varphi)\right) \overset{*}{=} \sum_{i=1}^n \xi_i^r\left(\varphi^r\right) = \sum_{i=1}^n \varphi_i^r \cdot \frac{\xi_i^r(\varphi^r)}{\varphi_i^r} \overset{**}{=} \frac{\xi_{i_0}^r(\varphi^r)}{\varphi_{i_0}^r} \cdot \ell_r(\varphi)$, $\forall i_0 \in N$, where we use the definition of a budget balanced cost sharing method in $(*)$ and Assumption (4) from Definition 3.2 in $(**)$. We obtain $\xi_{i_0}^r\left(\varphi^r\right) = \varphi_{i_0}^r \cdot \frac{C_r(\ell_r(\varphi))}{\ell_r(\varphi)}$, $\forall i_0 \in N$, proving the claim. □

EC.2. Proof of Lemma 3.1.

LEMMA 3.1 *Consider a resource allocation game G with basic cost sharing methods $\left(\xi_r, r \in R\right)$. The profiles ϑ and ψ are a Nash equilibrium and an optimal profile, respectively, if and only if for all players i the following conditions hold:*

$$\nabla \pi_i\big(\vartheta_i;\, \vartheta_{-i}\big) \cdot \big(\varphi_i - \vartheta_i\big) \le 0, \quad \textit{for all } \varphi_i \in \Phi_i\,,$$

$$U_i'\big(d_i(\vartheta)\big) = \hat{\xi}_{ij}(\vartheta), \quad \textit{for all } j \in M_i \textit{ with } \vartheta_{ij} > 0,$$

$$U_i'\big(d_i(\vartheta)\big) \le \hat{\xi}_{ij}(\vartheta), \quad \textit{for all } j \in M_i \textit{ with } \vartheta_{ij} = 0,$$

$$U_i'\big(d_i(\psi)\big) = \sum_{r \in R_{ij}} C_r'\big(\ell_r(\psi)\big), \quad \textit{for all } j \in M_i \textit{ with } \psi_{ij} > 0,$$

$$U_i'\big(d_i(\psi)\big) \le \sum_{r \in R_{ij}} C_r'\big(\ell_r(\psi)\big), \quad \textit{for all } j \in M_i \textit{ with } \psi_{ij} = 0.$$

Proof. The function π_i is differentiable and concave with respect to φ_i. Furthermore, the set of profiles Φ is convex. Since ϑ is a Nash equilibrium, the strategy ϑ_i solves $\max_{\varphi_i \in \Phi_i} \pi_i(\varphi_i;\, \vartheta_{-i})$. Thus, we can invoke the variational inequality as a necessary and sufficient optimality condition giving (1). Note that the derivative of π_i with respect to φ_{ij} is given by $\frac{\partial \pi_i}{\partial \varphi_{ij}}(\varphi_i;\, \varphi_{-i}) = U_i'\left(d_i(\varphi)\right) - \hat{\xi}_{ij}(\varphi)$. The conditions (2) and (3) follow directly from the Karush-Kuhn-Tucker conditions for the two problems $\max_{\varphi_i \in \Phi_i} \pi_i(\varphi_i;\, \vartheta_{-i})$ and $\max_{\varphi \in \Phi} \mathcal{U}(\varphi)$, respectively. □

EC.3. Proof of Lemma 4.1.

LEMMA 4.1 *Let G be a resource allocation games with n players, cost functions in $\mathcal{C}$, and basic cost sharing methods $\xi_r \in \mathcal{D}_n$ for all $r \in R$. Let Θ_G be the set of Nash equilibria. Then, $\mathcal{U}(\vartheta) \geq 0$ for all $\vartheta \in \Theta_G$.*

Proof. Let $\vartheta \in \Theta_G$ be a Nash equilibrium. We deduce the following inequalities.

$$\mathcal{U}(\vartheta) \geq \sum_{i=1}^{m}\sum_{j=1}^{m_i} \hat{\xi}_{ij}(\vartheta)\cdot\vartheta_{ij} - C(\vartheta) = \sum_{r\in R}\sum_{i=1}^{n} \frac{\partial \xi_i^r}{\partial \vartheta_i^r}(\vartheta)\cdot\vartheta_i^r - C(\vartheta) \geq \sum_{r\in R}\sum_{i=1}^{n} \xi_i^r(\vartheta) - C(\vartheta) \geq 0\,.$$

Here, the first inequality follows from (2) in Lemma 3.1. The second equality follows by rearranging terms. The third inequality follows from convexity of $\xi_i^r(\vartheta)$ and condition (3) of basic cost sharing methods implying $0 = \xi_i(0, \vartheta_{-i}^r) \geq \xi_i(\vartheta_i^r, \vartheta_{-i}^r) + \frac{\partial \xi_i^r}{\partial \vartheta_i^r}(\vartheta)\cdot(0 - \vartheta_i^r)$. The last inequality follows from the cost covering condition of ξ. □

EC.4. Proof of Lemma 4.3.

LEMMA 4.3 *Consider a game G with basic cost sharing methods and linear utility functions, that is, $U_i(x) = u_i \cdot x$, $u_i \geq 0$, $i \in N$. Let ψ be an optimal profile and ϑ be a Nash equilibrium. Then, ψ and ϑ generate a total surplus of*

$$\mathcal{U}(\psi) = \sum_{r\in R}\Big(\ell_r(\psi)\cdot C_r'\big(\ell_r(\psi)\big) - C_r\big(\ell_r(\psi)\big)\Big)$$
$$\mathcal{U}(\vartheta) = \sum_{i=1}^{n}\sum_{j=1}^{m_i} \hat{\xi}_{ij}(\vartheta)\,\vartheta_{ij} - C(\vartheta)\,.$$

Proof. Using condition (3) from Lemma 3.1, we get $u_i = \sum_{r\in R_{ij}} C_r'\big(\ell_r(\psi)\big)$ for all $i \in N$, $j \in M_i$, with $\psi_{ij} > 0$. By reordering the summation, we obtain

$$\mathcal{U}(\psi) = \sum_{i=1}^{n} u_i \cdot d_i(\psi) - \sum_{r\in R} C_r\big(\ell_r(\psi)\big) = \sum_{r\in R}\Big(\ell_r(\psi)\cdot C_r'\big(\ell_r(\psi)\big) - C_r\big(\ell_r(\psi)\big)\Big)\,,$$

proving the first claim. Using the optimality condition (2) in Lemma 3.1 we get $u_i = \hat{\xi}_{ij}(\vartheta)$ for all $i \in N$, $j \in M_i$, with $\vartheta_{ij} > 0$, proving the second claim. □

EC.5. Proof of Proposition 4.1.

PROPOSITION 4.1 *For incremental cost sharing, every Nash equilibrium is optimal.*

Proof. We use Theorem 4.1 as follows. We define $\lambda = 1$ and show that $\omega_n\big(\mathcal{C}, \mathcal{D}_n, \lambda\big) \leq 0$. To see this, we bound the nominator of (4):

$$\sum_{i=1}^{n} \hat{\xi}_i(x)\,(y_i - x_i) + C\big(\ell(x)\big) - C\big(\ell(y)\big) = C'\big(\ell(x)\big)\,\big(\ell(y) - \ell(x)\big) + C\big(\ell(x)\big) - C\big(\ell(y)\big) \leq 0\,,$$

where the last inequality follows from the convexity of C. □

EC.6. Proof of Lemma 5.2.

LEMMA 5.2 *Let ξ be a marginal cost pricing method for n players. Then, $\omega_n\big(C, \xi, \lambda\big) \leq \omega_n^{mcp}(C, \lambda)$, where*

$$\omega_n^{mcp}(C,\lambda) := \sup_{\substack{x,y\in\mathbb{R}_+\\ \mu\in\{0\}\cup[\frac{1}{n},1]}} \frac{C'(x)y + C''(x)\mu x y - \lambda\Big(C'(x)x + \big(\mu^2 + \frac{(1-\mu)^2}{n-1}\big)C''(x)x^2\Big) + \lambda C(x) - C(y)}{C'(y)\cdot y - C(y)}.$$

Proof. Using the definition of $\omega_n\big(C, \xi, \lambda\big)$, all we have to show is that

$$\begin{aligned}\sum_{i=1}^{n} \hat{\xi}_i(x)\,(y_i - \lambda\,x_i) \leq \sup_{\mu\in\{0\}\cup[\frac{1}{n},1]} &\Big\{C'(\ell(x))\,y + C''(\ell(x))\,\mu\,\ell(x)\,\ell(y)\\ &- \lambda\,(C'(\ell(x))\,\ell(x) + \Big(\mu^2 + \frac{(1-\mu)^2}{n-1}\Big)\,C''(\ell(x))\,\ell(x)^2)\Big\},\end{aligned}$$

where $x, y \in R_+^n$, $\ell(x) = \sum_{i=1}^n x_i$, $\ell(y) = \sum_{i=1}^n y_i$, and ξ is marginal cost pricing with n players and cost function C. Observe that $\hat{\xi}_i(x) = C''(\ell(x))\,x_i + C'(\ell(x))$. By defining $\mu = \max_{i\in N}\Big\{\frac{x_i}{\ell(x)}\Big\} \in \big[\frac{1}{n}, n\big]$, if $\ell(x) > 0$ and $\mu = 0$, otherwise, we obtain

$$\begin{aligned}&\sum_{i=1}^{n} \hat{\xi}_i(x)\,(y_i - \lambda\,x_i) = C'(\ell(x))\,\ell(y) - \lambda\,C'(\ell(x))\,\ell(x) + C''(\ell(x))\sum_{i=1}^{n}\Big(x_i\,y_i - \lambda\,x_i^2\Big)\\ &\leq C'(\ell(x))\,\ell(y) + C''(\ell(x))\mu\,\ell(x)\,\ell(y) - \lambda\big(C'(\ell(x))\,\ell(x) + \Big(\mu^2 + \frac{(1-\mu)^2}{n-1}\Big)\,C''(\ell(x))\,\ell(x)^2\big)\,,\end{aligned}$$

where the last inequality follows from $\sum_{i\in N} x_i\,y_i \leq \mu\,\ell(x)\,\ell(y)$ and $\sum_{i\in N} x_i^2 \geq (\mu^2 + \frac{(1-\mu)^2}{n-1})\,\ell(x)^2$. Thus, the claim is proven. □

EC.7. Proof of Lemma 5.3.

LEMMA 5.3 *Consider the class* $\mathcal{M}_d := \left\{C(z) = a_d\, z^d,\ a_d \geq 0,\ d \in \{2,3,\ldots\}\right\}$. *Then, it holds that*

$$\omega_\infty^{mcp}(\mathcal{M}_d;\lambda) \leq \left(\frac{1+\mu(d-1)}{\lambda(1+\mu^2 d)}\right)^{d-1} \left(\frac{d}{d-1} + \mu - 1\right) - \frac{1}{d-1},$$

where $\mu(d) = \frac{1}{\sqrt{d-1}+1}$.

Proof. Using the definition of $\omega_\infty^{mcp}(C;\lambda)$ for $C \in \mathcal{M}_d$ we get

$$\omega_\infty^{mcp}(C;\lambda) = \sup_{\mu\in[0,1]} \sup_{\beta\geq 0} d\,\beta^{d-1} \left(\frac{1}{d-1} + \mu\right) - \lambda\,\beta^d \left(1 + \mu^2\, d\right) - \frac{1}{d-1}.$$

The unique global maximizer with respect to β is $\beta^* = \frac{1+\mu(d-1)}{\lambda(1+\mu^2 d)}$. Thus, since $C \in \mathcal{M}_d$ was arbitrary, we get

$$\omega_\infty^{mcp}(\mathcal{M}_d;\lambda) \leq \sup_{\mu\in[0,1]} \left(\frac{1+\mu(d-1)}{\lambda(1+\mu^2 d)}\right)^{d-1} \left(\frac{d}{d-1} + \mu - 1\right) - \frac{1}{d-1}.$$

The unique maximizer for this supremum is given by $\mu(d) = \frac{1}{\sqrt{d-1}+1}$. □

EC.8. Proof of Proposition 5.1.

PROPOSITION 5.1 *Let* $\mathcal{C}^{convD}$ *be a class of cost functions with a convex derivative. Consider the set* $\mathcal{G}_n(\mathcal{C}^{convD})$ *of games with at most* $n \in \mathbb{N}^*$ *players. Then,* $\rho_n(\mathcal{C}^{convD}) \leq \frac{2(n-\sqrt{n})}{\sqrt{n}(n-1)}$ *which grows as* $O(1/\sqrt{n})$.

Proof. Consider the example in the proof of Theorem 5.2 with polynomial cost functions of degree $d \in \mathbb{N}$. Let $\vartheta^d(n)$ and $\psi^d(n)$be the Nash equilibrium and the optimum profile in the game with n players, respectively. We obtain

$$\rho_n(\mathcal{C}^{convD}) \leq \lim_{d\to\infty} \frac{\mathcal{U}(\vartheta^d(n))}{\mathcal{U}(\psi^d)} = \frac{1+n\,b^2-b^2}{1+n\,b-b},\ \forall b \in [0,1].$$

Since $\frac{1+n\,b^2-b^2}{1+n\,b-b}$ has a global minimum with respect to b with value $\frac{2(n-\sqrt{n})}{\sqrt{n}(n-1)}$, the proposition is proved. □

EC.9. Proof of Lemma 5.4.

LEMMA 5.4 *Consider a symmetric game with ξ being marginal cost pricing. Then, it holds that $\omega_n\big(C,\xi,\lambda\big) \le \omega_n^{mcp,\,sym}(C,\lambda)$, where*

$$\omega_n^{mcp,\,sym}(C;\lambda) := \sup_{x,y\in\mathbb{R}_+} \frac{C'(x)\,y + C''(x)\,\frac{x\,y}{n} - \lambda\left(C'(x)\,x + C''(x)\,\frac{x^2}{n} - C(x)\right) - C(y)}{C'(y)\cdot y - C(y)}.$$

Proof. The proof is analogous to the proof of Lemma 5.2, except that for symmetric games we use a symmetric optimal profile which implies $\sum_{i\in N} x_i y_i = \frac{\ell(x)\ell(y)}{n}$. Moreover, using $\sum_{i\in N} x_i^2 \ge \frac{\ell(x)^2}{n}$ the claim follows. □

EC.10. Proof of Proposition 5.2.

PROPOSITION 5.2 *Let $\mathcal{C}^{convD}$ be the class of cost functions with a convex derivative. Consider the set $\mathcal{G}_n(\mathcal{C}^{convD})$ of symmetric games with at most $n\in\mathbb{N}^*$ players. Then, $\rho_n(\mathcal{C}^{convD}) \ge \frac{2\,n}{2\,n+1}$.*

Proof. The proof proceeds along the lines of the proof of Theorem 5.1, except that $\lambda = \frac{1+2\,n}{2\,n}$ and the values μ is replaced by $\frac{1}{n}$. Then, the only interesting difference occurs for the case $x < y$ in evaluating the following maximum:

$$\max_{\beta\in[0,1)} \left(\frac{\beta}{n} - \frac{\lambda\,\beta^2}{n} - \frac{(1-\beta)^2}{2}\right) \le \frac{1+2\,n-2\,n\,\lambda}{2\,n\,(2\,\lambda+n)}.$$

Thus, since $\lambda = \frac{1+2\,n}{2\,n}$, the claim is proven. □

EC.11. Proof of Theorem 5.3.

THEOREM 5.3 *Let $\mathcal{C}_d$ be the class of polynomial cost function with non-negative coefficients and arbitrary degree $d\in\{2,3,\ldots\}$. Consider the set $\mathcal{G}_n(\mathcal{C}_d)$ of symmetric games with n players and cost functions in $\mathcal{C}_d$. Then, $\rho_n(\mathcal{C}_d) = \frac{3}{4}$.*

Proof. Let ϑ be a Nash equilibrium profile and ψ the system optimum. Using Remark 5.3, it is sufficient to consider monomial cost functions $C_{d'}(z) = a_{d'}\,z^{d'}$, $a_j \ge 0$, for some $d' \in \{2,3,\ldots\}$. Then, we obtain

$$\omega_n^{mcp,\,sym}(C_{d'};\lambda) \le \sup_{\beta\ge 0}\, d'\left(\frac{1}{d'-1} + \frac{1}{n}\right)\beta^{d'-1} - \lambda\,\beta^{d'}\left(1 + \frac{d'}{n}\right) - \frac{1}{d'-1}.$$

The unique maximizer is $\beta^* = \frac{n+d'-1}{\lambda\,(n+d')}$. Thus, we get

$$\omega_n^{mcp,\,sym}(C_{d'};\lambda) \leq \left(\frac{n+d'-1}{\lambda\,(n+d')}\right)^{d'-1} \left(\frac{n+d'-1}{n\,(d'-1)}\right) - \frac{1}{d'-1}\,.$$

We define $\lambda = \lambda(d',n) = \left(\frac{n+d'-1}{n+d'}\right)\left(\frac{n}{n+d'-1}\right)^{-\frac{1}{d'-1}}$ implying $\omega_n^{mcp,\,sym}\big(C_{d'};\lambda(d',n)\big) = 0$. Thus, applying Lemma 5.4 and Theorem 4.1 yields $\mathcal{U}(\psi) \leq \lambda(d',n)\,\mathcal{U}(\vartheta)$ for a Nash equilibrium ϑ and optimal profile ψ. We now observe that $\lambda(d',n)$ is a decreasing function in d' and n. Hence, the worst case occurs for $d'=2$ and $n=1$ leading to the desired bound of $3/4$.

To prove that the bound is tight, we consider a resource allocation game with a single resource and cost function $C(z) = \frac{1}{2}z^2$. We consider n players with utility functions $U(\varphi_i) = \varphi_i$. Then, the following conditions hold for a Nash equilibrium ϑ: $1 - \big(\ell(\vartheta) + \vartheta_i\big) = 0 \Rightarrow \vartheta_i = 1 - \ell(\vartheta)$. Hence, we have: $\ell(\vartheta) = n\,\vartheta_i = n\,\big(1 - \ell(\vartheta)\big) \Rightarrow \ell(\vartheta) = \frac{n}{n+1}$. The total surplus evaluates to $\mathcal{U}(\vartheta) = \frac{n}{n+1} - \frac{1}{2}\frac{n^2}{(n+1)^2}$. The optimal profile ψ has value 1 and its total surplus evaluates to $\mathcal{U}(\psi) = \frac{1}{2}$. Evaluating the ratio $\frac{\mathcal{U}(\vartheta)}{\mathcal{U}(\psi)}$ proves the claim. □

EC.12. Proof of Theorem 6.1.

In order to prove Theorem 6.1, we first present the following two lemmata. The first one establishes closed-form expressions of the total surplus of a Nash equilibrium and an optimal profile.

Lemma EC.12.1. *Consider a game G in which utility functions are linear, that is, $U_i(x) = u_i \cdot x$, $u_i \geq 0$, $i \in N$. Let ψ be an optimal profile and ϑ be a Nash equilibrium. Then, ψ and ϑ generate total surplus of $\mathcal{U}(\psi) = \sum_{r\in R}\big(\ell_r(\psi)\big)^2 \cdot c_r'\big(\ell_r(\psi)\big)$ and $\mathcal{U}(\vartheta) = \sum_{r\in R}\sum_{i=1}^n \big(\vartheta_i^r\big)^2 \cdot c_r'\big(\ell_r(\vartheta)\big)$, where $c_r(\ell(\varphi)) = \frac{C_r(\ell(\varphi))}{\ell(\varphi)}$ is the per-unit cost function.*

The proof follows from Lemma 4.3. Next, along the lines of Section 5, we derive an upper bound of the quantity $\omega_n\big(C,\xi,\lambda\big)$, where ξ corresponds to average cost sharing.

Lemma EC.12.2. *For ξ being average cost pricing, it holds $\omega_n\big(C,\xi,\lambda\big) \leq \omega_n^{avg}(C,\lambda)$, where*

$$\omega_n^{avg}(C,\lambda) := \sup_{x,y\in\mathbb{R}_+}\ \sup_{\mu\in\{0\}\cup[\frac{1}{n},1]} \frac{c(x)\,y + c'(x)\,x\,y\,\mu - c(y)\,y - \lambda\left(\mu^2 + \frac{(1-\mu)^2}{n-1}\right)c'(x)\,x^2}{c'(y)\,y^2}\,. \tag{EC.1}$$

Proof. The proof proceeds along the lines of the proof of Lemma 5.2. Using the definition of $\omega_n\big(C,\xi,\lambda\big)$, it remains to show that

$$\sum_{i=1}^{n}\hat{\xi}_i(x)\,(y_i-\lambda\,x_i)+\lambda\,\ell(x)\,c(\ell(x))$$
$$\leq \sup_{\mu\in\{0\}\cup[\frac{1}{n},1]} c(\ell(x))\,y+c'(\ell(x))\,\ell(x)\,\ell(y)\,\mu-\lambda\left(\mu^2+\frac{(1-\mu)^2}{n-1}\right)c'(\ell(x))\,\ell(x)^2.$$

where $x,y\in R^n_+$, $\ell(x)=\sum_{i=1}^n x_i$, $\ell(y)=\sum_{i=1}^n y_i$, ξ is average cost pricing with n players and cost function C, and $c(\ell(x))=\frac{C(\ell(x))}{\ell(x)}$. Using the definition of average cost pricing, this reduces to

$$\sum_{i\in N}x_i\,y_i-\lambda\sum_{i\in N}x_i^2\leq \sup_{\mu\in\{0\}\cup[\frac{1}{n},1]}\ell(x)\,\ell(y)\,\mu-\lambda\left(\mu^2+\frac{(1-\mu)^2}{n-1}\right)\ell(x)^2.$$

Observe that setting $\mu=\max_{i\in N}\left\{\frac{x_i}{\ell(x)}\right\}\in\left[\frac{1}{n},n\right]$, if $\ell(x)>0$ and $\mu=0$, otherwise, we obtain $\sum_{i\in N}x_i\,y_i\leq\mu\,\ell(x)\,\ell(y)$ and $\sum_{i\in N}x_i^2\geq\left(\mu^2+\frac{(1-\mu)^2}{n-1}\right)\ell(x)^2$ proving the claim. □

Now we prove the theorem.

THEOREM 6.1 *Let $\mathcal{C}^{conv}$ be a class of convex cost functions. Consider the set $\mathcal{G}_n(\mathcal{C}^{conv})$ of games with at most $n\in\mathbb{N}$ players. Then, $\rho_n(\mathcal{C}^{conv})\geq\frac{1}{n}$.*

Proof. We first need the following simple observation

$$\omega_n^{avg}(C;\lambda)\leq\sup_{x,y\in\mathbb{R}_+}\frac{c(x)\,y+c'(x)\,x\,y-c(y)\,y-\frac{\lambda}{n}\,c'(x)\,x^2}{c'(y)\,y^2}.$$

Then, we define $\lambda=n$ and prove the claim by showing $\omega_n^{avg}(C;\lambda)\leq 0$ for $C\in\mathcal{C}^{conv}$. Let $T_x(y)=c(x)\,x+(c(x)\,x)'\,(y-x)$ be the supporting tangent of $c(y)\,y$ in x. Using $T_x(y)\leq c(y)\,y$ for all $y\geq 0$ (note that $c(x)\,x$ is a convex function), we obtain

$$c(x)\,y+c'(x)\,x\,y-c(y)\,y-c'(x)\,(x)^2=T_x(y)-c(y)\,y\leq 0.$$

□

EC.13. Proof of Theorem 6.2.

THEOREM 6.2 *Let $\mathcal{C}^{convU}$ be the class of cost functions with convex unit costs. Then, $\rho(\mathcal{C}^{convU})=\frac{4}{n+3}$.*

Proof. First, we define $\lambda = \frac{n+3}{4}$ and prove $\rho(\mathcal{C}^{convU}) \geq \frac{4}{n+3}$ by showing $\omega_n^{avg}(C;\lambda) \leq 0$.

We denote the nominator of (EC.1) by $N(x,y,\mu;C,\lambda)$. Using $T_x(y) = c(x) + c'(x)\,(y-x) \leq c(y)$ (using that c is convex), we obtain

$$N(x,y,\mu;C,\lambda) \leq c'(x)\left(x\,y\,(1+\mu) - y^2 - \lambda\left(\mu^2 + \frac{(1-\mu)^2}{n-1}\right)x^2\right).$$

Then, since for $y \in \mathbb{R}_+$ it holds

$$x\,y\,(1+\mu) - y^2 \leq \frac{(1+\mu)^2}{4n}\,x^2,$$

we obtain

$$N(x,y,\mu;C,\lambda) \leq c'(x)\,x^2\left(\frac{(1+\mu)^2}{4n} - \lambda\left(\mu^2 + \frac{(1-\mu)^2}{n-1}\right)\right).$$

Finally, the inequality

$$\frac{(1+\mu)^2}{4n} - \lambda\left(\mu^2 + \frac{(1-\mu)^2}{n-1}\right) \leq \frac{\lambda\,(n-4\,\lambda+3)}{1+4\,\lambda\,n-n}$$

for $\lambda = \frac{n+3}{4}$ proves the claim.

To prove $\rho(\mathcal{C}^{convU}) \leq \frac{4}{n+3}$ consider the following example. Assume n users share a single resource with the cost function $C(\ell(\varphi)) = \ell(\phi)$. Further, assume user 1 has the utility function $U_1(\varphi_1) = a\,\varphi_1$, while users $i = 2,\ldots,n$ have utility functions $U_i(\varphi_i) = \frac{1-a}{n-1}\,\varphi_i$, with $a = \frac{n+3}{(n+1)^2}$. A system optimum is achieved when all of the resource is allocated to user 1 resulting in a total utility of $\mathcal{U}(\psi) = \frac{a^2}{4}$. A Nash equilibrium is $\vartheta_1 = a - \frac{1}{n+1}$, $\vartheta_i = \frac{1-a}{n-1} - \frac{1}{n+1}$ for $i = 2,\ldots,n$. We obtain a total utility of $\mathcal{U}(\vartheta) = a^2 + \frac{(1-a)^2}{n-1} - \frac{n+2}{(n+1)^2}$. Thus, relative efficiency is $\frac{\mathcal{U}(\vartheta)}{\mathcal{U}(\psi)} = \frac{4}{n+3}$. □

EC.14. Proof of Theorem 6.3.

THEOREM 6.3 *Let $\mathcal{C}^{convU}$ be the class of cost functions with convex unit costs. Consider the set $\mathcal{G}_n(\mathcal{C}^{convD})$ of symmetric games with at most $n \in \mathbb{N}^*$ players. Then, $\rho(\mathcal{C}^{convD}) = \frac{4\,n}{(n+1)^2}$.*

Proof. Using similar arguments as in the proof of Lemma 5.4, we obtain

$$\omega_n(C,\xi,\lambda) \leq \omega_n^{avg,\,sym}(C;\lambda) := \sup_{x,y\in\mathbb{R}_+^2} \frac{c(x)\,y + c'(x)\,\frac{x\,y}{n} - c(y)\,y - \frac{\lambda}{n}\,c'(x)\,x^2}{c'(y)\,y^2}.$$

We define $\lambda = \frac{(n+1)^2}{4}$ and prove the theorem by showing $\omega_n^{avg,\,sym}(C;\lambda) \leq 0$. With $T_x(y) = c(x) + c'(x)\,(y-x) \leq c(y)$ (using that c is convex), we obtain

$$c(x)\,y + c'(x)\,\frac{x\,y}{n} - c(y)\,y - \frac{\lambda}{n}\,c'(x)\,(x)^2 \leq c'(x)\,\left(x\,y\,\left(1+\frac{1}{n}\right) - y^2 - \frac{\lambda}{n}\,x^2\right).$$

Then, we use that for $y \in \mathbb{R}_+$ it holds that $x\,y\,(1+\frac{1}{n}) - y^2 \leq \frac{(1+1/n)^2}{4\,n}x^2$. The choice of λ, thus, proves the claim. The upper bound follows by a simple construction and is omitted.