One-Deviation Principle in Coalition Formation

Hannu Vartiainen†

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Abstract

We study equilibrium coalitional strategies in the general framework of Chwe (1994). A coalitional strategy assigns, for any history of play, at most one active coalition and, if no coalition is active, then the outcome on the table is implemented and the game ends. A one-deviation property is imposed on feasible strategies which requires that (i) the active coalition will not benefit from a single deviation, (ii) if there is no active coalition, then no coalition benefits from becoming active. Strategies meeting the equilibrium condition are characterized. Moreover, an equilibrium is shown to exist. Finally, the results are compared to some existing theories of coalition formation.

Keywords: one-deviation principle, coalition formation

JEL: C71, C72.

1 Introduction

Dating back to Harsanyi (1974), coalitional solutions have been criticized for not dealing with counterfactuals appropriately.¹ That is, in answering satisfactorily - without ad hoc assumptions - what will happen once a coalitional action has taken place. Should a deviation be followed by further deviations? Uncertainty of this leads to a well known prediction problem that threatens the validity of the solution concept itself: if the deviant coalition is not able to predict the consequences of its deviation, how should it know when to deviate?² The problem is particularly acute when the core is empty, which is often the case.

A coalitional theory should predict which coalitions should activate and how. The conceptual problem stems from the requirement that the players should anticipate any further deviations following the initial deviation.

¹I thank Hannu Salonen for useful discussions and the participants of the PCRC conference in Mariehamn for comments.
²Aumann and Maschler (1974) is a case in point.
Hence, they need to know where the theory predicts that the sequence of deviations leads to. But then there is a circularity in the definition of the theory since the same theory should predict which outcome were stable in the first place. Hence a well functioning theory of coalitional behavior necessarily has a form of a consistency criterion. A prime example of this type of a solution is the vNM stable set.

A natural approach is to demand dynamic consistency in the sense of dynamic non-cooperative solution concepts. But these consistency criteria do not seem to work well either. Consistency criteria seem to be too tightly formulated to guarantee existence (e.g. Xue, 1998; Mariotti, 1997), or they are very conservative by only removing outcomes with caution (Chwe 1998; Herings et al. 2008). Konishi and Rey (2003) is an exception. Their model assumes equilibrium reasoning and guarantees the existence of the solution. However, the existence requires randomization whose status is arguably less clear in coalition framework that in noncooperative theory. In fact, Barbera and Gerber (2007) argue that there is no internally consistent solution a natural class of cooperative models.

This paper develops a new coalitional solution concept that is based on the well known one-deviation principle. We focus on the canonical coalitional model by Chwe (1994), where coalitions of players may challenge the prevailing status quo outcome by demanding new outcomes. Our equilibrium argument is based on two observations: (i) The commitment problem underlying the coalitional model is analogous to the commitment problem in simple recursive games. Hence the same noncooperative solution is appropriate for both class of games. This guarantees the internal consistency of the coalitional solution. (ii) The previous models of coalition formation assume implicitly stationary, i.e. history independent coalitional strategies (including Barbera and Gerber, 2007). History dependency enlarges the set of possible equilibrium strategies in a natural way without relaxing tight consistency criteria.

Our aim is to fill the gap by studying coalitional strategies that meet a natural one-deviation restriction and are history dependent. Thus a new solution is suggested. Simple recursive games are a special case of our analysis.

We show that a strategy that a coalitional strategy that always implements an outcome in finite time and meets the one-deviation property exists. The equilibrium play paths are characterized and the equilibrium outcomes specified. Finally, the models are compared to some existing coalitional equilibrium concepts of Konishi and Ray (2003), Chwe’s (1994), and Xue (1998).

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3 Perfect information stochastic games where players’ payoffs depend on the node at which the play is stopped.
Literature  

Chwe’s (1994) framework allows description of coalitional game form in the sense that what may happen after a coalitional move is built into the structure of the game. The framework permits many interpretations, including classical cooperative games, networks, clubs, etc.. Chwe’s system is, however, difficult to analyze as it is a graph without a clear recursive structure. His solution, the largest consistent set, has been used widely (e.g. Page et al., 2005) even though it has been argued to be too permissive (Xue, 1998).

Ray and Vohra (1997) is a seminal paper analyzing commitment and consistency in a coalitional framework. Barbera and Gerber (2001), Diamantoudi and Xue (2007) are other recent contributions. Greenberg (1991) contains many insights that the later literature has greatly benefited and been inspired of. This paper is no exception.

2  Coalitional game

A coalitional game due to Chwe (1994) is defined by a list

\[ \Gamma = \langle N, X, (F_S)_{S \subseteq N}, (\succ_i)_{i \in N} \rangle, \]

where \( N \) is a finite set of players, \( X \) is a nonempty finite set of nodes or outcomes, an choice set \( F_S : X \to 2^X \) specifies the set of actions \( F_S(x) \subseteq X \) available to a coalition \( S \subseteq N \) at node \( x \in X \).

We assume \( x \in \cap_S F_S(x) \) for all \( x \), reflecting the idea that each \( S \) can choose to be not active. Each player \( i \in N \) has a preference relation \( \succ_i \) over the set of outcomes \( X \).

The set up is flexible and has many interpretations. In the language of cooperative game theory, an outcome \( x \) would be the description of a coalition structure as well as a vector of payoffs accruing to each player. In strategic form games, an outcome would represent a profile of actions taken by the players in the stage game.

The game is played in the following manner: There is the initial status quo \( x^* \). At any stage \( t = 0, 1, \ldots \), there is an outcome \( x \) that is the current status quo. The status quo can be changed by a coalition. Only one coalition may be active at a time. If coalition \( S \) is the active one, and chooses \( y \in F_S(x) \), then \( y \) becomes the new status quo at stage \( t + 1 \). If no coalition is active, then \( x \) is implemented.

The reason for why only one coalition may be active at a time is that game is meant to describe the coalition formation process. The graph \( (X, (F_S)_{S \subseteq N}) \) should reflect not only what are the feasible actions of a coalition at a given node, but also what the physical consequences of its activity. The graph should be interpreted as the primitive of the model, its "game form", and the activation plan of the coalitions as their strategy which should, presumably, meet the desired stability properties.

\(^4\)Chwe (1994) does not assume finite outcome space.
We make the following simplifying assumption: For all $x, y \in X$ such that $x \neq y$, there is at most one coalition such that $y \in F_S(x)$. Such coalition is denoted by $S(y, x)$.

The only role of the assumption is to reduce the expositional burden.\(^5\)

To see why the assumption is without loss of generality, note that it only requires that each outcome is indexed by the coalition whose activity brought the outcome on the table.\(^6\) Hence the assumption makes no restrictions on the underlying physical structure and, in particular, it does not affect the results.

**Paths** A path is a finite sequence $(x_0, ..., x_K) \in \bigcup_{k=1}^\infty X^k$ such that $x_{k+1} \in \bigcup F_S(x_k) \setminus \{x_k\}$, for all $k = 0, ..., K$. The length of the path $(x_0, ..., x_K)$ is $K$. Denote the set of all paths by $\mathcal{X}$, and the set of paths that start from node $y$ by $\mathcal{X}_y = \{(x_0, ..., x_K) \in \mathcal{X}: x_0 = y\}$.

A path is abbreviated by $\bar{x} = (x_0, ..., x_K)$. By our expositional assumption, a path $\bar{x}$ also implicitly defines the coalitions that activate along the play.

Our notational conventions concerning paths are: Generic components of $\bar{x}$, $\bar{y}$, and $\bar{z}$ are denoted $x$, $y$, and $z$, respectively. The path that is obtained by truncating $\bar{x} = (x_0, ..., x_K) \in X$ at the $k^{\text{th}}$ step is $\bar{x}_k = (x_0, ..., x_k)$. The path that is obtained by joining a truncated path $\bar{x}_k = (x_0, ..., x_k)$ and a path $\bar{y} = (y_0, ..., y_L)$ is $\bar{x}_k\bar{y} = (x_0, ..., x_k, y_0, ..., y_L)$. Then also $\bar{x}_k\bar{y}\bar{z} = (x_0, ..., x_k, y_0, ..., y_L, z_0, ..., z_M)$, where $\bar{z} = (z_0, ..., z_M)$, and so forth.

**Strategies** Denote the set of finite histories by $H := \mathcal{X}^*$. A coalitional strategy is a function $\sigma : H \to X$ such that $\sigma(h, x) \in \bigcup F_S(x)$ for all $(h, x) \in H$. By Assumption A, if $\sigma(h, x) = y \neq x$, then $y$ must have been proposed by coalition $S(y, x)$, and if $\sigma(h, x) = x$, then no coalition is active. Thus, due to Assumption A, a coalitional strategy implicitly specifies, for each history, which coalition is active and, if an active coalition exists, which choice it makes. The status quo outcome is chosen if no coalition is active.

Let $\sigma^0(h) = \sigma(h)$ and $\sigma^t(h) = \sigma(h, \sigma^0(h), ..., \sigma^{t-1}(h))$, for all $t = 1, ...$. Denote by $\bar{\sigma}(h)$ the sequence of nodes induced by strategy $\sigma$ from history $h$ onwards: $\bar{\sigma}(h) = (\sigma^0(h), \sigma^1(h), ...)$

If there is $T$ such that $\sigma^T(h) = \sigma^{T+1}(h)$, then $\bar{\sigma}(h, x_0) = (x_0, ..., x_T)$ where $x_t = \sigma^t(h)$, for $t = 0, ..., T$.

\(^5\)The assumption allows presenting the histories of play in terms of the nodes alone as they implicitly convey information of the active coalitions. Otherwise, a history should also specify which coalitions have been active along the play path.

\(^6\)Thus if $Y$ is the underlying physical outcome space, then $X = Y \times (2^N \setminus \{\emptyset\})$. 

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We concentrate on strategies that implement an outcome in finite time. Denote the final element of the path \((x_0, ..., x_K)\) by
\[
\mu([x_0, ..., x_K]) = x_K.
\]
We say that the strategy \(\sigma\) is well defined if \(\mu(\bar{\sigma}(h))\) exists for all \(h \in H\), i.e., an outcome is implemented in finite time after any history. Then \(\mu(\bar{\sigma}(h, x))\) is the outcome implemented when a well defined strategy \(\sigma\) is followed after history \((h, x)\), and \(\mu(\bar{\sigma}(h, x, a))\) is the outcome that becomes implemented if coalition \(S\) chooses action \(a \in F(x)\) and \(\sigma\) is followed thereafter:
\[
\mu(\bar{\sigma}(h, x, a)) = \begin{cases} \mu(\bar{\sigma}(h, x, y)), & \text{if } a = y \neq x, \\ x, & \text{if } a = x. \end{cases}
\]
(1)
In particular, if \(a = \sigma(h)\), then \(\mu(\bar{\sigma}(h, a)) = \mu(\bar{\sigma}(h))\).

**Solution**  We use the following notation for group preferences. For any \(S \subseteq N\), and \(x, y \in X\),
\[
y \succ_S x \text{ if } y \succ_i x, \text{ for all } i \in S.
\]
Our focus is on agents that can compute in finite time which outcome will become implemented given the coalitional strategies. Hence we confine attention on strategies that are well defined. Our primary question is whether the idea of implementing outcomes in finite time is consistent with equilibrium reasoning. Our equilibrium condition is the following.

**Definition 1 (ODP)** A well defined coalitional strategy \(\sigma\) satisfies the one-deviation property (ODP) if, for all \((h, x) \in H\),
1. \(\sigma(h, x) = x\) implies \(\mu(\bar{\sigma}(h, x, y)) \not\in S(y, x)\), for all \(y \in \cup S F_S(x)\).
2. \(\sigma(h, x) = y \neq x\) implies \(\mu(\bar{\sigma}(h, x, z)) \not\in S(y, x)\), for all \(z \in F_S(y, x)(x)\).

In words, after any history, 1. if the status quo outcome \(x\) becomes implemented, then it is not profitable for any coalition to become activated and to demand another outcome \(y\), given what follows after the activation. 2. If coalition \(S\) is active and demands \(y\), distinct from \(x\), then it is not profitable for all the members of \(S\) to demand \(a\) instead of \(y\), given what follows after the deviation. A consequence of the second requirement is that the payoff of the members of the active coalition cannot improve if the status quo is implemented.

A well defined strategy meeting the one-deviation property may be simply referred as the *equilibrium*. 7

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7 This is the key difference between our concept and that of Konishi and Ray (2003). They also require that the members of the active coalition weakly prefer the end point of the path over the status quo. For more on this see Section 4.
Inducable paths and implementable outcomes In the remainder we characterize equilibrium strategies, and study their existence. One problem with the equilibrium notion is that the play path as well as the eventually implemented outcome may be sensitive to the choice of the initial outcome. For that reason, it is natural to focus on paths that are inducable in equilibrium by varying the histories, and outcomes that can be implemented within those paths. More formally, paths that are induced in equilibrium are
\[ \bar{\sigma}(H) = \{ \bar{x} : \bar{\sigma}(h) = h, h \in H \} \]

Our main interest is in outcomes \( \mu[\bar{\sigma}(H)] \) that are implementable within an equilibrium. Denoting the set of outcomes that are implementable via paths in \( B \) by \( \mu[B] = \{ \mu[\bar{x}] : \bar{x} \in B \} \), the set of outcomes that are implementable in equilibrium is written as
\[ \mu[\bar{\sigma}(H)] = \{ \mu[\bar{x}] : \bar{x} \in \bar{\sigma}(H) \} \]

Any outcome inside \( \mu[\bar{\sigma}(H)] \subseteq X \), but nothing outside of it, can be implemented within an equilibrium by simply replacing the status quo (for finitely many times).

Simple recursive games In case there is one potentially active coalition under each status, and this coalition consists of a single player, the game structure specifies a simple recursive game or, equivalently, a deterministic perfect information stochastic game with absorbing states. In this class of games, the player in a decision making turn faces a question of whether to end the game or to move the turn to one of the other players in a subset of players (reflecting his choice set). The essential problem is that of commitment: how to assign players a strategy that allows them commit to ending the game rather than shifting the turn forward. This is not a trivial problem since no restrictions are put on players’ preferences as regards to who stops the game.

The appropriate restriction on strategies is the one-deviation principle. In this case, Definition 1 simplifies as part (i) of the definition can now be relaxed. To our knowledge, there is no known dynamically consistent solution to simple recursive games in the literature. Our results below not only shed light to this issue but they also hint that the core problem with describing behavior in the coalitional model is to describe behavior in simple recursive games in which the players cannot make binding commitments of ending the game. Hence once the coalitional set up is appropriately defined, the problem collapses back to simple recursive games, with standard (yet unknown) strategic properties.
3 Characterization

In this section, we aim to characterize coalitional equilibria in terms of the primitive data alone, i.e., in terms of walkable paths. To this end, we define a dominance relation over paths.

**Definition 2 (Path Dominance)** A path \( \bar{y} \in X \) dominates a path \( \bar{x} \in X \) at the \( k \)th step, denoted by \( \bar{y} \triangleright_k \bar{x} \), if \( y \in F_S(x_k), \bar{y} \in X_y, \) and \( \mu[\bar{y}] \succ_S \mu[\bar{x}] \), for \( S = S(x_{k+1}, x_k) \).

That is, if outcome \( x_k \) is reached along the path \( \bar{x} \), then there is an active coalition at \( x_k \) that benefits from moving the play to path \( \bar{y} \) under the hypothesis that the end point of \( \bar{y} \) is reached rather than that of \( \bar{x} \).

**Definition 3 (Consistent Path Structure)** A consistent path structure \( C \subseteq X \) satisfies:

(i) For all \( x \in X \), \( C \cap X_x \) is nonempty.

(ii) For all \( \bar{x} \in C \), there are no \( k \) and \( y \) such that \( \bar{y} \triangleright_k \bar{x} \), for all \( \bar{y} \in C \cap X_y \).

That is, for any initial status quo node, there is a feasible path in the consistent path structure that guides the play forward. Moreover, if an active coalition along the play path deviates from the path to a new node, then there is a path in the consistent path structure that starts from the new status quo, and that does not improve the payoffs of the members of the deviating coalition relative to what they would get if the original path is followed.

We now claim that a consistent path structure features stability in the sense described in the notion of coalitional equilibria. The core of the argument is the idea that paths of a consistent path structure is sustainable if any deviant coalition of any path can be punished. But, by Definition 3 (ii), punishment is feasible if it is true that any path of a consistent path structure can be played. Hence a consistent paths structure is stable in a consistent way. Next we verify this argument formally, i.e., that for any consistent path structure there exists a well defined strategy meeting the one-deviation property.

Let \( C \) be a consistent path structure. Identify a function \( \xi \) on \( C \times N \times X \) such that \( \xi(\bar{x}, k, y) \in C \cap X_y \) and \( \xi(\bar{x}, k, y) \triangleright_k \bar{x} \), for all \( (\bar{x}, k, y) \) such that \( y \in g_S(x_{k+1}, x_k)(x_k) \setminus \{x_k\} \). Since \( C \) satisfies Definition 3, such function exists.

We construct a strategy \( \sigma^* : H \rightarrow X \) that is well defined and satisfies the one-deviation property by using the function \( \xi \). It is convenient to describe the strategy as an automaton \( (\sigma^* : Q, g, \bar{x}^*) \), where \( Q \) is the set of states on which the strategy \( \sigma^* \) operates, \( g \) is the transition function from \( Q \times X \) to \( Q \), and \( \bar{x}^* \in C \cap X_{\bar{x}^*} \) is the initial path, which exists by Definition 3(i).
Define the set of states $Q$ by
\[ Q = \{(q^x, k) : \bar{x} = (x_0, ..., x_K) \in \mathcal{C} \text{ and } 0 \leq k \leq K\}. \] (2)

Start with the path $\bar{x}^*$. Let the transition function $g$ satisfy, for any $\bar{x} = (x_0, ..., x_K) \in \mathcal{C}$, for any $k = 0, ..., K - 1$, and for any $y \in X$,
\[ g(q^x, k, y) = \begin{cases} (q^x, k + 1), & \text{if } y = x_k + 1, \\ (q^{\bar{x}(x,k,y)}, 0), & \text{if } y \neq x_k + 1. \end{cases} \] (3)

Proceeding recursively from $x^*$, the set of histories $H$ is partitioned by the set $Q$.

The strategy $\sigma^*$ is conditional on the current state $(q^x, k) \in Q$, where $\bar{x} = (x_0, ..., x_K)$, in the following way:
\[ \sigma^*(q^x, k) = \begin{cases} x_{k+1}, & \text{if } k < K, \\ x_k, & \text{if } k = K. \end{cases} \] (4)

That is, the strategy calls the coalitions to continue along the path $\bar{x} = (x_0, ..., x_K)$ and implement $x_K$ when the end of the path is reached.

**Lemma 4** *Coalitional strategy $\sigma^*$ is well defined and satisfies the ODP.*

**Proof.** Well-definedness: Take $(q, l, y) \in Q \times X$ and let $g(q, l, y) = (q^x, k) \in Q$, where $\bar{x} = (x_0, ..., x_K)$. Then, applying (3) and (4) recursively, $\sigma^*$ implements an outcome in at most $K - k$ steps. Since $(q, l, y)$ was an arbitrary element of $Q \times X$, $\sigma^*$ is well defined.

One-deviation property: Take any path $\bar{x} = (x_0, ..., x_K)$ and state $(q^x, k) \in Q$. A deviation at step $k$ by coalition $S = S(x_{k+1}, x_k)$ to $y \in g_S(x_k) \setminus \{x_{k+1}\}$ induces a path $\xi(\bar{x}, k, y)$ such that $\xi(\bar{x}, k, y)|\bar{x}$. By the definition of dominance, $\mu[\xi(\bar{x}, k, y)] \not\in S \mu[\bar{x}]$. Thus a unilateral deviation to $y$ is not strictly profitable for all the members of the coalition $S$. \[ \Box \]

To fully characterize equilibrium strategies, consistent path structures need to be completed in the following sense: A consistent path structure $\mathcal{C}$ is complete if $(x_0, ..., x_K) \in \mathcal{C}$ implies $(x_k, ..., x_K) \in \mathcal{C}$, for all $0 \leq k \leq K$.

That is, following a path in a $\mathcal{C}$ is consistent with staying on a path in the $\mathcal{C}$. Note that completion is a purely expositional operation; existence of a complete $\mathcal{C}$ or its uniqueness is never an issue once the $\mathcal{C}$ is specified.

Given a complete consistent path structure $\mathcal{C}$ and strategy $\sigma^*$ defined on it, let $\bar{\sigma}^*(\chi)$ denote the path followed once $\chi \in Q$ has materialized. By construction,
\[ \bar{\sigma}^*(q^x, k) = (x_k, ..., x_K), \text{ for all } (q^x, k) \in Q. \]

In particular, by the construction of $\sigma^*$,
\[ \bar{\sigma}^*(q^x, 0) \in \mathcal{C}. \]
Thus, by the definition of completeness of $C$,
\[ \tilde{\sigma}^*(q^k, k) \in C, \text{ for all } (q^k, k) \in Q. \]

Thus $\tilde{\sigma}^*(Q) \subseteq C$. Moreover, since $\tilde{\sigma}^*(q^0, 0) = \bar{x}$, for all $\bar{x} \in C$, it follows that $C \subseteq \tilde{\sigma}^*(Q)$. Thus
\[ \tilde{\sigma}^*(Q) = C. \tag{5} \]

That is, for any consistent path structure $C$, we can find an equilibrium strategy that induces the corresponding complete $C$.

Now we prove the converse of Lemma 4 - that a consistent path structure characterizes behavior in any equilibrium, i.e., that any collection of equilibrium paths is equivalent to a consistent path structure.

**Lemma 5** Let a well defined coalitional strategy $\sigma$ satisfy the ODP. Then there is a complete consistent path structure $C \subseteq X$ such that $\tilde{\sigma}(H) = C$.

**Proof.** (i). Take any $(h, x) \in H$, and identify $\tilde{\sigma}(h, x)$. By construction, $\tilde{\sigma}(h, x) \in X$ and $\tilde{\sigma}(h, x) \in \tilde{\sigma}(H)$.

(ii). Take any $(h, x) \in H$. Let $\sigma(h, x) = x' \neq x$ and $y \in F_S(x) \setminus \{x'\}$ for $S = S(x', x)$. Since in equilibrium, $S$ chooses action $x'$ and not $y$, there must be a path $\bar{y} = \tilde{\sigma}(h, x, y) \in X$ that members of $S$ do not strictly prefer over the equilibrium path $\bar{x}$, i.e., $\mu[\bar{y}] \not\succeq_S \mu[\bar{x}]$. Since $\mu[\tilde{\sigma}(h, x, y)] \succeq_S x$ it follows that, in fact, $\bar{y} \in X$. Thus $\tilde{\sigma}(H)$ is a consistent path structure.

Finally, we argue that the consistent path structure is also complete. Let $(x_0, ..., x_K) = \tilde{\sigma}(h)$. It suffices to show that $(x_1, ..., x_K) = \tilde{\sigma}(h, x_0)$. But this follows from the recursive definition of $\tilde{\sigma}(h, x_0) = (\sigma(h), x_1, ..., x_K)$. $\blacksquare$

By Lemmata 4 and 5, a well defined equilibrium strategy induces behavior consistent with a consistent path structure and behavior in any consistent path structure can be supported by a well defined equilibrium strategy. We compound these observations in the following characterization.

**Theorem 6** A coalitional strategy $\sigma$ is well defined and satisfies the ODP if and only if there is a complete consistent path structure $C$ such that $\tilde{\sigma}(H) = C$.

Theorem 6 permits us to characterize the outcomes that are implementable within an equilibrium. By part 1 of the theorem, the set of outcomes that are implementable within a consistent path structure $C$ coincide with the set of outcomes that inducible within an equilibrium. Conversely, by part 2 of the theorem, the set of outcomes that are implementable within an equilibrium coincide with the set of outcomes that inducible within a $C$. Summing these results gives a theorem.
Theorem 7 A set $B \subseteq X$ of outcomes is implementable in a well defined coalitional strategy $\sigma$ that satisfies the ODP if and only if there is a consistent path structure $C$ such that $\mu[C] = B$.

This result does not, however, tell anything about the existence of a consistent path structure nor how it can be identified. The next section provides a procedure for identifying the maximal consistent path structure. The procedure also guarantees the existence of a solution.

3.1 Existence

The aim of this subsection is to prove that a consistent path structure and, à fortiori, a well defined strategy that meets the ODP does exist. To this end, we need to define the following relation between paths and nodes. The concept is inspired by its cousin in the social choice literature (cf. Fishburn, 1977; Miller, 1980; Dutta, 1988, or Laslier, 1991).

Let $B$ be subset of $X$.

Definition 8 Path $\bar{x} \in X$ is pseudo-covered in $B$ via node $y$ if there is $k$ such that $\bar{y} \triangleright_k \bar{x}$, for all $\bar{y} \in X_y \cap B$. If, furthermore, $\bar{x} \in B$, then $\bar{x}$ is covered in $B$ via $y$.

That is, a path $\bar{x}$ is pseudo-covered via $y$ in $B$ of paths if, at some particular node (the $k^{th}$) of $\bar{x}$ the members of the active coalition always profit by directing the play to node $y$ rather than continuing along $\bar{x}$, given the hypothesis that the continuation play after $y$ belongs to the set $B$, and that continuing along $\bar{x}$ means that $\bar{x}$ will be played. If, furthermore, $\bar{x}$ itself is an element of $B$, then $\bar{x}$ is said to be covered in $B$.

Denote by $uc(B)$ the uncovered set of $B$, i.e., the set of paths not covered in $B$. By construction, $uc(B) \subseteq B \cup \{\emptyset\}$.

We now strengthen of the uncovered set -concept by iterating the uncovered-operator until no paths are left to be covered. The ultimate uncovered set $UUC \subseteq X$ is defined recursively as follows. Set $UC^0 = X$, and let $UC^{t+1} = uc(UC^t)$, for all $t = 0, \ldots$. Then $UUC := UC^\infty$.

Since $UC^{t+1} \subseteq UC^t$ for all $t$, and since $X$ contains countably many elements, $UUC$ is uniquely defined. Our aim is to show that $UUC$ is a consistent path structure. One big problem in this is that, by Definition 3 (i), any consistent path structure is not only nonempty but also contains an element in $X_x$ for all $x \in X$. While $UUC$ always exists, it is not known a priori whether is satisfies this property (or even if it is nonempty). It needs to be shown.

Lemma 9 $UC^t \cap X_y$ is nonempty, for all $y \in X$, for all $t = 0, \ldots$. 

Proof. The proof is by contradiction. Suppose that $UC^t \cap \mathcal{X}_y$ is empty, for some $t \geq 1$.

First, index all the nodes in $X$ with natural numbers. Construct recursively a sequence $\bar{y}^0, \bar{y}^1, \ldots$ of paths in $\mathcal{X}_y$ as follows: Let $(y) = \bar{y}^0$. For any $n = 0, 1, \ldots$, let $k$ be the final step at which $\bar{y}^n$ is pseudo-covered in $UC^{t-1}$. Identify a node $x$ via which $\bar{y}^n$ is pseudo-covered at $k$. If there are many such nodes, pick the one with the lowest index. Choose

$$\bar{y}^{n+1} = (y_0^n, \ldots, y_k^n, x).$$

Now we prove the contradiction via a series of subclaims. The contradiction implies that there is $n$ such that $\bar{y}^n \in UC^t \cap \mathcal{X}_y$.

Claim 0. If $\bar{x} \in \mathcal{X}$ is covered via $x$ in $UC^t$ then $\bar{x} \in \mathcal{X}$ is pseudo-covered via $x$ in $UC^{t-1}$, for all $q = 0, \ldots, t$.

Proof: By the definition of covering, and since $UC^t \subseteq UC^q$.

Claim 1. The sequence $\bar{y}^0, \bar{y}^1, \ldots$ is infinite.

Proof: Suppose on the contrary that there is $n < \infty$ such that $\bar{y}^n = (y_0^n, \ldots, y_J^n)$ is not pseudo-covered in $UC^{t-1}$. By Claim 0, $\bar{y}^n$ is not covered in $UC^q$, for $q = 0, \ldots, t-1$. Then $\bar{y}^n \in UC^t$. But since $\bar{y}^n \in \mathcal{X}_y$, this contradicts the hypothesis that $UC^t \cap \mathcal{X}_y$ is empty.

Claim 2. The length of any path in $\bar{y}^0, \bar{y}^1, \ldots$ is at most $|X|$.

Proof: Let, to the contrary of the claim, $\bar{y}^n$ be a path with length $J > |X|$. By construction, $\bar{y}^n$ can be written, for some $0 = n_0 < n_1 < \ldots < n_J = n$,

$$\bar{y}^n = (y_0^n, \ldots, y_J^n) = (y_0^{n_0}, \ldots, y_J^{n_J}).$$

Since $J > |X|$ there is $0 \leq j < J$ such that $y_j^n = y_j^0$. Let $\bar{y}^n$ be the cycle $(y_{j+1}^{n_{j+1}}, \ldots, y_J^{n_J})$. Since the final pseudo-covering steps define the sequence $\bar{y}^0, \bar{y}^1$, it follows that, for any $j < k < J$,

$$\bar{y}^{n_{k-1} - n_j} = (y_0^{n_0}, \ldots, y_{j+1}^{n_{j+1}}, y_j^{n_j}, \ldots, y_k^{n_k}).$$

Of $n_{j}, \ldots, n_{J-1}$, find the $n_j$ with the property that $\bar{y}^{n_{j}^{*}}$ is pseudo-covered in $UC^q$, and no $\bar{y}^{n_k}$ is pseudo-covered in $UC^q$ for $p$ lower than $q$, for any $k = j, \ldots, J$. Then, by Claim 0, $(y_{j+1}^{n_{j+1}}, \ldots, y_{j}^{n_j}, \ldots, y_{J}^{n_{J}^{*}})$ is not covered in $UC^q$, for $q = 0, \ldots, t-1$, and hence it belongs to $UC^t$. But this contradicts the assumption that $\bar{y}^{n_{j}^{*}}$ is pseudo-covered in $UC^{t-1}$ via $y_{j+1}^{n_{j+1}}$.

Claim 3. There is a finite set of paths $\bar{Z}$ in which the sequence $\bar{y}^0, \bar{y}^1, \ldots$ ends up cycling.

Proof: By Claim 1, $\bar{y}^0, \bar{y}^1, \ldots$ is an infinite sequence. By construction, at the $n^{th}$ step the transition from $\bar{y}^n$ to $\bar{y}^{n+1}$ is contingent only on $\bar{y}^n$. Since, by Claim 2, the paths $\bar{y}^0, \bar{y}^1, \ldots$ are drawn from a finite set, we can view $\bar{y}^0, \bar{y}^1, \ldots$ as a (deterministic) finite state Markov process. Interpreting $\bar{Z}$ as the ergodic set of this process proves the claim.

Claim 4. The supposition that $UC^t \cap \mathcal{X}_y$ is empty does not hold.
Proof: By Claim 3, the elements in \( \bar{Z} \) agree until some maximal step \( k^* \). By the construction of the sequence \( \bar{y}^0, \bar{y}^1, \ldots \), no element in \( \bar{Z} \) is pseudo-covered at any step higher than \( k^* \). Hence, by Claim 0,

\[
\text{if } \bar{z} = (z_0, \ldots, z_K) \in \bar{Z}, \text{ then } (z_{k^*+1}, \ldots, z_K) \in UC^t. \tag{6}
\]

Let \( \bar{W}^0, \ldots, \bar{W}^r \) partition \( \bar{Z} \) into branches that start from step \( k^* + 1 \). Let, for each \( j = 0, \ldots, r - 1 \) there be \( \bar{w}^j \in \bar{W}^j \) that is pseudo-covered via \( w_{k^*+1} \) in \( UC^t \), and let \( \bar{w}^r \) be pseudo-covered via \( w_{0}^0 \). By (6) there is \( S \) such that \( \mu[\bar{w}^{j+1}] \succ_S \mu[\bar{w}^j] \) for all \( j = 0, \ldots, r - 1 \), and \( \mu[\bar{w}^0] \succ_S \mu[\bar{w}^r] \). But this violates the transitivity of \( \succ_S \), a contradiction.

Now we argue that the ultimate uncovered set is a well defined concept.

**Lemma 10** There is \( T < \infty \) such that \( UC^T = UUC \).

**Proof.** Call \( \{ (x_0, \ldots, x_K) \in X : \{ x_k \} = Y \text{ and } x_K = y \} \) a dominance class, parametrized by \( Y \) and \( y \), where \( Y \subseteq X \) and \( y \in X \). Since \( X \) is a finite set, the cardinality of distinct dominance classes is finite, and they partition \( X \).

A dominance class contains all the relevant information concerning dominance: If two paths \( \bar{x} \) and \( \bar{x}' \) belong to the same dominance class, then \( \bar{x} \) is covered in \( UC^t \) if and only if \( \bar{x}' \) is covered in \( UC^t \), for any \( t \). Since all paths in the same dominance class become covered at the same covering round \( t \), and since there are finitely many dominance classes, the number of covering rounds to reach \( UUC \) must be finite.

By induction on \( t \), Lemma 9, there is, for any \( y \in X \), an element in \( X_y \) is not covered in \( UC^T \). Moreover, since \( uc(UUC) = UUC \), no element in \( UUC \) is covered in \( UUC \). Thus we have proven that \( UUC \) satisfies the two properties of consistent path structure.

**Theorem 11** \( UUC \) is a consistent path structure.

**Proof.** (i). By Lemmata 9 and 10, \( UUC \cap X_y \) is well defined and nonempty, for all \( y \in X \).

(ii). By the construction of \( UUC \), \( \bar{x} \in UUC \) is not covered in \( UUC \). Thus there is no \( y \) and \( k \) such that \( \bar{y} \succ_k \bar{x} \), for all \( \bar{y} \in UUC \cap X_y \).

Note that Theorem 11(i) implies, since \( X \) is nonempty, that \( UUC \) is nonempty. The next result shows that \( UUC \) is the (unique) maximal consistent path structure in the sense of set inclusion.

**Theorem 12** \( UUC \) contains any consistent path structure.
Proof. Let $B$ be a consistent path structure. Take any $\bar{x} \in B$. Since $\bar{x}$ satisfies part (ii) of the definition of consistent path structure, it follows by the definition of covering in $X$ that $\bar{x} \in \text{uc}(X) = UC^1$. Since $\bar{x}$ was arbitrary, $B \subseteq UC^1$. By the definition of covering in $\text{uc}(X)$, $B \in \text{uc}(UC^1) = UC^2$. Again, $B \subseteq UC^2$. By induction, $B \subseteq UC^T =: UUC$. Since $UUC$ is a consistent path structure, it is the maximal consistent path structure. ■

By Theorems 11 and 6 1., one can construct on $UUC$ a well defined strategy that meets the one-deviation property. Since the $UUC$ is obtained via well defined recursive process that, by Lemma 10, terminates in finite time, there is no question about the existence. Hence we conclude that there is no question about the existence of an equilibrium, either.

Corollary 13 There is a well defined coalitional strategy that satisfies the ODP.

By Theorem 12, we have also characterized outcomes that are implementable with any equilibrium.

Corollary 14 The set of outcomes that are implementable via any well defined strategy that satisfies the ODP is contained in $\mu[UUC]$.

3.2 Algorithmic Considerations

The problem with game theoretic solution concepts is often their computability. Such questions are particularly acute here since the set of paths $\mathcal{X}$ contains typically infinitely many elements. A convenient algorithm for computing the relevant elements of the ultimate uncovered set, i.e. the largest consistent path structure is now provided. This algorithm, which terminates in finite time, thus generates a complete description of the outcomes that can be implemented in a coalitional equilibrium.

In order to establish the desired results, we develop some simplifying concepts. We say that $\bar{y} = (y_0, y_L) \in X$ is a reduction of $\bar{x} = (x_0, ..., x_K) \in X$ if $x_0 = y_0$, $x_K = y_L$, and $\{y_l\} \subseteq \{x_k\}$. Then $\bar{y}$ is a full reduction of $\bar{x}$ if it is a reduction of $\bar{x}$ and if the only reduction of $\bar{y}$ is $\bar{y}$ itself. That is, $\bar{y}$ contains only those nodes of $\bar{x}$ that are needed to travel from $x_0$ to $x_K$. A full reduction of $\bar{x}$ need not be unique.

For any set $B$ of paths, denote by $fr(B)$ the collection of all full reductions of the elements in $B$, i.e., the elements of $X$ that are full reductions of $B$. Note that the reduction -relation is transitive. Since the paths contain at most finitely many distinct elements, each $\bar{x}$ has a full reduction. The length of any full reduction is at most $|X|$.
Note that the set of fully reduced paths $fr(\mathcal{X})$ can be identified in finite time. Define recursively the uncovered-operation and its iterations on the set of fully reduced paths: $UC^t_{FR} = fr(\mathcal{X})$ and $UC^j_{FR} = uc(UC^{j-1}_{FR})$ for all $j = 1, \ldots$. Denote the ultimate uncovered set of fully reduced paths by $UUC_{FR} = UC^{\infty}_{FR}$. Since, by Lemma 10, only finitely many iterations are needed, all this can be done in finite time.

Now we argue that the algorithm identifies a subset of equilibrium strategies. Since these strategies are in the fully reduced form, they constitute the simple coalitional equilibria.

**Proposition 15** $fr(UUC) = UUC_{FR} \subseteq UUC$.

**Proof. Claim 1:** $UC^t_{FR} \subseteq UC^t$, for all $t$.

*Proof:* Let $t$ be the first stage when $\bar{x} \in UC^t_{FR} \setminus UC^t$, for some $\bar{x}$. Hence $UC^{t-1}_{FR} \subseteq UC^{t-1}$. But then, since $\bar{x}$ is covered in $UC^{t-1}$, it must be covered in $UC^{t-1}_{FR}$. But this contradicts $\bar{x} \in UC^{t}_{FR}$.

**Claim 2:** $fr(UC^t) \subseteq UC^t$, for all $t$.

*Proof:* Let $t$ be the first stage when there is $\bar{x} \in fr(UC^t) \setminus UC^t$. Since $\bar{x} \in fr(UC^t)$, $\bar{x}$ must be a full reduction of some $\bar{y} \in UC^t$. But by the definition of full reduction, the assumption that $\bar{x}$ is covered in $UC^{t-1}$ contradicts the assumption that $\bar{y}$ is not covered in $UC^{t-1}$.

**Claim 3:** $UC^t_{FR} = fr(UC^t)$ for all $t$.

*Proof:* Since full reduction of a set of fully reduced paths is the set itself, it follows by Claim 1 that $UC^t_{FR} = fr(UC^t_{FR}) \subseteq fr(UC^t)$, for all $t$. For the other direction, let $t$ be the first stage when $\bar{x} \in fr(UC^t) \setminus UC^t_{FR}$, for some $\bar{x}$. Hence, $fr(UC^{t-1}) \subseteq UC^{t-1}_{FR}$. But then, by the definition of covering, since $\bar{x}$ is covered in $UC^{t-1}_{FR}$ it must be covered in $fr(UC^{t-1})$. By the definition of full reduction, $\bar{x}$ is also covered in $UC^{t-1}$. Hence $\bar{x} \notin UC^t$. But then $\bar{x} \in fr(UC^t) \setminus UC^t$, which contradicts Claim 2. Thus $fr(UC^t) \subseteq UC^t_{FR}$, for all $t$.

**Claim 4:** $fr(UUC) = UUC_{FR} \subseteq UUC$.

*Proof:* Combining Claims 1 and 3, we have $fr(UC^t) = UC^t_{FR} \subseteq UC^t$ for all $t$.

Since, By Proposition 15, $fr(UUC) = UUC_{FR}$, and since the end point of a path is invariant with respect to the full reduction operation, i.e., $\mu(fr(UUC)) = \mu(UUC)$, it follows that the outcomes that can be implemented with $UUC$ coincide with the outcomes that can be implemented with $UUC_{FR}$.

**Proposition 16** $\mu[UUC_{FR}] = \mu[UUC]$.

The algorithm thus provides an unbiased prediction of the outcomes that can be implemented in any coalitional equilibria.

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8Since this set contains at most $|X|!$ elements.
4 Relation to Other Models

In this section, we compare our framework to some existing equilibrium notions of coalition formation. The key difference between our approach and the standard model is that the basic building block of the latter is the indirect dominance relation, not directly implied by the notion of ODP. More precisely, an outcome \( y \in X \) indirectly dominates \( x \) if there is a path \((x_0,...,x_K)\) such that \( x_0 = x \), \( y = x_K \), such that \( x_K \succ_S(x_{k+1},x_k) \) for all \( k = 0,...,K-1 \) (recall that there is a unique coalition that may induces \( x_{k+1} \) under \( x_k \)). That is, the following the path makes any member of active coalitions along the path better of relative what they would get by stopping the game, provided that the final outcome of the path will be reached.

Note that if \((x_0,...,x_K)\) is generated by a strategy that meets the ODP, then only \( x_k \not\in S(x_{k+1},x_k) \) for all \( k = 0,...,K-1 \). That is, the following the path makes any member of active coalitions along the path better of relative what they would get by stopping the game, provided that the final outcome of the path will be reached.

Definition 17 (CE) A well defined coalitional strategy \( \sigma \) is a coalitional equilibrium (CE) if, for all \((h,x)\) \( \in H \),

1. \( \sigma(h,x) = x \) implies \( \mu[\bar{\sigma}(h,x,y)] \not\in S(y,x) \) for all \( y \in \cup S(y,x) \).
2. \( \sigma(h,x) = y \neq x \) implies \( \mu[\bar{\sigma}(h,x,y)] \not\in S(y,x) \) for all \( z \in F_S(y,x) \) and \( \mu[\bar{\sigma}(h,x,y)] \succ S(y,x) \).

Call path \((x_0,...,x_K) \in X \) feasible if \( x_K \succ_S(x_{k+1},x_k) \) for all \( k = 0,...,K-1 \).

Denote the set of feasible paths by \( X^F \subseteq X \), and denote the set of feasible paths that start from node \( y \) by

\[ X^F_y = X^F \cap X_y. \]

Property 2 of Definition 17 hints that any coalitional equilibrium path is necessarily feasible. It will be convenient to work directly in terms of feasible paths.

Indeed, by defining the notions of domination and consistent path structure with respect to \( X^F \) rather than \( X \), and calling the latter a feasible consistent path structure \( CF \), it is easy to obtain a characterization analogous to Theorem ??.

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This is the weaker version of Chwe's (1994) indirect blocking relation, entertained e.g. by Konishi and Ray (2003) and Ray (2007).
Theorem 18 A coalitional strategy $\sigma$ is well defined and forms a CE if and only if there is a feasible complete consistent path structure $C^F$ such that $C^F = \bar{\sigma}(H)$.

Similarly the reasoning behind Theorem 19 can be replicated with respect to the domain of feasible paths, to obtain a characterization of implementable outcomes.

Theorem 19 A set $B \subseteq X$ of outcomes is implementable in a well defined coalitional strategy $\sigma$ that satisfies the ODP if and only if there is a consistent path structure $C$ such that $\mu[C] = B$.

The tricky part with feasible paths is the existence. Namely now the proof of Lemma 9 no longer suffices, given that the iteration of covering operation is conducted with respect to $X^F$ rather than $X$: the elements of sequence $\{\bar{y}_n\}$, which are always contained by $X$, need no longer belong to $X^F$. This needs to be shown.

Conjecture 20 $UC_t \cap X^F_y$ is nonempty, for all $y \in X$, for all $t = 0, ...$

This would imply that the corresponding ultimate uncovered set, denoted by $UUC^F_y$, is a feasible consistent path structure. This would guarantee the existence of a CE.

4.1 Equilibrium Process of Coalition Formation

In this section we interpret our results in the framework of dynamic coalition formation by Konishi and Ray (2003) (see also Ray, 2007).10 A model is captured by a tuple $(N, Z, u, (F_S)_{S \subseteq N}, \delta)$ where $X$ is now interpreted as a set of states and $u_i$ is the cardinal utility function of player $i \in N$ over $X$. Analogously to the model above, $F_S(x) \subseteq X$ such that $x \in F_S(x)$ is the set of states achievable by a one-step coalitional move in state $x$ by a coalition $S$, for all $x \in X$, and for all $S \subseteq N$.

Parameter $\delta \in (0, 1)$ is a discount factor, and player $i$’s payoff from a sequence of states $\bar{x} = (x_0, ..., x_t, ...)$ may be written as $\sum_{\tau=0}^{t} \delta^\tau u_i(x_\tau)$. Let $H$ be the set of all histories of states $(x_0, ..., x_t)$ such that $x_0 = x^*$. A deterministic process of coalition formation (PCF) is now a function $p : H \rightarrow X$, capturing the transitions from one history to another. These transitions will be induced by coalitions who stand to benefit from them. A PCF $p$ induces a value function $V_i$ for each $i \in N$. This value function captures the infinite horizon payoff to a player starting from any state history $(h, x)$ under the Markov process $p$. By the standard observation, the value function for $i$ is the unique solution to the functional equation

$$V_i(h, x) = (1 - \delta)[u_i(x) + \delta V_i(h, x, p(h, x))].$$

10For a model with similar approach, see Gomez and Jehiel (2005).
We are now in a position to define profitable moves. To this end, we adopt the following notation for vector inequalities. For any \( x, y \in X \),
\[
u_S(x) > \nu_S(y), \text{ if } u_i(x) > u_i(y), \text{ for all } i \in S.
\]

We study a slightly modified version of the solution concept of Konishi and Ray (2003). Define an equilibrium PCF as follows:

**Definition 21** \( p \) is a deterministic equilibrium process of coalition formation (EPCF) if, for all \((h, x) \in H\):

1. If \( p(h, x) \neq x \), then there is a coalition \( S \) such that \( p(h, x) \in F_S(x) \) and \( \nu_S(h, x, y) \neq \nu_S(h, x, p(h, x)) \), for all \( y \in F_S(x) \), and \( \nu_S(h, x, p(h, x)) \geq \nu_S(h, x, x) \).

2. If \( p(h, x) = x \), then \( \nu_S(h, x, y) \neq \nu_S(h, x, p(h, x)) \), for all \( y \in F_S(x) \), for any coalition \( S \).

Konishi and Ray (2003) showed the existence of a randomized stationary EPCF. Here we argue that there is also deterministic EPCF if one drops the stationarity assumption, i.e., allows \( p \) to be dependent on the history \( h \), and not only on the current state \( x \) as the stationary PCF does. This is established by demonstrating the equivalence of the deterministic, absorbing EPCF and the notion of a consistent path structure.

Let \( p^0(h) = p(h) \) and \( p^t(h) = p(h, p^0(h),...,p^{t-1}(h)) \), for all \( t = 1, ... \).

Denote by
\[
p(H) = (p^0(h), p^1(h),...)
\]

the chain of states that will materialize along the PCF \( p \), starting from history \( h \). All chains that are walkable, staring from any history of the past play is denote by \( p(H) \). To highlight the relationship of our model to Konishi and Ray (2003), we focus on PCFs that are absorbing in the following finitary sense:

\[
\text{11 there is an integer } T < \infty \text{ such that for any history } h \text{ there is } x \in X \text{ such that }
\]
\[
p^t(h) = x, \text{ for all } t > T.
\]
(7)

The absorbing state of the PCF \( p \), \( \alpha[p(H)] \), is then well defined for all \( h \). Denote the set of all absorbing states of the PCF \( p \) by \( \alpha[H] \).

Now we demonstrate the equivalence of the notions of EPCF and that of coalitional equilibria. As \( \delta \) becomes large, Definition 21 can be rewritten.

**Proposition 22** There is \( \delta^* \) such that for all \( \delta > \delta^* \), an absorbing, deterministic equilibrium process of coalition formation (EPCF) \( p \) is defined by the following condition: for all \((h, x) \in H\),

\[\text{Equivalently, we assume that } p \text{ is a Markov chain with bounded state space.}\]
1. If \( p(h, x) \neq x \), then there is a coalition \( S \) such that \( p(h, x) \in F_S(x) \) and \( u_S(\alpha[\bar{p}(h, x, y)]) \neq u_S(\alpha[\bar{p}(h, x)]) \), for all \( y \in F_S(x) \), and \( u_S(\alpha[\bar{p}(h, x)]) \geq u_S(\alpha[\bar{p}(h, x, x)]) \).

2. If \( p(h, x) = x \), then \( u_S(\alpha[\bar{p}(h, x, y)]) \neq u_S(\alpha[\bar{p}(h, x)]) \), for all \( y \in F_S(x) \), for any coalition \( S \).

Proof. By continuity, and since \( p \) is absorbing, \( V_i(h, p(h)) \) converges uniformly to \( u_i(\alpha[\bar{p}(h)]) \) as \( \delta \) tends to unity, for all \( h \in H \) and for all \( i \in N \). The result now follows since the inequalities in Definition 21 are strict.

Noting Assumption A and the notational convention concerning payoffs over \( X \), and replacing the \( \alpha \) with \( \mu \), it follows that for high enough \( \delta \), the notion of an absorbing deterministic equilibrium process of coalition formation and that of a well defined coalitional strategy meeting the one-deviation property are the same. Hence, collecting the results in Theorems 6 and 7, and Corollaries 13 and 14, we see that the former can be characterized as follows.

For \( \delta \) higher than \( \delta^* \):

- A deterministic absorbing PCF \( p \) is an EPCF if and only if there is a feasible complete consistent path structure \( C^F \) such that \( \{(x_0, \ldots, x_{K-1}, x_K, x_K, \ldots) : (x_0, \ldots, x_K) \in C^F \} = \bar{p}(H) \).

- There is an absorbing EPCF \( p \) such that \( \alpha[\bar{p}(H)] = \mu[UUC^F] \). Moreover, any absorbing EPCF \( p \) satisfies \( \alpha[\bar{p}(H)] \subseteq \mu[UUC^F] \).

Moreover, under Conjecture 20, a deterministic absorbing EPCF exists if \( \delta > \delta^* \).

4.2 Largest Consistent Set

The now we develop the solution concept suggested by Chwe (1994). Recall that an outcome \( y \) directly dominates \( x \) if there is a coalition \( S \) such that \( y \in F_S(x) \) and \( y \succ_S x \). Set \( C \subseteq X \) is a consistent set if \( C \) consists of all \( x \) for which the following holds: if \( z \in F_S(x) \), then there is \( y \in C \) such that either \( y = z \) or \( y \) indirectly dominates \( z \) and \( y \not\prec_S x \). Chwe showed that a consistent set exists and the largest consistent set is unique.

Chwe’s solution, even though one of the most used coalitional solutions, has been subject to criticism that the indirect dominance need not be credible: the path of blockings may be deviated by a subset of an active coalition. The following example is due to Xue (1998): (where \( x \xrightarrow{S} y \) reflects the re-
Consider the game in Figure 3, where \( N = \{1, 2\} \), \( X = \{a, b, c, d\} \), and 
\[
F_{\{1\}}(a) = \{a, b\}, \ F_{\{2\}}(b) = \{b, c\}, \ F_{\{1, 2\}}(b) = \{b, d\}, \text{ and in all other cases } 
\]
\( F_S(x) \) is just a singleton. Recall that in our model \( x \) is implemented if 
\( x \in F_{\{i\}}(x) \) is chosen. Numerical payoffs (in the order of players’ indices) from each choice \( a, b, c, \) and \( d \) are, respectively, \((6, 0)\), \((7, 4)\), \((5, 10)\), \((10, 5)\). In the set up of Figure 3, the largest consistent set chooses \( \{a, c, d\} \). However, the largest consistent set is too large since it is not reasonable to predict that \( d \) is ever chosen: when \( a \) is the status quo, the "predicted" outcomes are \( \{a, d\} \), the latter in the case when coalition \( \{1, 2\} \) forms in status quo \( b \). But note that once node \( b \) is reached and the coalition \( \{1, 2\} \) is about to form, player 2 would renege and choose the option \( c \) instead. Hence, \( d \) should not be considered as a conceivable outcome.

This example suggests that indirect dominance over outcomes does not suffice but also dominance over paths must be taken under consideration.

**Proposition 23** \( \mu[UUC^F] \) is a consistent set.

**Proof.** Suppose, on the contrary, that \( \mu[UUC^F] \) is not a consistent set. Then there is an \( x \in \mu[UUC^F] \), an \( S \), and a \( y \in F_S(x) \) such that \( \mu[y] \succ_S x \), for all \( y \in X_p^F \cap UUC^F \). By construction, there is \( \bar{x} \in UUC^F \) such that \( \mu[\bar{x}] = x \). But this contradicts the assumption that \( UUC^F = uc(UUC^F) \), i.e., that \( \bar{x} \) is not covered in \( UUC^F \). □

Thus outcomes that are inducable in any equilibrium strategy are contained in the largest consistent set. Importantly, however, \( \mu[UUC^F] \) need not coincide with the largest consistent set: \( \mu[UUC^F] \) is a subset of the largest consistent set. One example when \( \mu[UUC^F] \) is a strict subset of the
largest consistent set is the game in Figure 3: $UUC^F$ consists of \{(a), (b, c), \{2\}, (c), (d)\}, thus if a is the status quo, then a is implemented, if b is the status quo, then c is implemented, and if c (or d) is the status quo, then c (or d) is implemented.

The following example is more dramatic.

\[ \begin{array}{c}
\text{a} \\
\text{\{1\}} \\
\text{\text{\{2\}}} \\
\text{\text{\{3\}}} \\
\text{c} \\
\text{\text{\{3\}}} \\
\text{\text{\text{d}}} \\
\end{array} \]

Figure 4

Here $N = \{1, 2, 3\}$, $X = \{a, b, c, d\}$, and $F_{\{1\}}(a) = \{a, b\}$, $F_{\{2\}}(b) = \{b, c\}$, $F_{\{3\}}(c) = \{a, c, d\}$. Numerical payoffs from choices a, b, c, and d are, respectively, (0, 0, 1), (0, 1, 0), (1, 0, 0), and (2, 2, 2). The largest consistent set is \{a, b, d\}. However, the largest consistent path structure $UUC^F$ consists only of the path (a, b, c, d), and hence $\mu[UUC^F] = \{d\}$.

4.3 Path Stability

Xue (1998) is based on the same idea as this paper; what is crucial is the stability of paths rather than outcomes. He uses von Neumann-Morgenstern stable set approach to identify paths that are robust against deviations. In the analysis that proceeds, perfect foresight is captured explicitly by the "situation with perfect foresight".\footnote{Referring to Greenberg’s (1991) Theory of Social Situations.} Assume that the alternative $x \in X$ is status quo. Consider a path $\bar{x}$ and some of its node $x_k$. Assume that a coalition $S$ can replace $x_k$ by some alternative $y \neq x_{k+1}$. In doing so, $S$ is aware of that the set of feasible paths from $y$ is $\mathcal{X}_y^F$. In contemplating such a deviation from $y$, however, members of $S$ base their decision on comparing paths that might be followed by rational and farsighted individuals at $y$. Let $SB(y) \subseteq \mathcal{X}_y^F$ denote this "standard of behavior". The following definition, which is due to Xue (1998), describes in our notation the conservative approach to stable standard of behavior.
Definition 24 An SB is conservatively stable if it is:

Internally stable: for all $x \in X$, if $\bar{x} \in SB(x)$, then there is no $y$, coalition $S$, and $k$ such that $y \in F_S(x_k) \setminus \{x_{k+1}\}$, and $\mu[\bar{y}] \succ_S \mu[\bar{x}]$, for all $\bar{y} \in SB(y)$.

Externally stable: for all $x \in X$, if $\bar{x} \in X^F \setminus SB(x)$, then there is $y$, coalition $S$, and $k$ such that $y \in F_S(x_k) \setminus \{x_{k+1}\}$, and $\mu[\bar{y}] \succ_S \mu[\bar{x}]$, for all $\bar{y} \in SB(y)$.

To see most clearly the relationship between our solution concept and the conservatively stable standard of behavior, let us rewrite the definition of a consistent path structure in the following form:

Definition 25 The consistent path structure $C^F$ satisfies

(i) For all $x \in X$, the set $X^F_x \cap C^F$ is nonempty.

(ii) If $\bar{x} \in C^F$, then there is no $y$ and $k$ such that $y \in F_S(x_k) \setminus \{x_{k+1}\}$ and $\mu[\bar{y}] \succ_S \mu[\bar{x}]$, for $S(x_k, x_k) = S$, and for all $\bar{y} \in X^F_y \cap C$.

From this definition it is clear that à la Xue (1998) is more stringent a solution concept. Definition 25 mainly covers the internal stability -part of Definition 24 with the difference that the latter’s stability requirement concerns all possible coalitions along the play path whereas for our solution, only the active coalition is tested.

Definition 26 The ultimate uncovered set $UUC^F$ satisfies

Internal stability: if $\bar{x} \in UUC^F$, then there is no $y$ and $k$ such that $y \in F_S(x_k) \setminus \{x_{k+1}\}$ and $\mu[\bar{y}] \succ_S \mu[\bar{x}]$, for $S(x_k, x_k) = S$, and for all $\bar{y} \in X^F_y \cap UUC^F$.

External stability: if $\bar{x} \in X^F \setminus UUC^F$, then there is $y$ and $k$ such that $y \in F_S(x_k) \setminus \{x_{k+1}\}$ and $\mu[\bar{y}] \succ_S \mu[\bar{x}]$, for $S(x_k, x_k) = S$, and for all $\bar{y} \in X^F_y \cap UUC^F$.

Internal stability follows from the definition of consistent path structure. External stability, in turn, is a consequence of the construction of $UUC^F$; no path outside $UUC^F$ is pseudo-covered in $UUC^F$. From Definition 26 it is clear that $UUC^F$ is a stable standard of behavior in the sense of Definition 24. Again, it is easy to see that the key difference between the solution concepts is that the notion of stable standard of behavior requires stability against arbitrary coalitional deviations whereas the ultimate uncovered set only with respect to the active coalition. Since the restriction affects both internal and external stability but to the different directions, there is no straightforward relationship between the concepts.

From these definitions it is clear that the notion of stable standard of behavior and the ultimate uncovered set coincide in games where the collection of possibly active coalitions is always singleton, i.e. only one coalition may change the status quo at a time. Roughly, this corresponds to
the scenarios where the coalitional game can be exhibited in the form of a simple recursive game (vis-a-vis interpreting coalitional preferences as single individual preferences).

The more stringent requirement for acceptable deviations in the context of ultimate uncovered set prevents pathological blocking relationships and guarantees the nonemptiness of the solution. Conversely, the leeway provided by arbitrary coalitional deviations in the stable standard of behavior is to blame for the occasional emptiness of the solution, as is the case in the following example (due to Xue, 1998):

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure5.png}
\caption{Figure 5}
\end{figure}

where payoffs from \(a, b\) and \(c\) are, respectively, \((0, 0)\), \((2, 1)\), and \((1, 2)\). The unique conservative standard of behavior is empty, and hence the solution gives no guidance of how the play evolves starting from \(a\). However, the ultimate uncovered set contains both the arcs: \(X^F_a \cap UUC^F = \{(a, b), (a, c)\}\) and hence \(\mu[UUC^F] = \{b, c\}\).

5 Efficiency

A recurrent theme in coalitional analysis concerns efficiency. An classic argument that goes under the label of Coase theorem says that an outcome that results from unrestricted coalitional bargaining will always be efficient: otherwise a coalition would block the outcome by proposing another outcome that all the players prefer. This intuition is not insufficient in the current framework.

Consider a game in Figure 6. The payoffs to three players \(N = \{1, 2, 3\}\) as depicted in different nodes. Each shaded node is Pareto dominated by some other node, reflected by the arrow from the node to another node. At each node all subcoalitions are entitled to move the game to any node. Thus there are no \(a \text{ priori}\) restrictions on what the coalitions can achieve. Nevertheless there is a \(C\) such that \(\mu[C]\) consists only of the shaded nodes \((2, 0, 1)\), \((1, 2, 0)\), or \((0, 1, 2)\). The construction is as follows: if, say, \((2, 0, 1)\) is blocked by \(N\) to \((3, 1, 2)\), then \((3, 1, 2)\) is blocked by \(\{2\}\) to \((1, 2, 0)\), which remains unblocked.
and becomes implemented. A similar consideration applies to blocking from 
$(1, 2, 0)$ or $(0, 1, 2)$.

Figure 6

Thus there is nothing inconsistent with the idea that an inefficient out-
come becomes implemented, even if bargaining opportunities are unrestricted.

6 Conclusion

References

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