Multimarket Oligopolies with Restricted Market Access

Tobias Harks[1] and Max Klimm[2]

[1] Department of Quantitative Economics, Maastricht University, the Netherlands.
t.harks@maastrichtuniversity.nl
[2] Department of Mathematics, Technische Universität Berlin, Germany.
klimm@math.tu-berlin.de

Abstract. We study the existence of Cournot equilibria in multimarket oligopolies under the additional restriction that every firm may sell its product only to a limited number of markets simultaneously. This situation naturally arises if market entry is subject to a valid license and each firm holds a fixed number of licenses only, or, equivalently, if the firms' short-term assets only suffice to serve up to a certain number of markets. We allow for firm-specific market reaction functions modeling heterogeneity among products. As our main result, we show the existence of a Cournot equilibrium under the following assumptions stated informally below: (i) cost functions are convex; (ii) the marginal return functions strictly decrease for strictly increased own quantities and non-decreased aggregated quantities; (iii) for every firm, the firm-specific price functions across markets are identical up to market-specific shifts. While assumptions (i) and (ii) are frequently imposed in the literature on single market oligopolies, only assumption (iii) seems limiting. We show, however, that if it is violated, there are games without a Cournot equilibrium.

1 Introduction

Cournot's work on industrial organization [3] doubtlessly represents a landmark of economic theory and is one of the earliest reference points of game theory. To date, his model of oligopolistic competition remains a corner stone of empirical and mathematical analysis in these fields. Most of the work on the existence of equilibria in Cournot oligopolies has to make strong assumptions on the topological properties of the firms' strategy sets and their utility functions. Commonly it is assumed that the strategy space of each firm corresponds to a closed interval on the real line (and, thus, forms a convex and compact subset of a one-dimensional Euclidean space) and utilities are continuous and quasi-concave. This way, classical fixed point theorems of Kakutani [7] and adapted versions (cf. Debreu [4], Glicksberg [5]) can be applied. In the past decades, the assumptions on the quasi-concavity of the utility functions have been considerably relaxed, see Vives [18] for an excellent survey. E.g., Novshek [11] only requires that the marginal revenue of each firm is decreasing in the aggregate quantities of other firms. Starting with Topkis [15] several works (cf. Amir [1], Kukushkin [8], Milgrom and Roberts [9], Milgrom and Shannon [10], Topkis [16], Vives [17]) discovered that Tarski's fixed-point theorem (cf. [14]) yields the existence of an equilibrium if the underlying game is supermodular, that is, the strategy space forms a lattice and the marginal utility of each firm is increasing in any other firm's output. Like this, one can obtain existence results without requiring quasi-concavity of utilities.

In *multimarket oligopolies*, firms may produce quantities for a *set* of markets; see Bulow et al. [2] and Topkis [16, §4.4.3]. In the classical model of Bulow et al., each firm has a firm-specific set of markets on which positive quantities of a homogeneous good can be offered. The utility of each firm equals the profit from selling the produced goods on the markets minus the total production cost. Similar to the single market case, under the assumption that the utilities of the firms are continuous and quasi-concave in the outputs, the existence of an equilibrium follows by standard fixed-point theorems in the spirit of Kakutani. Analogously, if the underlying game is supermodular, the application of Tarski's fixed-point theorem yields the existence of an equilibrium; see Topkis [16, §4.4.3].

In this paper, we study multimarket oligopolies in which firms may only offer positive quantities on a *limited* number of markets. Such situations arise for instance if governmental policies oblige each firm to be engaged in at most a fixed number of markets at a time, e.g., by issuing a limited number of licenses to enter a market (see, e.g., Stähler and Upmann [13]). Another typical situation in which support constraints occur is when the firms' short-term assets only suffice to serve a certain number of markets.[3] We model these situations by assuming that every firm i may only choose positive production quantities for up to $k_i \in \mathbb{N}$ many markets out of a firm-specific set of markets. Formally, the restriction of serving at most k_i markets at a time with positive quantities imposes a support restriction on the vector of production quantities of firm i.

In previous work (cf. Harks and Klimm [6]), we considered a class of games in which a strategy of a player can be represented as a tuple consisting of an action and a (one-dimensional) demand quantity. Under certain regularity assumptions on the allowable class of utility functions, the main result establishes the existence of a pure Nash equilibrium. As a special case of this result it is shown that multimarket oligopolies in which each firm procures a homogeneous product only on a *single* market at a time possess an equilibrium provided that market price functions are *equal* across markets, see [6, §4]. Regarding multimarket oligopolies, in this paper we prove a much more general result showing that there exist (pure) equilibria even for *general support constraints* and *player-specific* market reaction functions (allowing for heterogeneous products). These generalizations also require a substantially different proof technique. The main proof idea of the result in [6] crucially relies on the decoupled structure of strategies. As each firm uses only a single market at a time, there are only two local effects whenever a firm changes its strategy: only the quantities of the "new"

[3] As an illustration, think of a company running several ice cream vans that sell ice cream on local beaches. In the short term, the number of vans is fixed and their number imposes an upper bound on the number of beaches that can be visited.

and the "old" market change. In the case of multimarket oligopolies with *general* support constraints, however, whenever a firm changes the current support set (even if it adds only a single market at a time to its support set), the deviating firm will adapt the quantities on all markets contained in its support set because of the coupling in the production cost. This adaption may trigger global cascading effects on possibly all markets since those firms having support sets intersecting with that of the deviating firm will change their quantities, which, in turn, triggers the adaption of the production quantities by further firms.

Our Results We study multimarket oligopolies with restrictions on the number of markets on which each firm can offer positive quantities. In our model, every firm has access to a firm-specific set of markets and is associated with a firm-specific non-decreasing and convex cost function. We allow for firm-specific market price functions modeling the price effect of heterogeneous products. The main result of this paper is an existence theorem for Cournot equilibria in multimarket oligopolies assuming that (i) the firm-specific price functions are non-increasing; (ii) the marginal return functions strictly decrease for strictly increased own quantities and non-decreased aggregated quantities; (iii) for every firm the firm-specific price functions per markets are identical up to market specific shifts. The proof of our existence result relies on a combination of ideas. We first show that for any strategy profile, if a firm can improve, there is always a *restricted improvement* that only adds a single new market to the support but also yields an improvement. We further introduce the notion of a *partial equilibrium*, a strategy profile that is stable against unilateral quantity deviations assuming *fixed* support sets. We show (using Kakutani's fixed point theorem) that partial equilibria always exist. Based on these two properties, we design an algorithm that computes an equilibrium. Our algorithm relies on iteratively computing a *partial equilibrium* and, whenever a firm can improve, this firm deviates to a *restricted best reply* defined as the best restricted improvement. After such a restricted best reply, the algorithm recomputes the partial equilibrium and reiterates. We prove that a firm-specific load vector of the partial equilibria lexicographically decreases in every iteration and, thus, the algorithm terminates. The key for proving this is to derive certain monotonicity properties of the computed partial equilibrium after executing a restricted best reply. It might seem surprising that there is enough structure on the thus computed partial equilibria given they are computed only implicitly using Kakutani's fixed-point theorem as a black box. We finally show that our existence result is "tight" in the sense that if the requirement of having essentially "identical markets per firm" is dropped, there is a game without an equilibrium. We conclude the paper by outlining an important generalization of our model.

2 The Model

In a multimarket oligopoly, there is a non-empty and finite set N of firms and a non-empty and finite set M of markets each endowed with a non-increasing firm-specific market reaction function $p_{i,m}, m \in M, i \in N$. A strategy of firm $i \in N$ is to choose a production quantity $x_{i,m}$ for each market m. Given a vector $\mathbf{x}_i = (x_{i,m})_{m \in M}$ of production quantities of firm i, the support of firm i is $S(\mathbf{x}_i) = \{m \in M : x_{i,m} > 0\}$. We impose two restrictions on the support of each firm i in each strategy profile. First, we assume that each firm is associated with a subset $M_i \subseteq M$ of markets that it can potentially procure, i.e., we require that $S(\mathbf{x}_i) \subseteq M_i$. Furthermore, we assume that there is an upper bound $k_i \in \mathbb{N}$ with $k_i \leq |M_i|$ on the number of markets that firm i may serve in a strategy profile, i.e, we require that $|S(\mathbf{x}_i)| \leq k_i$.

Formally, we derive a strategic game as follows. The set X_i of feasible strategies of firm i is defined as

$$X_i = \big\{\mathbf{x}_i = (x_{i,m})_{m \in M} \in \mathbb{R}^m_{\geq 0} : S(\mathbf{x}_i) \subseteq M_i \text{ and } |S(\mathbf{x}_i)| \leq k_i\big\}.$$

The Cartesian product $X = \bigtimes_{i \in N} X_i$ of the firms' sets of feasible strategies is the joint strategy space. An element $\mathbf{x} = (\mathbf{x}_i)_{i \in N} \in X$ is called a strategy profile. With a slight abuse of notation, for a firm i and one of its strategies $\mathbf{x}_i \in X_i$, we write $x_i = \sum_{m \in M} x_{i,m}$ for the total production quantity of firm i. Analogously, for a market m, and a strategy profile $\mathbf{x} \in X$, we write $x_m = \sum_{i \in N} x_{i,m}$ for the total quantity offered on market m under strategy profile $\mathbf{x}$. The utility of firm i under strategy profile $\mathbf{x} \in X$ is then defined as $u_i(\mathbf{x}) = \sum_{m \in M} p_{i,m}(x_m)\, x_{i,m} - c_i(x_i)$. In the remainder of the paper, we will compactly represent the strategic game by the tuple

$$G = (N, (M_i)_{i \in N}, (p_{i,m})_{i \in N, m \in M_i}, (c_i)_{i \in N}, (k_i)_{i \in N}).$$

We call G a *multimarket oligopoly with support constraints.*

We use standard game theory notation. For a player $i \in N$ and a strategy profile $\mathbf{x} \in X$, we write $\mathbf{x}$ as $(\mathbf{x}_i, \mathbf{x}_{-i})$. A Cournot equilibrium is a strategy profile $\mathbf{x} \in X$ such that no firm can improve its utility by a unilateral deviation, i.e., $u_i(\mathbf{x}) \geq u_i(\mathbf{y}_i, \mathbf{x}_{-i})$ for all $i \in N$ and $\mathbf{y}_i \in X_i$.

We impose the following assumptions on the market reaction and cost functions, respectively.

Assumption 1 *For each firm $i \in N$ the cost function $c_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is non-decreasing, convex and differentiable.*

Assumption 2 *For all $i \in N, m \in M_i$, the market reaction function $p_{i,m} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ has the following properties:*

(a) *The market reaction function $p_{i,m}$ is non-increasing.*
(b) *The return function $x \mapsto p_{i,m}(x + x_0)\, x$ is differentiable with respect to x for all residual quantities $x_0 \in \mathbb{R}_{\geq 0}$.*
(c) *There exists $\bar{x}_{i,m} > 0$ with $p_{i,m}(\bar{x}_{i,m}) = 0$.*
(d) *For all $x, x', x_0, x_0' \in \mathbb{R}_{\geq 0}$ with $x < x'$ and $x + x_0 \leq x' + x_0' \leq \bar{x}_{i,m}$ we have $\frac{\partial}{\partial x}\big(p_{i,m}(x + x_0)\, x\big) > \frac{\partial}{\partial x'}\big(p_{i,m}(x' + x_0')\, x'\big)$.*

The above Assumption 2 implies that the marginal profits of every firm are decreasing in both, the own quantity and the aggregate quantities of the competitors. Bulow et al. [2] call this property *strategic substitutes*: A more aggressive play of one firm leads to a quantity reduction of the other competing firms. Assumption 2(c) implies that for every firm, there is an upper bound on the quantity that a firm will produce, hence, the space of feasible quantity vectors can be bounded. It is a simple observation that Assumption 2(d) is, e.g., satisfied if the market reaction functions are concave, decreasing and differentiable.

Finally, we require that the set of firm-specific market reaction functions consists of "compatible" functions, that is, up to market-specific shifts the firm-specific market reaction functions must be identical.

Assumption 3 *For every market $m \in M$ there is a constant $x_{0,m} \in \mathbb{R}_{\geq 0}$ such that for all $i \in N$, there is a function $p_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $p_{i,m}(x) = p_i(x+x_{0,m})$ for all $m \in M_i$.*

This last assumption is restrictive since it requires that different markets have identical player-specific responses for equal aggregated quantities (or identical player-specific responses up to market-specific shifts). In Section 5, we show, however, that this assumption is necessary in the sense that, if it is relaxed, there are multimarket oligopolies with support constraints not possessing a Cournot equilibrium.

In the remainder of this paper, whenever a game satisfies Assumption 3, we slightly abuse notation as we set $x_m = x_{0,m} + \sum_{i \in N} x_{i,m}$ for all $m \in M$. This allows us to write the utility of player i in strategy profile $\mathbf{x}$ as $u_i(\mathbf{x}) = \sum_{m \in M} p_i(x_m)\, x_{i,m} - c_i(x_i)$.

3 Existence of a Cournot Equilibrium

To show the existence of a Cournot equilibrium in multimarket oligopolies with support constraints we first introduce a relaxation of the Cournot equilibrium concept which we call *partial equilibrium*. Roughly speaking, a strategy profile is a partial equilibrium, if it is a Cournot equilibrium in a game in which the set of accessible markets of each firm i is restricted to a certain *active set* $Q_i \subseteq M_i$ of cardinality $|Q_i| = k_i$. We show – using standard fixed point arguments – that partial equilibria always exist and are essentially unique when fixing the underlying active set vector $(Q_i)_{i \in N}$.

For a given partial equilibrium, we further show that if a firm can improve, there is always a *restricted improvement* that exchanges only one market in the deviating firm's active set. On these two ingredients we design an algorithm that iteratively computes a *partial equilibrium* and, whenever possible let a firm deviate to a restricted improvement. The main result of this paper shows that the algorithm terminates after finitely many iterations and outputs a Cournot equilibrium.

Partial Equilibria. For a firm i, we call a set $Q_i \subseteq M_i$ of k_i markets an *active set* of firm i.

Definition 1 (Partial Equilibrium). *A strategy profile* $\mathbf{x}$ *is a* partial equilibrium*, if for each* $i \in N$ *there is an active set* Q_i *such that* $S_i(\mathbf{x}_i) \subseteq Q_i$ *and* $u_i(\mathbf{y}_i, \mathbf{x}_{-i}) > u_i(\mathbf{x})$ *for all* $\mathbf{y}_i \in X_i$ *with* $S(\mathbf{y}_i) \subseteq Q_i$*.*

If $\mathbf{x}$ and $(Q_i)_{i\in N}$ satisfy the conditions of Definition 1, we active sets say that $\mathbf{x}$ is a partial equilibrium for the active set vector $(Q_i)_{i\in N}$. We proceed to prove that for each active set vector $(Q_i)_{i\in N}$ there is a partial equilibrium for $(Q_i)_{i\in N}$.

Lemma 1 (Existence of a Partial Equilibrium). *Let* G *be a multimarket oligopoly with support constraints for which Assumptions 1 and 2 hold. For each vector of active sets* $(Q_i)_{i\in N}$*, there is a partial equilibrium* $\mathbf{x}$ *for* $(Q_i)_{i\in N}$*.*

The proof follows by applying classical fixed point results for concave games with convex and compact strategy spaces [5,7].

We proceed to prove that for a fixed active set vector $(Q_i)_{i\in N}$, the partial equilibria for $(Q_i)_{i\in N}$ are essentially unique. In order to prove this result, we need the following lemma that expresses necessary optimality conditions for a partial equilibrium.

Lemma 2. *Let* G *be a multimarket oligopoly with support constraints for which Assumptions 1 and 2 hold. Let* $\mathbf{x}$ *be a partial equilibrium for* $(Q_i)_{i\in N}$*. Then, the following conditions hold for all* $i \in N$ *and all* $m \in Q_i$*:*

(a) $\frac{\partial}{\partial x_{i,m}} u_i(\mathbf{x}) \leq 0$.
(b) $\frac{\partial}{\partial x_{i,m}} u_i(\mathbf{x}) = 0$*, if* $x_{i,m} > 0$.

We are now ready to prove that a given vector $(Q_i)_{i\in N}$ of active sets, the partial equilibrium for the active sets $(Q_i)_{i\in N}$ is essentially unique in the sense that all such equilibria give rise to the same aggregated production quantities on all markets.

Lemma 3 (Uniqueness of Partial Equilibria). *Let* G *be a multimarket oligopoly with support constraints for which Assumptions 1 and 2 hold. Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two partial equilibria for the active set vector* $(Q_i)_{i\in N}$*. Then,* $x_m = y_m$ *for all* $m \in M$*.*

The proof can be found in the full version.

Restricted Improvements. For a partial equilibrium $\mathbf{x}$ for the active sets vector $(Q_i)_{i\in N}$, we introduce the notion of a *restricted improvement* from $\mathbf{x}$.

Definition 2 (Restricted Improvement, Restricted Best Reply). *Let* G *be a multimarket oligopoly with support constraints and let* $\mathbf{x}$ *be a partial equilibrium for the vector of active sets* $(Q_i)_{i\in N}$*.*

1. *A* restricted improvement *for firm i is a strategy $\mathbf{z}_i \in X_i$ with $|S(\mathbf{z}_i) \setminus Q_i| \leq 1$ and $u_i(\mathbf{z}_i, \mathbf{x}_{-i}) > u_i(\mathbf{x})$.*
2. *A* restricted best reply *maximizes $u_i(\cdot, \mathbf{x}_{-i})$ among all restricted improvements.*

Note that a restricted best reply of player $i \in N$ to $\mathbf{x}$ need not always exist. If it exists, say $\mathbf{z}_i \in X_i$, it always satisfies $u_i(\mathbf{z}_i, \mathbf{x}_{-i}) > u_i(\mathbf{x})$.

Equilibrium Existence. We are now ready to state an algorithm that actually computes an equilibrium for multimarket oligopolies with support constraints provided Assumptions 1–3 are satisfied.

The algorithm starts with arbitrary active set vector $(Q_i)_{i \in N}$ and computes a partial equilibrium $\mathbf{x}$ for $(Q_i)_{i \in N}$. Here, we assume that an oracle outputs an equilibrium (or, we apply Rosen's continuous best response dynamics, which are guaranteed to converge under rather mild conditions on utility functions [12]).

As long as there is a player i that can improve its utility by deviating from $\mathbf{x}$, the algorithm computes a restricted best reply in which only one market enters the active set of firm i. Then, a partial equilibrium is recomputed and the algorithm reiterates.

Algorithm 1
Input: $G = (N, (M_i)_{i \in N}, (p_i)_{i \in N}, (c_i)_{i \in N}, (k_i)_{i \in N})$
Output: Cournot equilibrium $\mathbf{x}$

1. Choose an active set vector $(Q_i)_{i \in N}$ arbitrarily.
2. Compute a partial equilibrium $\mathbf{x}$ for $(Q_i)_{i \in N}$.
3. If there is a firm $i \in N$ who can improve unilaterally,
   (a) compute a restricted best reply $\mathbf{z}_i \in X_i$
   (b) choose an active set $Q_i' \supseteq S(\mathbf{z}_i)$ with $|Q_i' \setminus Q_i| = 1$ arbitrarily
   (c) $Q_i \leftarrow Q_i'$
   (d) proceed with (2).
4. Else, output $\mathbf{x}$.

Theorem 4. *Let G be a multimarket oligopoly with support constraints for which market reaction functions and cost functions satisfy Assumptions 1–3. Then, Algorithm 1 computes a Cournot equilibrium for G.*

4 Proof of the Theorem

In this section, we present a formal proof of Theorem 4. The proof consists of two steps showing that Algorithm 1 is correct and that it terminates.

For the remainder of this section we consider a multimarket oligopoly with support constraints G that satisfies Assumptions 1–3. Recall that by Assumption 3, for all $i \in N$ and $m \in M_i$ we can represent the market price function $p_{i,m}$ by a single function p_i (see the input of Algorithm 1).

Correctness of the Algorithm. For the correctness of the algorithm, we only have to show that Step (3a) is well-defined, i.e., whenever there is a unilateral improvement for some firm $i \in N$, then, there is also a restricted improvement for firm i. The proof can found in the full version.

Lemma 4 (Existence of restricted improvements). *Let* $\mathbf{x}$ *be a partial equilibrium of* G *for the active set vector* $(Q_i)_{i\in N}$*. If firm* i *can improve unilaterally, then there exists a restricted improvement* $\mathbf{z}_i \in X_i$*.*

Termination of the Algorithm. We finally have to show that Algorithm 1 terminates. We do this by proving a series of lemma showing that whenever a partial equilibrium or a restricted best response is computed, a vector-valued potential monotonically decreases.

First, we show that whenever a firm plays a restricted best reply in which one market enters its support, then the total production quantity on the market that entered the support after the best reply is strictly smaller than the production quantity on the market that left the active set. Furthermore, for all markets that are contained in the support set of the deviating firm both before and after the best reply, the quantity offered by the deviating firm may only decrease.

Lemma 5. *Let* $\mathbf{x}$ *be a partial equilibrium of* G *for the active set vector* $(Q_i)_{i\in N}$*. Let* $\mathbf{y}_i$ *be a restricted improvement of firm* i *and let* $r \in M_i$*,* $s \in Q_i$ *be such that* $S(\mathbf{y}_i) \subseteq (Q_i \setminus \{s\}) \cup \{r\}$*. Then, the following properties hold:*

(a) $x_r - x_{i,r} + y_{i,r} < x_s$,
(b) $x_m - x_{i,m} + y_{i,m} \leq x_m$ *for all* $m \in M \setminus \{r, s\}$.

Proof. We first prove (a). For a contradiction, assume $x_r - x_{i,r} + y_{i,r} \geq x_s$. We distinguish the following three cases:

First case $y_{i,r} > x_{i,s}$. As $\mathbf{x}$ is a partial equilibrium Lemma 2 implies $0 \geq \frac{\partial}{\partial x_{i,s}}\big(p_i(x_s)\, x_{i,s}\big) - c_i'(\mathbf{x}_i)$. Since the strategy $\mathbf{y}_i$ is a restricted best reply and $y_{i,r} > x_{i,s} \geq 0$, we get $\frac{\partial}{\partial y_{i,r}}\big(p_i(x_r - x_{i,r} + y_{i,r}) y_{i,r}\big) = c_i'(y_i)$. We obtain

$$c_i'(x_i) \geq \frac{\partial}{\partial x_{i,s}}\big(p_i(x_s)\, x_{i,s}\big) > \frac{\partial}{\partial y_{i,r}}\big(p_i(x_r - x_{i,r} + y_{i,r})\, y_{i,r}\big) = c_i'(y_i). \quad (1)$$

Using that c_i is convex, we derive that $y_i < x_i$. If $k_i = 1$, this is a contradiction to $y_{i,r} > x_{i,s}$. If, on the other hand, $k_i > 1$ there is another market $\tilde{m} \in Q_i \setminus \{s\}$ with $y_{i,\tilde{m}} < x_{i,\tilde{m}}$. We then obtain along the same lines

$$c_i'(y_i) \geq \frac{\partial}{\partial y_{i,\tilde{m}}}\big(p_i(x_{\tilde{m}} - x_{i,\tilde{m}} + y_{i,\tilde{m}})\, y_{i,\tilde{m}}\big) > \frac{\partial}{\partial x_{i,\tilde{m}}}\big(p_i(x_{\tilde{m}})\, x_{i,\tilde{m}}\big) = c_i'(x_i),$$

which contradicts (1).

Second case $y_{i,r} = x_{i,s}$. We first show that firm i does not change its supplied quantity on all markets used in both strategies $\mathbf{x}_i$ and $\mathbf{y}_i$, i.e., $x_{i,m} = y_{i,m}$ for all markets $m \in Q_i \setminus \{s\}$. For the sake of a contradiction, let us assume that there is a market $\tilde{m} \in Q_i \setminus \{s\}$ with $x_{i,\tilde{m}} \neq y_{i,\tilde{m}}$. We distinguish two cases.

If $x_{i,\tilde{m}} < y_{i,\tilde{m}}$, we use that $\mathbf{x}$ is a partial equilibrium and that $\mathbf{y}_i$ is a restricted improvement and obtain

$$\begin{aligned} 0 &\geq \frac{\partial}{\partial x_{i,\tilde{m}}} \big(p_i(x_{\tilde{m}})\, x_{i,\tilde{m}}\big) - c_i'(x_i) \\ &> \frac{\partial}{\partial y_{i,\tilde{m}}} \big(p_i(x_{\tilde{m}} - x_{i,\tilde{m}} + y_{i,\tilde{m}})\, y_{i,\tilde{m}}\big) - c_i'(y_i) = 0, \end{aligned} \tag{2}$$

which is a contradiction! If, on the other hand, $x_{i,\tilde{m}} < y_{i,\tilde{m}}$, we obtain the same contradiction as in (2), but with all inequality signs reversed. We conclude that $x_{i,m} = y_{i,m}$ for all markets $m \in Q_i \setminus \{s\}$. This implies

$$u_i(\mathbf{y}_i, \mathbf{x}_{-i}) - u_i(\mathbf{x}) = p_i(x_r - x_{i,r} + y_{i,r})\, y_{i,r} - p_i(x_s)\, x_{i,s} \leq 0.$$

Thus, firm i does not improve, a contradiction to the fact that $\mathbf{y}_i$ is a restricted best response of firm i.

Third case $y_{i,r} < x_{i,s}$. Consider the strategy $\mathbf{w}_i = (w_{i,m})_{m \in M}$ in which firm i plays as in strategy $\mathbf{y}_i$ except that the quantity $y_{i,r}$ is put on market s instead of market r and market r is not served at all. Formally,

$$w_{i,m} = \begin{cases} y_{i,m}, & \text{if } m \in Q_i \setminus \{s\} \\ y_{i,r}, & \text{if } m = s \\ 0, & \text{otherwise.} \end{cases}$$

We observe that $x_s - x_{i,s} + w_{i,s} < x_s$ as $w_{i,s} = y_{i,r} < x_{i,s}$. We obtain

$$\begin{aligned} u_i(\mathbf{w}_i, \mathbf{x}_{-i}) &= \sum_{m \in M} p_i(x_m - x_{i,m} + w_{i,m})\, w_{i,m} - c_i(\mathbf{w}_i) \\ &= u_i(\mathbf{y}_i, \mathbf{x}_{-i}) - p_i(x_r - x_{i,r} + y_{i,r})\, y_{i,r} + p_i(x_s - x_{i,s} + w_{i,s})\, w_{i,s} \\ &> u_i(\mathbf{y}_i, \mathbf{x}_{-i}) > u_i(\mathbf{x}), \end{aligned} \tag{3}$$

where the first inequality in (3) follows from

$$x_s - x_{i,s} + w_{i,s} < x_s \leq x_r - x_{i,r} + y_{i,r}$$

and the assumption that market reaction functions are non-increasing. As $S(\mathbf{y}_i') \subseteq S(\mathbf{x}_i)$ this is a contradiction to the fact that $\mathbf{x}$ is a partial equilibrium.

We proceed to show part (b) of the statement of the lemma. Let us assume for a contradiction that there is a market $\tilde{m} \in M \setminus \{r, s\}$ with $x_{\tilde{m}} - x_{i,\tilde{m}} + y_{i,\tilde{m}} > x_{\tilde{m}}$ and, hence, $y_{i,\tilde{m}} > x_{i,\tilde{m}}$. It follows that

$$c_i'(x_i) \geq \frac{\partial}{\partial x_{i,\tilde{m}}} \big(p_i(x_{\tilde{m}})\, x_{i,\tilde{m}}\big) > \frac{\partial}{\partial y_{i,\tilde{m}}} \big(p_i(x_{\tilde{m}} - x_{i,\tilde{m}} + y_{i,\tilde{m}})\, y_{i,\tilde{m}}\big) = c_i'(y_i), \tag{4}$$

which implies together with the convexity of c_i that $y_i < x_i$. This implies that at least one of the following two cases holds: (i) $y_{i,r} < x_{i,s}$; or (ii) there is a market $m \in Q_i \setminus \{s\}$ with $y_{i,m} < x_{i,m}$.

We proceed to derive contradictions for both cases. First, suppose that case (i) holds. Using $x_r - x_{i,r} + y_{i,r} < x_s$ from the first part of the statement of this lemma, we obtain

$$c_i'(y_i) \geq \frac{\partial}{\partial y_{i,r}}\big(p_i(x_r - x_{i,r} + y_{i,r})\, y_{i,r}\big) > \frac{\partial}{\partial x_{i,s}}\big(p_i(x_s)\, x_{i,s}\big) = c_i'(x_i), \qquad (5)$$

a contradiction to (4).

Next, suppose that (ii) holds, i.e., there is a market $m \in Q_i \setminus \{s\}$ with $y_{i,m} < x_{i,m}$ and, thus, $x_m - x_{i,m} + y_{i,m} < x_m$. The same calculations as in (5) where we replace r and s by m give a contradiction to (4). □

Second, we show that whenever a firm plays a restricted improvement in which one market enters its active set, then after recomputing a partial equilibrium, the total quantity offered on each market may only decrease. The proof can be found in the full version.

Lemma 6. *Let* $\mathbf{x}$ *be a partial equilibrium of* G *for the active set vector* $(Q_i)_{i \in N}$*,* $\mathbf{y}_i$ *be a restricted improvement of firm* i *with* $S(\mathbf{y}_i) \subseteq Q_i'$*,* $Q_i' = (Q_i \cup \{r\}) \setminus \{s\}, s \in Q_i$*,* $r \in M_i \setminus Q_i$*. Let* $(\tilde{\mathbf{y}}_i, \tilde{\mathbf{x}}_{-i})$ *be a partial equilibrium for the active set vector* $(Q_i')_{i \in N}$ *where* $Q_j' = Q_j$ *for all* $j \in N \setminus \{i\}$*. Then, the following two properties hold:*

(a) $\tilde{x}_r - \tilde{x}_{i,r} + \tilde{y}_{i,r} \leq x_r - x_{i,r} + y_{i,r}$.
(b) $\tilde{x}_m - \tilde{x}_{i,m} + \tilde{y}_{i,m} \leq x_m$ *for all* $m \in M \setminus \{r\}$.

We are now ready to prove that Algorithm 1 terminates.

Lemma 7 (Termination). *Algorithm 1 terminates.*

Proof. To prove that Algorithm 1 terminates, we consider the function

$$L : \bigtimes_{i \in N} 2^{M_i} \to \bigtimes_{i \in N} \mathbb{R}^2_{\geq 0},\ (Q_j)_{j \in N} \mapsto \big(L_i((Q_j)_{j \in N})\big)_{i \in N},\ \text{with}$$

$$L_i\big((Q_j)_{j \in N}\big) = \Big(L_i^1\big((Q_j)_{j \in N}\big), L_i^2\big((Q_j)_{j \in N}\big)\Big) = \Big(\max_{m \in Q_i} x_m,\ \big|\arg\max_{m \in Q_i} x_m\big|\Big),$$

for all $i \in N$, where $\mathbf{x}$ is an arbitrary partial equilibrium for the active set vector $(Q_i)_{i \in N}$. In words, L maps each active set vector $(Q_i)_{i \in N}$ to the vector that contains for each player (under a partial equilibrium $\mathbf{x}$ for the active set vector $(Q_i)_{i \in N}$) the tuple of the maximum aggregated supply that firm i experiences among the markets contained in Q_i and the number of markets for which this maximum is attained. Note that L is well-defined as, by Lemma 3, for a given active set vector $(Q_i)_{i \in N}$ the aggregated demands for all markets are unique for all partial equilibria for $(Q_i)_{i \in N}$.

Let us denote by $\tilde{L}\big((Q_j)_{j \in N}\big)$ the vector that contains the $|N|$ tuple of $L\big((Q_j)_{j \in N}\big)$ in non-decreasing lexicographical order, i.e.,

$$\begin{aligned} &\tilde{L}_i^1\big((Q_j)_{j \in N}\big) \geq \tilde{L}_{i+1}^1\big((Q_j)_{j \in N}\big), \\ \text{and}\quad &\tilde{L}_i^2\big((Q_j)_{j \in N}\big) \geq \tilde{L}_{i+1}^2\big((Q_j)_{j \in N}\big),\ \text{if } \tilde{L}_i^1\big((Q_j)_{j \in N}\big) = \tilde{L}_{i+1}^1\big((Q_j)_{j \in N}\big). \end{aligned}$$

We claim that $\tilde{L}$ decreases lexicographically during the execution of Algorithm 1. To see this, fix an active set vector $(Q_i)_{i\in N}$ with $Q_i \subseteq M_i$ for all $i \in N$ and an arbitrary partial equilibrium $\mathbf{x}$ for $(Q_i)_{i\in N}$. If there is no firm with a profitable unilateral deviation, then there is nothing left to show as we have reached a Cournot equilibrium. So, let us assume that there is a firm i with a strategy $\mathbf{y}_i \in X_i$ such that $u_i(\mathbf{y}_i, \mathbf{x}_{-i}) > u_i(\mathbf{x})$. Lemma 4 implies that the strategy $\mathbf{z}_i$ chosen in Line (3a) of Algorithm 1 yields also an improvement of firm i. We denote the partial equilibrium recomputed in Line (2) of Algorithm 1 by $(\tilde{\mathbf{z}}_i, \tilde{\mathbf{x}}_i)$.

For all $m \in M \setminus \{r\}$ we obtain $\tilde{x}_m - \tilde{x}_{i,m} + \tilde{z}_{i,m} \leq x_m$ using Lemma 6. Furthermore, we obtain

$$\tilde{x}_r - \tilde{x}_{i,r} + \tilde{z}_{i,r} z \leq x_r - x_{i,r} + y_{i,r} < x_s, \tag{6}$$

where the first inequality follows from Lemma 6 and the second inequality follows from Lemma 5. For firm i, the tuple $L_i\big((Q_i)_{i\in N}\big)$ does decrease because firm i leaves market s with maximal aggregated quantity and using (6) the aggregated quantity on the new market r settles strictly below the old aggregated quantity on s. For all firms $j \in N \setminus \{i\}$ with $\max_{m\in Q_j} x_m \geq x_s$, we conclude that the tuple $L_j\big((Q_i)_{i\in N}\big)$ does not increase since the maximum was not attained at s and s is the only market on which the aggregated quantity increases. Finally, for all firms $j \in N \setminus \{i\}$ with $\max_{m\in Q_j} x_m < x_s$ we observe that $\max_{m\in Q_j} x_m < x_s$ since the aggregated quantity on r does not increase beyond x_s as shown in (6). We conclude that $\tilde{L}$ decreases lexicographically.

Since there are only finitely many active set vectors $(Q_i)_{i\in N}$, Algorithm 1 terminates after a finite number of steps and outputs a Cournot equilibrium. □

5 Violation of Assumptions

In this section, we show that our assumptions on the market price functions are necessary conditions in the sense that if one of them is violated, a Cournot equilibrium may fail to exist. Since Assumption 2 frequently appears in the literature on Cournot equilibria and some kind of regularity of the market price functions is already necessary for games with a single market (cf. Novshek [11]), we here show only the necessity of the critical Assumption 3.

Proposition 1. *There is a multimarket oligopoly with support constraints for which market reaction functions and cost functions satisfy Assumptions 1–2 that does not admit a Cournot equilibrium.*

The proof can be found in the full version.

6 Conclusions

We studied multimarket oligopolies in which players face a bound on the number of markets they can be engaged in simultaneously. We assumed that the firms' cost functions are convex and the player-specific market reaction functions are concave. We proved that a Cournot equilibrium is guaranteed to exist provided that the player-specific market reaction functions on the markets are identical up to a market-specific shift in the argument. While this condition seems may seem very demanding, we further showed that if this assumption is violated, a Cournot equilibrium need not exist. We see this as a first step towards a better understanding of multimarket oligopolies with market access restrictions.

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