The Probabilistic Serial Assignment Mechanism

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Abstract

This paper studies the problem of assigning a set of indivisible objects to a set of agents when monetary transfers are not allowed. We offer two characterizations of the prominent lottery assignment mechanism called the probabilistic serial. We show that it is the only mechanism satisfying non-wastefulness and ordinal fairness. Our second result shows that a direct ordinal mechanism satisfies ordinal efficiency, envy-freeness, and upper invariance if and only if it is the probabilistic serial.

Keywords: Random assignment; Probabilistic serial; Ordinal efficiency; Fairness

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1 Introduction

A wide range of resource allocation economies involves the assignment of indivisible objects without the use of monetary transfers. Well-known examples include college admissions (Gale and Shapley, 1962; Balinski and Sönmez, 1999), the assignment of students to public schools (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005), transplantation of organs through exchange of live donors (Roth, Sönmez, and Ünver, 2004) or allocation of deceased donors (Su and Zenios, 2005), assignment of college course seats to students (Sönmez and Ünver, 2010), distribution of on-campus housing units to college students (Abdulkadiroğlu and Sönmez, 1999), assignment of tasks to workers (Hylland and Zeckhauser, 1979), assignment of time slots to users of a public good, etc. Most of these markets use ordinal matching mechanisms (where the participants reveal only their ordinal preferences, i.e., their preference rankings over given choices, to the central authority rather than cardinal von Neumann-Morgenstern utility functions).

In such applications ensuring fairness in an ex-post sense can entail significant efficiency losses (see, for example, Kesten and Yazici (2009)). Therefore, random assignment mechanisms are commonly used to restore fairness ex ante.¹

A random assignment mechanism, or simply a mechanism, is a systematic way of selecting a random assignment for any given problem. A random assignment specifies the probability with which each agent receives each object. Two prominent mechanisms have been the focus of a growing literature on assignment problems: the random priority mechanism² and the probabilistic serial mechanism.

The random priority, which is probably the most commonly used real-life mechanism, works as follows:³ Draw a random ordering of agents from the uniform distribution, and let the first agent choose her favorite object; next, let the second agent choose her favorite object among the remaining objects; and so on. The random priority has two important properties. First, it is strategy-proof, i.e., it is a dominant strategy for each agent to state her preferences truthfully. Second, it induces an ex-post efficient lottery over assignments, i.e., any assignment that may be produced by the

¹For example, the assignment mechanisms used in the context of school choice operate through a collection of strict priority orders of students over schools. In practice, determining these orders often involves randomization (Abdulkadiroğlu and Sönmez, 2003; Pathak, 2006; Erdil and Ergin, 2008; Kesten and Ünver, 2010). In allocating courses to students, some colleges also follow a similar approach (Budish and Cantillon, 2010). In the exchange of live-donor kidneys among kidney patients for transplantation, the egalitarian approach requires the design of a random mechanism (Roth, Sönmez, and Ünver, 2005).

²It is also commonly referred as the random serial dictatorship mechanism.

³For example, the school-choice mechanism recently adopted in the Boston and New York public school districts reduces to the random priority if all students have equal priorities at all schools.
random priority is Pareto efficient. In a seminal paper Bogomolnaia and Moulin (2001) show that the random priority fails to satisfy a compelling notion of efficiency called ordinal efficiency, which is stronger than ex-post efficiency. A mechanism is ordinally efficient if its outcome is not (first-order) stochastically dominated by an alternative random assignment. This implies, for instance, that the ex-ante outcome of the random priority may not be Pareto efficient whenever agents are expected utility maximizers regardless of the utility functions representing their preferences.

The probabilistic serial mechanism, proposed by Bogomolnaia and Moulin (2001), does not suffer from the same drawback as the random priority. Its outcome is computed through the simultaneous eating algorithm, which works as follows: Think of each object as an infinitely divisible commodity. Each agent “eats” from her favorite object at the same speed until one of the objects is completely consumed. Next, each agent “eats” from her favorite object among the remaining objects at the same speed until one of the objects is completely consumed; and so on. The amount of an object eaten by an agent throughout the process is interpreted as the probability with which the agent is assigned this object by the probabilistic serial mechanism. The probabilistic serial has two important properties. First, it is ordinally efficient, i.e., its outcome is Pareto efficient regardless of the von Neumann-Morgenstern utility functions of the agents. Second, it is envy-free, that is, each agent’s random assignment under the probabilistic serial stochastically dominates the random assignment of any other agent. It is also weakly strategy-proof, that is, no agent by misstating her preferences can obtain an alternative assignment that stochastically dominates her assignment by truth-telling. In large economies, it becomes strongly strategy-proof, i.e., truth-telling stochastically dominates any misstatement of preferences for each agent (Kojima and Manea, 2010). Moreover, in continuum economies with a finite number of types of objects, it is equivalent to the random priority. As Che and Kojima (2009) show, in the limit of discrete economies with finite object types, this equivalence also holds, i.e., the probabilistic serial and random priority converge to each other.

1.1 Our Contribution

In this paper we provide two characterizations of the probabilistic serial mechanism. As probabilistic serial and random priority are the same mechanisms for large economies, our characterizations can also be viewed essentially as the first characterizations of both mechanisms for large problems. We refer to a random discrete resource assignment problem as a house allocation problem and to the indivisible objects as houses, following the terminology of Shapley and Scarf (1974). We consider generalized house allocation problems with strict preferences and voluntary participation for agents. There can be multiple identical copies of a house, as in school choice where a school has multiple
Our first main result is a characterization of the probabilistic serial mechanism based on two properties, non-wastefulness and ordinal fairness. Non-wastefulness is a standard and weak efficiency property. A mechanism is non-wasteful if no agent ever has a positive chance of being assigned an object while a copy of another object to which she would rather be matched has a positive chance of remaining vacant. On the other hand, ordinal fairness is a new property.

A mechanism is ordinally fair if its outcome satisfies the following requirement: Take an agent \( i \) who receives some object \( x \) with positive probability. Then there should be no other agent who receives a weakly better object than \( x \) for herself with a smaller probability than the probability with which agent \( i \) receives a weakly better object for herself. Ordinal fairness is a notion of equity from the viewpoint of a benevolent social planner who apriori sees agents as having equal rights over each object. Our first result shows that a mechanism satisfies non-wastefulness and ordinal fairness if and only if it is the probabilistic serial mechanism (Theorem 1). An immediate corollary of this result is that in economies with perfect supply (i.e., when the number of agents is exactly equal to the total number of copies of the goods available), a mechanism is ordinally fair if and only if it is equivalent to the probabilistic serial mechanism.

Ordinal fairness is quite different from existing notions of fairness in the literature. While existing notions of fairness, such as envy-freeness, focus on random assignments from the viewpoint of each agent in an aggregate fashion, ordinal fairness is a notion from the viewpoint of each agent on each object individually. Since ordinal fairness respects fairness individually for each object, aggregate consumptions also become fair. That is, ordinal fairness implies envy-freeness.

If two agents are offered the same object with some positive (but possibly different) probabilities at a random assignment, ordinal fairness requires that they should then receive an object at least as good as this particular one with the same probability; i.e., their individual upper-contour sets at this object should have the same cumulative probability. Fairness is perceived in an ordinal way: Both agents a priori have equal property rights over this object. So if they are both given a chance of ever receiving it, they should also receive a weakly better object with equal chances.

On the other hand, if neither of a given pair of agents has a chance of receiving a particular object, say object \( a \), with positive probability, ordinal fairness does not place any restriction on comparing the two agents’ chances of getting weakly better objects. However, if only one of two agents, say agent 1, is consuming an object with positive probability while the other, say agent 2, is not, then ordinal fairness restricts agent 1’s probability of getting a weakly better object than \( a \) to not be

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\[4\]This is a generalization of the Bogomolnaia and Moulin (2001) model that only allowed for assignment with perfect supply with single copies of houses and no outside options.
larger than that of agent 2 who is not consuming it at all. Note that not offering any chance to agent 2 of getting object $a$ does not imply a more favorable treatment toward the first agent. This is because ordinal fairness allows agent 2 to have a probability share over strictly better objects than $a$ that is no smaller than the probability share of agent 1 of object $a$.

In this sense, the ordinal fairness requirement takes the normative viewpoint that all agents a priori have equal rights over houses, and allows them to redistribute the assignment probabilities among themselves in any way so long as no agent is “disadvantaged” by the redistribution.

For three-agent problems Bogomolnaia and Moulin (2001) show that the probabilistic serial is the only mechanism satisfying ordinal efficiency, envy-freeness, and weak strategy-proofness. We show that this result no longer holds if there are at least five agents (Proposition 3).

For our main characterization, we introduce an auxiliary robustness axiom called upper invariance. A mechanism is upper invariant if whenever the ranking of a particular object is improved by demoting in the preferences of a particular agent some better objects that the agent has zero probability of receiving (while the relative ranking of worse objects can be arbitrarily changed among each other), the assignment probability of this object stays the same for every agent in the problem. It is a natural concept, as it diminishes the roles of houses that have no chance of assignment to an agent. Listing or not listing them in the upper contour set of a specific house makes no difference for the assignment chances of this house. Upper invariance is satisfied by a wide class of deterministic strategy-proof and Pareto-efficient mechanisms such as the hierarchical exchange (Papai, 2000) mechanisms. Therefore, the random priority is also upper invariant, since this property is combinative over lotteries and every serial dictatorship lies in the class of hierarchical exchange mechanisms.

Our second main result shows that a mechanism satisfies ordinal efficiency, envy-freeness, and upper invariance if and only if it is the probabilistic serial mechanism (Theorem 2).

We have already defined and discussed ordinal efficiency. Envy-freeness is the most common fairness axiom in the literature. It views fairness from each agent’s perspective. A random assignment is envy-free if each agent’s allocation stochastically dominates (according to her own preferences) any other agent’s allocation, i.e., each agent unambiguously prefers her own allocation to that of any other agent (Foley, 1967). Besides its fairness appeal, it is an important stability and robustness property. For example, any exchange economy competitive equilibrium allocation obtained from equal endowments is envy-free, implying that such an outcome can be autonomously achieved in markets.

On the other hand, upper invariance has close connections with incentive properties. It implies strategy-proofness for deterministic mechanisms. For deterministic mechanisms, upper invariance and non-bossiness (Satterthwaite and Sonnenschein, 1981) imply coalitional strategy-proofness or
equivalently Maskin monotonicity (Maskin, 1999; Takamiya, 2001). Non-bossiness requires that if an agent cannot change her allocation by changing her preferences, then she should not be able to change anybody else’s, either. The probabilistic serial, which is upper invariant, also satisfies non-bossiness, although it is only weakly strategy-proof, i.e., truth-telling is not stochastically dominated by any untruthful revelation of preferences. Yet, it is not difficult to see a deep relationship between upper invariance and incentive properties of assignment mechanisms.

1.2 Related Literature

There are very few papers discussing the random assignment problem prior to the new millennium. The earliest account of the problem is due to Hylland and Zeckhauser (1979), who proposed a pseudo-market mechanism that relies on cardinal preferences of agents. Much later, Zhou (1990) proved an important impossibility result for the cardinal domain: There exists no strategy-proof, Pareto-efficient, and symmetric mechanism.

Another negative result, which is similar in spirit to Zhou (1990), was later proven by Chambers (2004). He showed that the uniform rule, the distribution of objects with an even lottery without considering any preference information, is the unique mechanism that satisfies equal treatment of equals and a strong version of probabilistic consistency.

Following the ground breaking work of Bogomolnaia and Moulin (2001) that introduced the probabilistic serial, the literature on the random assignment problem is growing rapidly. Contrary to the early literature, the new strand of literature restricts attention to the case when agents’ preferences are ordinal.\(^5\)

The probabilistic serial was initially proposed by Crès and Moulin (2001) for a simple model where agents have the same rankings over objects. A characterization for this special context is given by Bogomolnaia and Moulin (2002). Kojima and Manea (2010) show that the probabilistic serial recovers strategy-proofness when the market size becomes sufficiently large. Manea (2009) shows that ordinal inefficiency of the random priority prevails even for large assignment problems. Katta and Sethuraman (2006) extend the probabilistic serial to the domain of weak preferences. Yılmaz (2006, 2009) adapt it to environments where there may be initial property rights over some of the objects. Athanassoglou and Sethuraman (2007) further extend this model and mechanism to the case with probabilistic endowments. Kojima (2009) offers a generalization of the probabilistic serial to multiple assignment problems.

\(^5\)Three common justifications for the ordinal approach are as follows: First, since agents are boundedly rational, cardinal preferences are difficult to elicit. Second, ordinal mechanisms are relatively simpler and more practical than cardinal ones. Third, real-life matching markets function mostly through elicitation of ordinal preferences.
Abdulkadiroğlu and Sönmez (1998) have shown that the random priority is equivalent to a core mechanism that uniformly randomly selects an initial assignment of objects and then utilizes Gale’s celebrated *top trading cycles* (Shapley and Scarf, 1974) procedure. Sönmez and Ünver (2005), Pathak (2006), Sethuraman (2010), and Carroll (2010) extend this result to different random matching domains. Kesten (2009) shows a similar connection between the probabilistic serial mechanism and the top trading cycles procedure: Probabilistic serial is equivalent to a particular top trading cycles mechanism that initially endows each agent with an equal share of each object. He also provides a “modified” random priority mechanism that becomes equivalent to the probabilistic serial in the limit. Budish, Che, Kojima, and Milgrom (2010) characterize the constraints on random assignments that can also be satisfied by each of the deterministic assignments in the support of a lottery inducing it.

The compelling notion of ordinal efficiency has also been the main focus of other related papers. Abdulkadiroğlu and Sönmez (2003) offer a characterization of ordinally efficient random assignments. McLennan (2002) proves an interesting result on the relationship between ordinal efficiency and ex-ante efficiency. Manea (2008) provides a constructive proof of this result.

## 2 The Model

A *house allocation problem* (Hylland and Zeckhauser, 1979) is a list \((N, A, q, \succ)\) where

- \(N = \{1, \cdots, n\}\) is a finite set of agents;
- \(A\) is a finite set of objects; following Shapley and Scarf (1974)’s terminology, we refer to objects as *houses*;
- \(q = (q_a)_{a \in A}\) is a positive integer vector, where \(q_a\) is the number of copies of house \(a\) available (to which we sometimes refer as the *quota* of house \(a\)) such that \(\sum_{a \in A} q_a \geq |N|\); and
- \(\succ := (\succ_i)_{i \in N}\) is a preference profile, where \(\succ_i\) is the strict preference relation of agent \(i\) on \(A\). Let \(\succeq_i\) denote the weak preference relation induced by \(\succ_i\). We assume that preferences are linear orders, i.e., for all \(a, b \in A\), \(a \succeq_i b \iff a = b\) or \(a \succ_i b\).

Observe that this model is general enough to allow different interpretations: For example, we can have one house, which we will refer to as the *null house*, and assign it quota \(|N|\) to represent voluntary participation. Agents who are assigned the null house take their outside options by interpretation, or, using the matching jargon, they *remain unassigned*. The outside option can be ranked at any place in the rankings. Another example is the setting of Bogomolnaia and Moulin: Each house has
quota 1 and there are exactly \(|N|\) houses. This reflects perfect supply (i.e., total house quota is equal to the number of agents) with unit quotas.\(^6\)

A centralized authority shall assign houses to agents. We assume that probabilistic assignments are possible. A **random allocation** for agent \(i\) is a vector \(P_i = (p_{ia})_{a \in A}\) where \(p_{ia} \in [0, 1]\) denotes the probability that agent \(i\) receives house \(a\), and \(\sum_{a \in A} p_{ia} = 1\). A **random assignment**, denoted as \(P = [P_i]_{i \in N} = [p_{ia}]_{i \in N, a \in A}\), is a substochastic matrix, the rows of which correspond to the random allocations of agents such that probabilities along each row of a random assignment sums to one and the sum of probabilities along the column referring to each house \(a\) does not exceed \(q_a\), i.e., \(\sum_{i \in N} p_{ia} \leq q_a\).

Observe that a random assignment only gives the marginal probability distribution according to which each agent will be assigned a house. It does not specify the distribution according to which the houses should jointly be assigned to the agents. In order to define this joint probability distribution, we need to define deterministic assignments and probability distributions over them. An **assignment** is a random assignment \(P\) such that \(p_{ia} \in \{0, 1\}\) for all \(i \in N\) and \(a \in A\). Let \(\mathcal{M}\) be the set of assignments. A **lottery** \(\lambda = (\lambda_{\mu})_{\mu \in \mathcal{M}}\) is a probability distribution over assignments, i.e., \(\lambda_{\mu} \in [0, 1]\) for all \(\mu \in \mathcal{M}\) and \(\sum_{\mu \in \mathcal{M}} \lambda_{\mu} = 1\).

Clearly, each lottery induces a random assignment. Let \(P^\lambda\) be the random assignment induced by lottery \(\lambda\), i.e., \(p^\lambda_{ia} = \sum_{\mu: p_{ia}=1} \lambda_{\mu}\) for all \(i \in N\) and \(a \in A\). It turns out that the converse statement is also true: For each random assignment \(P\) there exists a lottery \(\lambda\) that induces it, i.e., \(P^\lambda = P\) (Birkhoff, 1946; von Neumann, 1953).\(^7\)

Thanks to the Birkhoff-von Neumann theorem, the centralized authority can simply restrict attention to random assignments rather than lotteries. Once a random assignment is determined, finding a lottery that induces it is a computationally easy task using the constructive proof of Birkhoff-von Neumann theorem and simple combinatorial search techniques such as the algorithm proposed by Edmonds (1965).

We refer to a procedure that can be used to find a random assignment for a given problem as a **mechanism**. Formally, a **mechanism** is a function from the set of problems to the set of random assignments. The centralized authority uses a mechanism to assign houses to agents based on their reported preferences for a given set of houses with certain quotas and agents. Throughout the paper, whenever it is not ambiguous, we will fix \(N\), \(A\), and \(q\) and denote a (house allocation) problem by a preference profile.

\(^6\)In this latter setting, one of the properties, which we will define below and use in our first characterization, non-wastefulness will be redundant, as all allocations will be non-wasteful.

\(^7\)See also Kojima and Manea (2010) for an extension of this result to economies with multiple house copies and substochastic random assignments.
3 The Probabilistic Serial Assignment Mechanism

Bogomolnaia and Moulin (2001) introduced the probabilistic serial mechanism (PS hereafter) which is based on an iterative algorithm. They referred to this algorithm as the simultaneous eating algorithm. First, we describe the simultaneous eating algorithm:

We fix a problem \((N,A,q,\succeq)\) throughout this section. Think of each house as an infinitely divisible good.

**Step 1:** Each agent eats from her favorite house at the same speed. Proceed to the next step when a house is completely exhausted.

**Step s, for \(s \in \{2, \ldots, S\}\):** Each agent eats from her favorite house among the remaining houses at the same speed. Proceed to the next step when a house is completely exhausted.

The procedure terminates after \(S \leq n\) steps when each agent has eaten exactly 1 total unit of houses. For an agent having eaten \(p_{ia}\) units of house \(a\) is interpreted as agent \(i\) receiving house \(a\) with probability \(p_{ia}\). The random allocation of agent \(i\) by the PS mechanism is then determined by the amount of each house she has eaten until the algorithm terminates. Let \(PS(\succeq)\) be the outcome random assignment of the PS mechanism for problem \(\succeq\).

Before formally describing the simultaneous eating algorithm we first introduce some new notation. Given \(a \in B\), let \(M(a,B)\) be the set of agents whose most preferred house in \(B\) is \(a\), i.e., \(M(a,B) = \{i \in N | a \succeq_i b \text{ for all } b \in B\}\). Given a preference profile \(\succeq\), the random assignment of the PS mechanism is defined by the following recursive procedure:

**The Simultaneous Eating Algorithm:**

Let \(A^0 = A, y^0 = 0, P^0 = [0]\), the \(n \times n\) matrix of zeros.

**Step s, for \(s \in \{1, 2, \ldots, S\}\):** For each \(a \in A^{s-1}\), define

\[
y^s(a) = \begin{cases} 
\min \left\{1, y^{s-1} + \frac{q_a - \sum_{i \in N} p_{ia}^{s-1}}{|M(a,A^{s-1})|}\right\} & \text{if } M(a,A^{s-1}) \neq \emptyset, \\
\infty & \text{otherwise}.
\end{cases}
\]

Then, define

\[
y^s = \min_{a \in A^{s-1}} y^s(a)
\]

\[
B^s = \{a \in A^{s-1} | y^s(a) = y^s\}
\]

\[
A^s = A^{s-1} \setminus B^s
\]

\[
p_{ia}^s = \begin{cases} 
p_{ia}^{s-1} + (y^s - y^{s-1}) & \text{if } i \in M(a,A^{s-1}), \\
p_{ia}^{s-1} & \text{otherwise}.
\end{cases}
\]
Since \( A \) is a finite set, there is a step \( S \) such that \( y^S = 1 \). We always take the minimum of such \( S \)'s. By construction, \( \emptyset \subseteq A^S \subseteq A^{S-1} \subseteq \cdots \subseteq A^0 \), and \( \{B^s\}_{s=1}^S \) is a partition of \( A \setminus A^S \). We define \( PS(\succ) := P^S \) to be the probabilistic serial assignment for the preference profile \( \succ \).

We define one additional concept regarding the eating algorithm: Given \( PS_{ia}(\succ) > 0 \), let \( s_i(a) \) be the step of the eating algorithm at which \( i \in N \) starts eating \( a \in A \), i.e., \( s_i(a) = \min\{t|i \in M(a, A^{t-1})\} \) whenever \( PS_{ia}(\succ) > 0 \).

Below, we provide an example to demonstrate the simultaneous eating algorithm using both its informal and formal descriptions.

We provide two useful lemmas about the properties of the \( PS \) mechanism. First, we introduce some new notation. Given house \( a \) and a preference \( \succ_i \) of agent \( i \), let \( U(\succ_i, a) := \{a'' \in A|a'' \succeq a\} \) be the upper contour set of house \( a \) at \( \succ_i \). Given a random allocation \( P_i \), let 

\[
F(\succ_i, a, P_i) := \sum_{a'' \in U(\succ_i, a)} p_{ia''}
\]

be the probability that agent \( i \) obtains a house at least as good as \( a \) under \( P_i \).

**Lemma 1** For all \( s = 1, \cdots, S \), \( a \in A^{s-1} \), and \( i \in N \),

(a) if \( i \notin M(a, A^{s-1}) \), then \( p^s_{ia} = 0 \),

(b) if \( i \in M(a, A^{s-1}) \), then \( p^s_{ia} = y^s - y^{s_i(a)} \).

The following is an immediate consequence of Lemma 1:

**Corollary 1** For all \( i \in N \),

(a) for all \( s = 1, \cdots, S \), \( a \in B^s \), \( PS_{ia}(\succ) \equiv p^s_{ia} = \begin{cases} p^s_{ia} & \text{if } i \in M(a, A^{s-1}), \\ 0 & \text{otherwise.} \end{cases} \)

(b) for all \( a \in A^S \), \( PS_{ia}(\succ) = 0 \).

The second lemma is as follows:

**Lemma 2** For all \( s = 1, \cdots, S \), \( a \in B^s \), and \( i \in M(a, A^{s-1}) \),

\[
y^s = F(\succ_i, a, PS_i(\succ)).
\]

The above lemma says that the cumulative assigned probability to each agent until the end of step \( s \), \( y^s \), coincides with the cumulative assigned probability by the PS mechanism of the houses in the upper contour set of a completely eaten house for an agent who eats from it in step \( s \).\(^8\)

\(^8\)This enables us to understand how ordinal fairness, our new fairness axiom that is characterized by the PS mechanism, is satisfied in the simultaneous eating algorithm. The proofs of the lemmas are in Appendix A.1.
Example 1 (The execution of the simultaneous eating algorithm.) Consider the following preference profile in a problem where houses have single copies:

\[
\begin{align*}
  a & \succ _1 b \succ _1 c \\
  a & \succ _2 b \succ _2 c \\
  b & \succ _3 c \succ _3 a
\end{align*}
\]

Step \( s = 1 \): Each agent eats from her favorite house at the same speed. That is, agents 1 and 2 eat from house \( a \); and agent 3 eats from house \( b \) at the same speed. We stop when house \( a \) is exhausted.

At this point, agents 1 and 2 have eaten \( 1/2 \) of \( a \) each, and agent 3 has eaten \( 1/2 \) of \( b \).

Formally, \( A^0 = A, y^0 = 0, P^0 = [0] \). We have \( M(a, A^0) = \{1, 2\} \) and \( M(b, A^0) = \{3\} \). Hence, \( y^1(a) = 1/2, y^1(b) = 1, \) and \( y^1(c) = \infty \). Then \( y^1 = 1/2, B^1 = \{a\}, \) and \( A^1 = \{b, c\} \). Thus the partial assignments are \( p^1_{1a} = p^1_{2a} = 1/2, \) and \( p^1_{3b} = 1/2 \).

Step \( s = 2 \): Each agent eats from her remaining favorite house at the same speed. That is, all agents eat from house \( b \) (which is available with \( 1/2 \) probability) at the same speed. We stop when house \( b \) is exhausted. At this point, each agent has eaten \( 1/6 \) of \( b \).

Formally, we now have \( M(b, A^1) = \{1, 2, 3\} \). Hence, \( y^2(b) = 1/2 + \frac{1-1/2}{3} = 2/3, \) and \( y^2(c) = \infty \). Then \( y^2 = 2/3, B^2 = \{b\}, \) and \( A^2 = \{c\} \). Thus the partial assignments are \( p^2_{1b} = p^2_{2b} = 1/6, \) and \( p^2_{3b} = 1/2 + 1/6 = 2/3 \).

Step \( s = 3 \): The only remaining house is \( c \) (which has not been eaten before). Thus all agents eat from house \( c \) at the same speed. We stop when house \( c \) is exhausted. At this point, each agent has eaten \( 1/3 \) of it.

Formally, we now have \( M(c, A^2) = \{1, 2, 3\} \). Hence, \( y^3(c) = 2/3 + 1/3 = 1. \) Then \( y^3 = 1, B^3 = \{c\}, \) and \( A^3 = \emptyset \). Thus all partial assignments are now complete. From this step we have \( p^3_{1c} = p^3_{2c} = p^3_{3c} = 1/3 \).

Since each agent is assigned a total of probability 1 at the end of step 3, the procedure stops.

The procedure leads to the following random assignment:

\[
P = \frac{1}{6} \begin{pmatrix} 3 & 1 & 2 \\ 3 & 1 & 2 \\ 0 & 4 & 2 \end{pmatrix}.
\]

\( \diamond \)

In the next section we introduce the axioms that will be used in characterizing the PS mechanism and vice versa.
4 Axioms

4.1 Ordinal Efficiency

We start with an appealing efficiency property satisfied by the PS mechanism. Bogomolnaia and Moulin (2001) propose quite a powerful notion of efficiency called ordinal efficiency – based on ordinal preference information – which is much stronger than the commonly used requirement of *ex-post efficiency*, i.e., having an equivalent lottery whose support contains only Pareto-efficient assignments.

To define this efficiency property, we need to define a partial preference relation over random allocations. Given a preference profile $\succ$ and random assignments $P$ and $Q$, $P$ **stochastically dominates** $Q_i$ at $\succ$ if

$$F(\succ_i, a, P_i) \geq F(\succ_i, a, Q_i) \quad \text{for all } a \in A$$

In addition, $P$ **stochastically dominates** $Q$ at $\succ$ if $P_i$ stochastically dominates $Q_i$ at $\succ_i$ for all $i \in N$.

We are now ready to introduce our central efficiency notion: A random assignment is **ordinally efficient** if it is not stochastically dominated by another random assignment (Hylland and Zeckhauser, 1979; Bogomolnaia and Moulin, 2001).

A mechanism is **ordinally efficient** if it always selects an ordinally efficient random assignment for any problem.

Bogomolnaia and Moulin (2001) show that the PS mechanism is ordinally efficient.

4.2 Envy-freeness

Our first fairness notion is a fundamental fairness principle in mechanism design theory originally proposed by Foley (1967). Given a preference profile $\succ$, a random assignment $P$ is **envy-free** at $\succ$ if for all $i \in N$, $P_i$ stochastically dominates $P_j$ for all $j \in N$ at $\succ_i$ (Foley, 1967).

A mechanism is **envy-free** if it always selects an envy-free random assignment for any problem.

Envy-freeness is satisfied by many important allocation rules in different domains, such as the competitive equilibrium from equal endowments (Hylland and Zeckhauser, 1979), the uniform rule for single-peaked preferences over a divisible resource (Benassy, 1982), and the max-min allocation rule for quasi-linear preferences over indivisible objects and a fixed amount of money (Alkan, Demange, and Gale, 1991). It is an important robustness property in mechanism design.
4.3 Upper Invariance

Next, we will formally introduce our natural auxiliary axiom that will be used to characterize ordinal efficiency and envy-freeness.

First, we define $\succ_i |_B$ as the restriction of $\succ_i$ to the set of houses in $B \subseteq A$, that is, $\succ_i |_B$ is a preference relation over $B$ such that for all $a, b \in B$, $a \succ_i |_B b \iff a \succ_i b$.

**Definition 1** For a given random assignment $P$ and a given problem $\succ_i$, $\succ'_i$ is an upper invariant transformation of $\succ_i$ for agent $i$ at house $a$ under $P$, if for some $Z \subseteq \{c \in A | P_{ic} = 0\}$, $U(\succ'_i, a) = U(\succ_i, a) \setminus Z$ and $\succ'_i |_{U(\succ'_i, a)} = \succ_i |_{U(\succ'_i, a)}$.

In an upper invariant transformation at $a$ under $P$, the upper contour set at $a$ shrinks by removing houses that have no assignment chance. However, the relative rankings of houses weakly preferred to $a$ in $\succ'_i$ stay the same as in $\succ_i$. On the other hand, the relative ranking of houses that are strictly worse than $a$ in $\succ'_i$ can be arbitrary and does not need to match with that in $\succ_i$. Below, we give some examples of upper invariant transformations:

**Example 2 (Upper invariant transformations)** Consider a preference relation $\succ_i$ for agent $i$ such that $b \succ_i c \succ_i a \succ_i d \succ_i e$. Suppose $p_{ib} = p_{ic} = 0$ at random assignment $P$. The following are examples of upper invariant transformations of $\succ_i$ at house $a$:

- $a \succ'_i b \succ'_i c \succ'_i d \succ'_i e$,
- $b \succ'_i c \succ'_i a \succ'_i e \succ'_i d$,
- $b \succ'_i a \succ'_i d \succ'_i c \succ'_i e$,
- $c \succ'_i a \succ'_i e \succ'_i d \succ'_i b$.

◊

We are now ready to define our invariance concept:

**Definition 2** A mechanism $\phi$ is upper invariant if for all $\succ$, $i \in N$, $\succ'_i$, and $a \in A$, if $\succ'_i$ is an upper invariant transformation of $\succ_i$ at $a$ under $\phi(\succ)$, then for all $j \in N$, $\phi_{ja}(\succ'_i, \succ_{-i}) = \phi_{ja}(\succ_i, \succ_{-i})$.

The PS mechanism also satisfies upper invariance.

**Proposition 1** The PS mechanism is upper invariant.
Proof. Take any preference profile \( \succ \), any agent \( i \in N \), and any house \( a \). Let \( P = PS(\succ) \). Let \( \tilde{\succ}_i \) be an upper invariant transformation of \( \succ_i \) at \( a \) under \( P \). Let \( \tilde{P} = PS(\tilde{\succ}) \) where \( \tilde{\succ} = (\tilde{\succ}_i, \succ_{-i}) \). Then, \( U(\tilde{\succ}_i, a) = U(\succ_i, a) \setminus Z \) for some \( Z \subseteq \{ c \in A | p_{ic} = 0 \} \). We need to show \( \tilde{p}_{ja} = p_{ja} \) for all \( j \in N \).

First, consider the case where \( Z = \emptyset \). Let \( t(a) \) be the step in the simultaneous eating algorithm for \( \succ \) at which \( a \) is fully eaten. Moreover, in the simultaneous eating algorithm, let the variables \( A^{s-1}, B^s \) be determined for \( \succ \), and \( \tilde{A}^{s-1}, \tilde{B}^s \) for \( \tilde{\succ} \). Let \( M(a, C, \succ) \) be the set of agents whose most preferred house in \( \succ \) is \( a \) among houses in any set \( C \).

It is sufficient to show that for each step \( s \) with \( 1 \leq s \leq t(a) \), \( A^{s-1} = \tilde{A}^{s-1} \), \( M(c, A^{s-1}, \succ) = M(c, \tilde{A}^{s-1}, \tilde{\succ}) \) for each \( c \in A^{s-1} = \tilde{A}^{s-1} \), and \( B^s = \tilde{B}^s \). We prove by induction on step \( s \).

Step \( s = 1 \): First, \( A^0 \equiv A \equiv \tilde{A}^0 \).

Take any house \( c \in A^0 = \tilde{A}^0 \). We show for all \( j \in N \), \( j \in M(c, A^0, \succ) \Leftrightarrow j \in M(c, \tilde{A}^0, \tilde{\succ}) \). Fix any agent \( j \in N \). This holds for any agent \( j \neq i \) as \( \tilde{A}^0 = A^0 \) if \( \succ \) and \( \tilde{\succ}_j = \succ_j \). Consider agent \( j = i \). Suppose \( i \in M(c, A^0, \succ) \). Then, \( c \succ_i a \). Since houses are ranked in the same order in upper-contour sets of \( a \) for both \( \succ_i \) and \( \tilde{\succ}_i \), \( c \) is the most preferred in \( A^0 = \tilde{A}^0 \) under \( \tilde{\succ}_i \). Thus, \( i \in M(c, \tilde{A}^0, \tilde{\succ}) \). The converse can be obtained by the symmetric argument.

Because \( M(c, A^0, \succ) = M(c, \tilde{A}^0, \tilde{\succ}) \) for each \( c \in A^0 = \tilde{A}^0 \), the simultaneous eating algorithm implies \( B^1 = \tilde{B}^1 \), i.e., the same houses are consumed in the first step.

Step \( s \leq t(a) \): Suppose the claim is true up to step \( s - 1 \). First, \( A^{s-1} = A^{s-2} \setminus B^{s-1} = \tilde{A}^{s-2} \setminus \tilde{B}^{s-1} = \tilde{A}^{s-1} \) by the induction hypothesis.

Take any house \( c \in A^{s-1} = \tilde{A}^{s-1} \). We show for all \( j \in N \), \( j \in M(c, A^{s-1}, \succ) \Leftrightarrow j \in M(c, \tilde{A}^{s-1}, \tilde{\succ}) \). Fix any agent \( j \in N \). This clearly holds for any agent \( j \neq i \) as \( \tilde{A}^{s-1} = A^{s-1} \) if \( \succ \) and \( \tilde{\succ}_j = \succ_j \). Consider agent \( j = i \). Suppose \( i \in M(c, A^{s-1}, \succ) \). Since houses are ranked in the same order in upper-contour sets of \( a \) for both \( \succ_i \) and \( \tilde{\succ}_i \), \( c \) is the most preferred house in \( \tilde{A}^{s-1} = A^{s-1} \) under \( \tilde{\succ}_i \). Thus, \( i \in M(c, \tilde{A}^{s-1}, \tilde{\succ}) \). The converse can be obtained by the symmetric argument.

Because \( M(c, A^{s-1}, \succ) = M(c, \tilde{A}^{s-1}, \tilde{\succ}) \) for each \( c \in A^{s-1} = \tilde{A}^{s-1} \), the simultaneous eating algorithm implies \( B^s = \tilde{B}^s \). This completes the case for \( Z = \emptyset \).

Next, consider the case where \( Z \neq \emptyset \). Since agent \( i \) does not eat any house \( b \in Z \) in \( \succ \) and such a house is less preferred to \( a \) in \( \tilde{\succ} \), we can see from the previous proof that an upper invariant transformation does not change which house agent \( i \) eats until \( a \) is eaten. Thus, the previous induction still holds.

The induction also implies that all other variables of the algorithm under \( \succ \) and \( \tilde{\succ} \) are identical to each other until \( a \) is fully eaten. Thus, we conclude that \( p_{ja} = \tilde{p}_{ja} \) for all \( j \in N \). ■

Despite the fact that upper invariance is an auxiliary axiom, it is a mild axiom satisfied by a
variety of plausible mechanisms, such as the random priority or a hierarchical exchange. It is an independence of irrelevant alternatives (IIA) type of axiom: The houses that an agent does not receive with any positive probability and the houses ranked below the house in question have no effect in the allocation of this house to any agent so long as those are ranked below the house in question by this agent. We already discussed its link to incentive properties in the Introduction. Nota bene, we define one other concept to give a better sense of the incentive properties of the PS mechanism:

A mechanism $\phi$ is weakly strategy-proof if for all $\succ$ and $i \in N$, there is no $\succ'_{i}$ such that $\phi_{i}(\succ'_{i}, \succ_{-i})$ stochastically dominates $\phi_{i}(\succ_{i}, \succ_{-i})$ and $\phi_{i}(\succ'_{i}, \succ_{-i}) \neq \phi_{i}(\succ_{i}, \succ_{-i})$. That is, a mechanism is weakly strategy-proof if by manipulating her preferences, an agent can never receive a random allocation that is strictly better than her original one (under stochastic dominance).

Indeed, it turns out that the PS mechanism is weakly strategy-proof (Bogomolnaia and Moulin, 2001). However, it turns out that weak strategy-proofness is not enough to characterize the PS mechanism (see Proposition 3).

4.4 Non-wastefulness

We next introduce a very weak standard efficiency property:

**Definition 3** Given a preference profile $\succ$, a random assignment $P$ is non-wasteful at $\succ$ if for all $i \in N$ and $a \in A$ such that $p_{i,a} > 0$, $\sum_{j \in N} p_{j,b} = q_{b}$ for all $b \in A$ with $b \succ_{i} a$.

All random assignments are non-wasteful when the total quota of houses is equal to the number of agents. When agents are allowed to consume the null house, an example of a wasteful assignment is the one that leaves all agents unassigned with probability 1 even when there are agents who prefer some real house to the null house.

Ordinal efficiency implies non-wastefulness. Thus, the PS mechanism also satisfies non-wastefulness.

4.5 Ordinal Fairness

Our second new property is a fairness axiom:

**Definition 4** Given a preference profile $\succ$, a random assignment $P$ is ordinally fair at $\succ$ if for all $a \in A$, and $i, j \in N$ with $p_{ia} > 0$, we have $F(\succ_{i}, a, P_{i}) \leq F(\succ_{j}, a, P_{j})$.

A mechanism is ordinally fair if it always selects an ordinally fair random assignment for any problem.
Although we have extensively motivated this axiom in the Introduction, a couple of remarks are in order here: A random assignment is ordinally fair if, whenever an agent consumes a positive probability from a house, her total assignment probability in her upper contour set cannot exceed that of any other agent at the same house. That is, if two agents are assigned a positive probability at some house, their upper contour set consumptions should be equal to each other; on the other hand, if only one agent is assigned a positive probability, then the other agent should not envy her for this particular house, i.e., the other agent’s upper counter set consumption should not be smaller than the first. This requirement ensures that the only time one agent is not given any assignment chance at a house is when she has the chance of assignment to a better house.

Through the adoption of an ordinally fair mechanism, the mechanism designer, who thinks that all agents have equal consumption rights over the houses, is making sure that all agents are treated fairly for each house individually. Previous formulations of fairness rely on aggregate notions of equity. For example, envy-freeness implies that each agent should be treated fairly in terms of her aggregate assignment so that she does not prefer other agents’ allocations. It turns out that we still preserve envy-freeness if we adopt ordinal fairness: Achieving fairness housewise in this fashion entails aggregate fairness of assignments, i.e., ordinal fairness implies envy-freeness.

**Example 3 (Examples of ordinally fair random assignments)** We will state two examples:

First, consider the preference profile in Example 1. The random assignment $P$ that we calculated for the PS mechanism under this preference profile is ordinally fair, ordinally efficient, and envy-free. For the same preference profile, consider the following random assignment:

$$P = \frac{1}{6} \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 0 & 2 & 4 \end{pmatrix}.$$  

This is ordinally efficient and envy-free. However, it is not ordinally fair: $p_{1b} > 0$ and $F(\succ_1, b, P_1) = 5/6 > F(\succ_3, b, P_3) = 2/6$.

Second, consider a problem with outside options and a random assignment that leaves all agents unassigned with probability 1. This assignment is ordinally fair, but not necessarily ordinally efficient, as it might be wasteful. ♦

It is not difficult to determine that whether the PS mechanism satisfies this fairness property.

**Proposition 2** The PS mechanism is ordinally fair.

---

9This example is due to Bogomolnaia and Moulin (2001).
Proof. Fix a preference profile $\succ$. Let $P = PS(\succ)$ be the random assignment induced by the PS mechanism for $\succ$. Let $P_i = PS(\succ)$ be the random assignment induced by the PS mechanism for $\succ$. Let $a \in A$ and $i \in N$ with $p_{ia} > 0$. Fix $j \in N$. We need to show $F(\succ_i, a, P_i) \leq F(\succ_j, a, P_j)$. Since $p_{ia} > 0$, there exists some step $s = 1, \ldots, S$ such that agent $i$ eats house $a$ in the simultaneous eating algorithm until it disappears at step $s$, i.e., $a \in B^s$ and $i \in M(a, A^{s-1})$. Then, it follows from Lemma 2 that

$$F(\succ_i, a, P_i) = y^s.$$  \hspace{1cm} (1)

Depending on whether or not agent $j$ wants to eat house $a$ at step $s$, we consider two cases. First, consider the case where she wants to eat it, i.e., $j \in M(a, A^{s-1})$. Then, $F(\succ_j, a, P_j) = y^s$ by Lemma 2. Thus, $F(\succ_i, a, P_i) = y^s = F(\succ_j, a, P_j)$ by (1). This is the desired equality.

Consider the other case where agent $j$ does not want to eat house $a$ at step $s$, i.e., $j \notin M(a, A^{s-1})$. Then, she eats some other house $a'$ instead of $a$ at step $s$, i.e., $a' \in B^s$. Since $a'$ is available at step $s$, we have $s \leq s'$. Also, she continues to eat $a'$ until step $s'$, that is, $j \in M(a', A^{s'-1})$. Thus, Lemma 2 implies $y^{s'} = F(\succ_j, a', P_j)$. Hence,

$$F(\succ_i, a, P_i) = y^s \quad (\because (1))$$
$$\leq y^{s'} \quad (\because s \leq s')$$
$$= F(\succ_j, a', P_j)$$
$$\leq F(\succ_j, a, P_j) \quad (\because a' \succ_j a)$$

This is the desired inequality. ■

5 A Characterization of Ordinal Fairness

Ordinal fairness emphasizes the equal rights of agents over the houses when they are viewed as social endowments. In our first main result, we prove that, for each problem, there is a unique ordinally fair and non-wasteful random assignment: This random assignment is the outcome of the simultaneous eating algorithm. In other words, the PS mechanism fully characterizes ordinal fairness with non-wastefulness, and vice versa.

Theorem 1 A mechanism is ordinally fair and non-wasteful if and only if it is the PS mechanism.

Before we explain the philosophy behind our proof, one immediate corollary to the theorem can be stated for economies with perfect supply, where non-wastefulness is vacuous:
Corollary 2  For problems with perfect supply, i.e., problems with \( \sum_{a \in A} q_a = |N| \), a mechanism is ordinally fair if and only if it is the PS mechanism.

One direction of the theorem follows from Proposition 2 and the fact that the PS mechanism is non-wasteful. To prove the other direction, we first introduce a recursive indexing procedure for any given random assignment \( \hat{P} \). Given a problem \( \succ \), this procedure indexes houses, agents, and cumulative assignment probabilities in \( \hat{P} \). Then, our proof strategy for Theorem 1 will be as follows: Through the indexing procedure, we define new variables \( \bar{S}, \{\bar{A}^s\}, \{\bar{B}^s\}, \) and \( \{\bar{y}^s\} \), and then show that if \( \hat{P} \) is ordinally fair, these variables must coincide with the corresponding variables (i.e., those without the upper bar) defined through the simultaneous eating algorithm of the PS mechanism associated with the preference profile \( \succ \), i.e. we show that \( S = \bar{S}, y^s = \bar{y}^s, A^s = \bar{A}^s, B^s = \bar{B}^s, \) and \( \bar{y}^s = y^s \) for each \( s = 1, \ldots, \bar{S} \). This, in turn, will show that when the ordinal fairness requirement is coupled with non-wastefulness, both algorithms are outcome equivalent and \( PS(\succ) = \hat{P} \).

The Indexing Procedure of \( \hat{P} \) under \( \succ \):
Let \( \bar{A}^0 = A, \bar{y}^0 = 0, \) and \( \bar{y}^0_i = 0 \) for each \( i \in N \).

Step \( s, \) for \( s \in \{1, 2, \ldots, n\} \): For each \( a \in \bar{A}^{s-1} \), define

\[
\bar{y}^s_i(a) = \begin{cases} 
\bar{y}^{s-1}_i + \bar{p}_{ia} & \text{if } i \in M(a, \bar{A}^{s-1}), \\
\infty & \text{otherwise}.
\end{cases}
\]

Then, define

\[
\bar{y}^s = \min \{ \bar{y}^s_i(a) | a \in \bar{A}^{s-1}, i \in N \}, \\
\bar{B}^s = \{ a \in \bar{A}^{s-1} | \bar{y}^s_i(a) \text{ for some } i \in N \}, \text{ and} \\
\bar{A}^s = \bar{A}^{s-1} \setminus \bar{B}^s.
\]

Also, for each \( i \in N \), define

\[
\bar{y}^s_i = \begin{cases} 
\bar{y}^s_i(a) & \text{if } i \in M(a, \bar{A}^{s-1}) \text{ for some } a \in B^s, \\
\bar{y}^{s-1}_i & \text{otherwise}.
\end{cases}
\]

Since \( A \) is finite, the indexing procedure is well defined. This procedure stops at the step \( \bar{S} \) when all houses whose sums of probabilities across agents are positive, i.e., houses \( a \) with \( \bar{q}_a := \sum_{i \in N} \bar{p}_{ia} > 0 \), disappeared from step 1 to \( \bar{S} \). This occurs either when there is no house left or when any remaining house has the zero probability for all agents, i.e., \( \hat{P}|_{\bar{A}^\bar{S}} = 0 \). By construction, \( \emptyset \subseteq \bar{A}^{\bar{S}} \subsetneq \bar{A}^{\bar{S}-1} \subsetneq \cdots \subsetneq \bar{A}^0 = A \) and \( \{B^s\}_{s=1}^{\bar{S}} \) is a partition of \( A \setminus \bar{A}^{\bar{S}} \). Note that in each step \( s, \) \( \bar{y}^s_i(a) \leq 1 \) for each agent \( i \in M(a, \bar{A}^{s-1}) \), and so \( \bar{y}^s \leq 1 \).
The indexing procedure is a way of partitioning the set of houses based on a given random assignment. It is similar to the simultaneous eating algorithm in that at each step agents’ favorite houses are considered and some of these disappear at the end of that step. The procedure records agent-specific “cumulative probabilities” that agents accumulate throughout the steps based on the order in which the houses disappear. It differs, however, from the simultaneous eating algorithm in that agents may accumulate different probability shares of their favorite houses within a given step.

Intuitively, at a given step \( s \) of the indexing procedure, we consider every house that is demanded as the favorite house of some agent from the set of available houses (denoted by \( \bar{A}^{s-1} \)) of that step. We index and record an associated probability for each such house as follows. Among these houses we determine which ones will disappear at the end of step \( s \) (denoted as \( \bar{B}^s \)). To do this we find those houses that face the smallest individual demand from an agent (denoted by \( \{\bar{y}^s_i(a)\} \)), which is calculated as the sum of the demanding agent’s assignment probability of the house under \( (\bar{A}^{s-1}) \) and the agent-specific cumulative probability (denoted by \( \{\hat{y}^s_i\} \)) that the demanding agent has accumulated until step \( s \). We then remove the houses that disappear at the end of step \( s \), and apply the same method at the next step to the set of remaining houses in order to index and record an associated probability for each house that is the favorite house of some agent among the available houses at step \( s+1 \), and so on. In this case, we say that this agent consumes the disappearing house.

A noteworthy observation about the indexing procedure is that if a disappearing house, say \( a \), of some step \( s \) is not the favorite house of an agent, say \( i \), at step \( s \), then even if \( \bar{P} \) may assign her this house with positive probability, i.e., \( \bar{p}_{ia} > 0 \), this probability will never matter for the indexing procedure, i.e., it will not be recorded.

**Example 4 (Examples of the execution of the indexing procedure.)** Consider the preference profile in Example 1. We consider two examples. Consider first an ordinally unfair random assignment

\[
\bar{P} = \frac{1}{6} \begin{pmatrix} 2 & 4 & 0 \\ 4 & 1 & 1 \\ 0 & 1 & 5 \end{pmatrix}.
\]

Step \( s = 1 \):

| \( \bar{A}^0 \) | \( M(\cdot, \bar{A}^0) \) | \( \hat{y}^1_1(\cdot) \) | \( \hat{y}^1_2(\cdot) \) | \( \hat{y}^1_3(\cdot) \) | \( \hat{y}^1 \) | \( \bar{B}^1 \) | \( \bar{A}^1 \) |
|---|---|---|---|---|---|---|
| \( a \) | 1, 2 | 2/6 | 4/6 | \( \infty \) | \( \infty \) | \( 1/6 \) | \( 1/6 \) | \( \checkmark \) |
| \( b \) | 3 | \( \infty \) | \( \infty \) | 1/6 | \( \checkmark \) |
| \( c \) | \( \emptyset \) | \( \infty \) | \( \infty \) | \( \infty \) | \( \checkmark \) |

\( \hat{y}^1 = \begin{pmatrix} 0 \\ 0 \\ 1/6 \end{pmatrix} \)
Observe that all barred variables in the indexing procedure (i.e., \( \{\tilde{A}^s\}, \{\tilde{B}^s\}, \{\tilde{y}^s\}, \tilde{S} \)) coincide with the corresponding unbarred variables of the simultaneous eating algorithm under \( \succ \) that we calculated in Example 1.

\( \diamond \)

We are now ready to prove Theorem 1:

**Proof of Theorem 1.** (\( \Leftarrow \)) It follows from Proposition 2 and the fact that PS is non-wasteful.

(\( \Rightarrow \)) Fix a preference profile \( \succ \). Let a random assignment \( \tilde{P} \) be ordinally fair and non-wasteful at this preference profile. The proof consists of three parts:
I. We state three claims that hold for the indexing procedure at \( \bar{P} \) under \( \succ \).

II. We show that all barred variables in the indexing procedure (i.e., \{\bar{\bar{A}}^{s-1}\}, \{\bar{B}^s\}, \{\bar{y}^s\}, \bar{S}) coincide with the corresponding unbarred variables of the simultaneous eating algorithm under \( \succ \).

III. We show that \( \bar{P} \) coincides with \( P \) that is the random assignment induced by the PS mechanism.

\textbf{Part I.}

We first introduce some notation: For each house \( a \in A \), let \( \bar{q}_a := \sum_{i \in N} \bar{p}_{ia} \) be the sum of given probabilities of \( a \) among all agents. Also, let

\[
\bar{B}^s = \left\{ a \in \bar{\bar{A}}^{s-1} \mid \sum_{i \in M(a, \bar{\bar{A}}^{s-1})} \bar{p}_{ia} = \bar{q}_a > 0 \right\}
\]

for each step \( s \).

The three claims needed are as follows for all \( s = 1, \ldots, \bar{S} \):

\textbf{Claim 5.1} For all \( a \in \bar{\bar{A}}^{s-1} \) and \( i \in N, i \in M(a, \bar{\bar{A}}^{s-1}) \Rightarrow \bar{p}_{ia} > 0 \); and if \( a \in \bar{B}^s \), then \( \bar{p}_{ia} > 0 \Rightarrow i \in M(a, \bar{\bar{A}}^{s-1}) \).

Claim 5.1 says that agent \( i \) demands house \( a \) that is fully consumed in step \( s \) if and only if she has a positive probability of getting \( a \) at \( \bar{P} \).

\textbf{Claim 5.2} \( \bar{B}^s \subseteq \hat{B}^s \). Moreover,

1. \( \bar{y}^s = \min\{\bar{y}^s_i(a) \mid a \in \bar{B}^s, i \in M(a, \bar{\bar{A}}^{s-1})\} \), and

2. for all \( a \in \bar{B}^s \) and \( i \in M(a, \bar{\bar{A}}^{s-1}) \), \( \bar{y}^s_i(a) = \bar{y}^s \).

Claim 5.2 says that any disappearing house in step \( s \) is actually assigned in full share to the agents who demand it in step \( s \) at \( \bar{P} \).

\textbf{Claim 5.3} Suppose \( i \in M(a, \bar{\bar{A}}^{s-1}) \) for some \( a \in \bar{\bar{A}}^{s-1} \). Let \( \bar{s}_i(a) \) be the first step in which agent \( i \) demands house \( a \), i.e., \( \bar{s}_i(a) := \min\{t \mid i \in M(a, \bar{\bar{A}}^{t-1})\} \).\(^{10}\) Then, \( \bar{y}^s_{\bar{s}_i(a)-1} = \bar{y}^\bar{s}_i(a)^{-1} \).

\(^{10}\) \( \bar{s}_i(a) \) corresponds to \( s_i(a) \) defined in the PS algorithm, and we will prove that they are equal.
Claim 5.3 says that when agent \(i\) demands house \(a\) at step \(s\), her indexed cumulative probability \(\hat{y}_i^{s-1}\) at step \(s-1\) coincides with the one at the step before she first demands it. This is because the indexing procedure records all assigned probability of a house immediately in one step, once it is indexed. Before the house is indexed, in the interim steps, this agent demands it; however, this house has to wait its turn to be indexed and disappear. Thus, in the meantime agent \(i\)’s indexed cumulative probability does not increase.

**Part II.**

We prove this part through the following claim:

**Claim 5.4** For all \(s = 1, \ldots, \min\{S, \bar{S}\}\), \(y^s = \bar{y}^s, B^s = \bar{B}^s, A^{s-1} = \bar{A}^{s-1}\).

Proofs of the above claims are in Appendix A.2.

**Part III.**

Finally, we show \(P = \bar{P}\). Take any \(a \in A\). Then, since \(\{B^s\}_{s=1}^{S}\) is a partition of \(A \setminus \bar{A}^S\), either \(a \in B^s\) for some \(s\) or \(a \in \bar{A}^S\).

Suppose the latter case holds, i.e., \(a \in A^S = \bar{A}^S\). We have \(p_{ia} = 0\) by Corollary 1. Also, since \(a \in \bar{A}^S\), it follows from the stopping condition in the indexing procedure that \(\bar{p}_{ia} = 0\). Hence, \(p_{ia} = \bar{p}_{ia} = 0\).

On the other hand, suppose the former case holds, i.e., \(a \in B^s\) for some \(s\). From Part II, \(B^s = \bar{B}^s\). Consider any agent \(i \in N\). First, consider the case where \(i \in M(a, A^{s-1}) = M(a, \bar{A}^{s-1})\). Then, we have

\[
p_{ia} = p_{ia}^s \quad (\because \text{Corollary 1 with } a \in B^s \text{ and } i \in M(a, A^{s-1}))
\]

\[
= y^s - y^{s_i(a)-1} \quad (\because \text{Lemma 1 with } i \in M(a, A^{s-1}))
\]

\[
= \bar{y}^s - \bar{y}^{\bar{s}_i(a)-1} \quad (\because y^s = \bar{y}^s, s_i(a) = \bar{s}_i(a), y^{s_i(a)-1} = \bar{y}^{\bar{s}_i(a)-1}).
\]

Now, because \(a \in \bar{B}^s\) and \(i \in M(a, \bar{A}^{s-1})\), Claim 5.2 implies \(\bar{y}^s = \bar{y}^s_i(a)\). Moreover, since \(i \in M(a, \bar{A}^{s-1})\), Claim 5.3 implies \(\bar{y}^{\bar{s}_i(a)-1} = \bar{y}^{\bar{s}_i(a)-1}\). Thus, it follows from (2) that

\[
p_{ia} = \bar{y}^{s_i(a)}_i - \bar{y}^{s_i-1}_i = \bar{p}_{ia} \quad (\because \text{definition}).
\]

On the other hand, consider the case where \(i \notin M(a, A^{s-1}) = M(a, \bar{A}^{s-1})\). Since \(i \notin M(a, A^{s-1})\) and \(a \in B^s\), Corollary 1 implies \(p_{ia} = 0\). On the other hand, since \(i \notin M(a, \bar{A}^{s-1})\) and \(a \in \bar{B}^s \subseteq \bar{B}^s\), Claim 5.1 implies \(\bar{p}_{ia} = 0\). Hence, \(p_{ia} = \bar{p}_{ia} = 0\).

Therefore, we have \(P = \bar{P}\), concluding the proof of Theorem 1. \(\blacksquare\)
6 A Characterization of Ordinal Efficiency and Envy-freeness

Ordinal efficiency and envy-freeness are appealing properties of assignment mechanisms. In this section, we prove that the PS mechanism is characterized by these two properties together with the natural robustness property we have introduced, upper invariance.

Ours is not the first attempt at this kind of a characterization. Bogomolnaia and Moulin (2001) give a full characterization of these two axioms together with weak strategy-proofness, a natural incentive property. However, the characterization holds when there are at most three agents.

Our first result shows that the characterization fails with five or more agents:

Proposition 3 If the number of agents is greater than or equal to five, the PS mechanism is not characterized by ordinal efficiency, envy-freeness, and weak strategy-proofness.

We prove this proposition through a counterexample, i.e., by constructing a mechanism that satisfies all three properties but is not equal to the PS mechanism. We show that this mechanism has these three properties in Appendix A.4. It turns out that this mechanism is not upper invariant, which we also prove in Appendix A.4.

Example 5 An ordinally efficient, envy-free, and weakly strategy-proof mechanism different from PS: Suppose that there are five agents and five houses each with quota 1.

Let $\succ^*$ be defined as follows, when $N = \{1, 2, ..., 5\}$ for five houses $a, b, c, d, e$ each with unit quota:

\[
\begin{align*}
  a &\succ^*_1 c \succ^*_1 d \succ^*_1 e \succ^*_1 b & i = 1, 2, 3 \\
  b &\succ^*_4 c \succ^*_4 d \succ^*_4 e \succ^*_4 a \\
  b &\succ^*_5 a \succ^*_5 c \succ^*_5 e \succ^*_5 d
\end{align*}
\]

We find the outcome of the PS mechanism for this problem as

\[
PS(\succ^*) = \frac{1}{720} \begin{pmatrix}
  240 & 0 & 192 & 180 & 108 \\
  240 & 0 & 192 & 180 & 108 \\
  240 & 0 & 192 & 180 & 108 \\
  0 & 360 & 72 & 180 & 108 \\
  0 & 360 & 72 & 0 & 288 \\
\end{pmatrix}
\]
Also, define

\[
P^\star = \frac{1}{720} \begin{pmatrix}
220 & 0 & 210 & 185 & 105 \\
220 & 0 & 210 & 185 & 105 \\
220 & 0 & 210 & 185 & 105 \\
0 & 360 & 75 & 165 & 120 \\
60 & 360 & 15 & 0 & 285
\end{pmatrix}
\]

We construct a mechanism \( \phi \) as follows:

\[
\phi(\succ) = \begin{cases} 
P^\star & \text{if } \succ = \succ^\star, \\
PS(\succ) & \text{otherwise}.
\end{cases}
\]

This mechanism is ordinally efficient, envy-free, weakly strategy-proof, but not upper invariant (see Appendix A.4).

\[\Diamond\]

Our main result of this section is the PS mechanism characterization of ordinal efficiency and envy-freeness together with upper invariance:

**Theorem 2** A mechanism is ordinally efficient, envy free, and upper invariant if and only if it is the PS mechanism.

**Proof.** \((\Leftarrow)\) The ordinal efficiency and envy-freeness of the PS mechanism are proved by Bogo- molnaia and Moulin (2001). Its upper invariance follows from Proposition 1.

\((\Rightarrow)\) Let a mechanism \( \phi \) be ordinally efficient, envy-free, and upper invariant. Because ordinal efficiency implies non-wastefulness, in order to prove that it is the PS mechanism, we show that it is ordinally fair, and by Theorem 1 the result follows. Fix a preference profile \( \succ \). For notational simplicity, let \( P = \phi(\succ) \). To simplify the proof, we slightly modify the indexing procedure: In the original, the set \( \bar{B}^s \) of disappearing houses is not always a singleton but may contain several houses. We modify this point so that it is a singleton. If it contains several houses, we pick any one house and remove it so that we have \( S \leq |A| \) steps in the indexing procedure. Let \( a^s \) be the house that disappears in step \( s \). Then, \( \bar{A}^{s-1} = A \setminus \{a^1, \ldots, a^{s-1}\} \) and \( \bar{B}^s = \{a^s\} \).

To show that \( \bar{P} \) is ordinally fair at \( \succ \), we need to compare the probabilities of the upper contour sets of a house for two arbitrary agents. To this end, we will construct a new preference profile from \( \succ' \) to \( \succ'' \): given a preference \( \succ'_i \) of agent \( i \), suppose she prefers house \( a \) to house \( b \). Then, we say that a new preference \( \succ''_i \) *upgrades* \( b \) just below \( a \) if \( \succ''_i \) ranks \( b \) just below \( a \) and keeps the relative ranking of the other houses the same.
In particular, we construct preferences \( \succ_s \) for each agent \( i \in N \) using upgrading. Set \( \succ^0 = \succ_i \).

For each step \( s = 1, \ldots, \bar{S} \),

- if \( i \in M(a^s, \bar{A}^{s-1}) \),\(^{11}\) \( \succ^s_i = \succ^{s-1}_i \).
- If \( i \in M(b, \bar{A}^{s-1}) \) with \( b \neq a^s \),
  - if \( \bar{s}_i(b) = s \),\(^{12}\) \( \succ^s_i \) upgrades \( a^s \) just below \( b \) in \( \succ^{s-1}_i \),
  - if \( \bar{s}_i(b) < s \), \( \succ^s_i \) upgrades \( a^s \) just below \( a^{s-1} \) in \( \succ^{s-1}_i \).

Below, we give an example of this construction:

**Example 6 (Construction of \( \{\succ^s_i\}\})**

<table>
<thead>
<tr>
<th>( \succ^s_i )</th>
<th>( a^3 )</th>
<th>( a^6 )</th>
<th>( a^2 )</th>
<th>( a^5 )</th>
<th>( a^1 )</th>
<th>( a^4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \succ^1_i )</td>
<td>( a^3 )</td>
<td>( a^1 )</td>
<td>( a^6 )</td>
<td>( a^2 )</td>
<td>( a^5 )</td>
<td>( a^4 )</td>
</tr>
<tr>
<td>( \succ^2_i )</td>
<td>( a^3 )</td>
<td>( a^1 )</td>
<td>( a^2 )</td>
<td>( a^6 )</td>
<td>( a^5 )</td>
<td>( a^4 )</td>
</tr>
<tr>
<td>( \succ^3_i )</td>
<td>( a^3 )</td>
<td>( a^1 )</td>
<td>( a^2 )</td>
<td>( a^6 )</td>
<td>( a^5 )</td>
<td>( a^4 )</td>
</tr>
<tr>
<td>( \succ^4_i )</td>
<td>( a^3 )</td>
<td>( a^1 )</td>
<td>( a^2 )</td>
<td>( a^6 )</td>
<td>( a^4 )</td>
<td>( a^5 )</td>
</tr>
<tr>
<td>( \succ^5_i )</td>
<td>( a^3 )</td>
<td>( a^1 )</td>
<td>( a^2 )</td>
<td>( a^6 )</td>
<td>( a^4 )</td>
<td>( a^5 )</td>
</tr>
<tr>
<td>( \succ^6_i )</td>
<td>( a^3 )</td>
<td>( a^1 )</td>
<td>( a^2 )</td>
<td>( a^6 )</td>
<td>( a^4 )</td>
<td>( a^5 )</td>
</tr>
</tbody>
</table>

Notice that at the end of the process, agent \( i \) actually consumes houses \( a^3 \) and \( a^6 \), houses \( a^1 \) and \( a^2 \) are reordered in an increasing way after \( a^3 \) until the next consumed house \( a^6 \), and houses \( a^4 \) and \( a^5 \) are reordered in an increasing way after \( a^6 \) until the end. ♦

Observe the pattern of the constructed preferences in this example. We prove in Claim A.5 in Appendix A.3 that this pattern holds in general.

For any step \( s = 1, \ldots, \bar{S} \), we also have the following claims. Their proofs are in Appendix A.3.

**Claim 6.1** For all \( a \in \bar{A}^{s-1} \) and \( i \in M(a, \bar{A}^{s-1}) \), \( y^i_s(a) = F(\succ_i, a, \bar{P}_i) \).\(^{13}\)

---

\(^{11}\)Recall that the set \( M(a^s, \bar{A}^{s-1}) \) of agents whose most preferred house is \( a^s \) in \( \bar{A}^{s-1} \) is defined for the original preference profile \( \succ \).

\(^{12}\)Recall that \( \bar{s}_i(b) \) is the first step in the indexing procedure at which agent \( i \) demands house \( b \).

\(^{13}\)Recall that \( y^i_s(a) \) is defined in the indexing procedure as the cumulative individual demand in step \( s \) including \( a \).
That is, for each agent, in the indexing procedure, the cumulative individual demand (i.e., the indexed cumulative probability of the houses that disappeared before plus the probability of house \( a \) that she demands at step \( s \)) is equal to the assigned cumulative probability of the upper contour set at \( a \) under \( \bar{P} \). Hence, the indexing procedure at \( \phi(\succ) \) indexes all assigned probabilities correctly when \( \phi \) is ordinally efficient, envy-free, and upper invariant.

Claim 6.2  
1. \( \phi(\succ^s) = \bar{P} \),

2. For all \( t = 1, \ldots, \bar{S} \) and \( i \in M(a^t, \bar{A}^{t-1}) \), \( F(\succ^t_i, a^t, \phi_i(\succ^s)) = F(\succ^t_i, a^t, \bar{P}_i) \).

3. For all \( i \in M(a^s, \bar{A}^{s-1}) \), \( F(\succ^s_i, a^s, \phi_i(\succ^s)) = \bar{y}^s \).

4. \( a^s \in \bar{B}^s \), i.e., \( \sum_{i \in M(a^s, \bar{A}^{s-1})} \bar{p}_{ia} = \bar{q}_a \).

Claim 6.2 states in part (1) that the random assignment of the mechanism for an updated preference profile stays the same as the original one, \( \bar{P} \equiv \phi(\succ) \). The rest of the statements pertain to an arbitrary step of the indexing procedure at \( \bar{P} \) under \( \succ^s \): Part (2) states that each agent has been assigned at \( \bar{P} \) the same cumulative probability for the upper contour set at the available house that she demands; part (3) says that \( \bar{y}^s \) coincides with the assigned probability of the upper-contour set at the disappearing house under the updated preference profile for an agent who demands it; and finally, part (4) states that the disappearing house is assigned at \( \bar{P} \) in full to the agents who demand it, hence any other agent has no chance of getting it.

Now, we are ready to conclude the proof and prove that \( \bar{P} \) is ordinally fair at \( \succ \) based on the above claims. Consider any agent \( i \) with \( \bar{p}_{ia} > 0 \) for some house \( a \). By the stopping condition of the indexing procedure, \( a \) is not in \( \bar{A}^\bar{S} \). Then, since \( A = \{a^1, \ldots, a^\bar{S}\} \cup \bar{A}^\bar{S} \), we have \( a = a^s \) for some \( s \) and thus \( i \in M(a^s, \bar{A}^{s-1}) \) by Part (4) of Claim 6.2. Now, we take any agent \( j \in N \). If \( j \in M(a^s, \bar{A}^{s-1}) \),

\[
F(\succ^s_i, a^s, \phi_i(\succ^s)) = \bar{y}^s = F(\succ^s_j, a^s, \phi_j(\succ^s)) \quad (\because \text{Part (3) of Claim 6.2})
\]

\[
\Rightarrow \quad F(\succ^s_i, a^s, \bar{P}_i) = F(\succ^s_j, a^s, \bar{P}_j) \quad (\because \text{Part (2) of Claim 6.2}).
\]

On the other hand, if \( j \notin M(a^s, \bar{A}^{s-1}) \), then \( j \in M(b, \bar{A}^{s-1}) \) for some house \( b \neq a^s \). Since \( \bar{y}^s \leq \bar{y}^s_j(b) \) by definition,

\[
F(\succ^s_i, a^s, \phi_i(\succ^s)) \leq F(\succ^s_j, b, \bar{P}_j) \quad (\because \text{Part (3) of Claim 6.2 and Claim 6.1})
\]

\[
\Rightarrow \quad F(\succ^s_i, a^s, \bar{P}_i) \leq F(\succ^s_j, b, \bar{P}_j) \quad (\because \text{Part (2) of Claim 6.2})
\]

\[
\leq F(\succ^s_j, a^s, \bar{P}_j) \quad (\because b \succ_j a^s).
\]
7 Independence of Axioms

First, we check the logical independence of axioms in Theorem 1.

Example 7 (An ordinally fair but wasteful mechanism) If the total quota of houses is equal to the number of agents, we have an assignment problem with perfect supply. Thus, non-wastefulness holds vacuously, as all random assignments are non-wasteful. In this case, there is no wasteful mechanism. When the total quota of houses exceeds the number of agents, consider the following mechanism: Fix $q'_a \leq q_a$ for all $a \in A$ such that $\sum_{a \in A} q'_a = n$, where $n$ is the number of agents. The mechanism that assigns agents according to the PS mechanism for artificial quota vector $(q'_a)_{a \in A}$ is ordinally fair but wasteful. ♦

Example 8 (A non-wasteful but ordinally unfair mechanism.) The random priority mechanism. ♦

Now we check the independence of axioms in Theorem 2. We have determined that the mechanism in Proposition 3 is ordinally efficient and envy-free, but not upper invariant. We now consider the other two cases.

Example 9 (An ordinally efficient and upper invariant mechanism that is not envy-free.) Consider a deterministic priority (or serial dictatorship) mechanism, $PM(\cdot)$. This mechanism is defined through a given priority ordering of agents: For a preference profile, the agent with the highest priority is assigned the most preferred house; the agent with the second highest priority is assigned the most preferred house among the remaining houses; and so on. This mechanism is Pareto efficient but not envy-free.

We show that it is upper invariant: Take any preference profile $\succ$, any agent $i \in N$, and any house $a$. Let $\tilde{\succ}_i$ be an upper invariant transformation of $\succ_i$ at $a$ under $PM(\succ)$. Let $\tilde{\succ} = (\tilde{\succ}_i, \succ_{\neg i})$.

Suppose $PM_{ia}(\succ) = 1$. Any house $c$ with $c \succ_i a$ is assigned to agents with higher priority than agent $i$ under $PM(\succ)$. Thus, any house $c$ with $c \succ_i a$ is also assigned to agents with higher priority than agent $i$ under $PM(\tilde{\succ})$, because $\tilde{\succ}_j \Rightarrow \succ_j$ for all $j \neq i$. So, agent $i$ is assigned $a$ under $PM(\tilde{\succ})$, and hence $PM_{ja}(\succ) = PM_{ja}(\tilde{\succ})$ for all $j \in N$.

Suppose $PM_{ia}(\succ) = 0$. Take the house $b \in A$ with $PM_{ib}(\succ) = 1$. Consider first the case where $a \succ_i b$. The house $a$ is assigned to agents with higher priority than agent $i$ under $PM(\succ)$. Since $\tilde{\succ}_j \Rightarrow \succ_j$ for all $j \neq i$, $a$ is assigned to the same agent under $PM(\tilde{\succ})$ as $PM(\succ)$. Hence, $PM_{ja}(\succ) = PM_{ja}(\tilde{\succ})$ for all $j \in N$. Consider next the case where $b \succ_i a$. Note that $b \tilde{\succ}_i a$. Any house $c$ with $c \succ_i b$ is assigned to agents with higher priority than agent $i$ under $PM(\succ)$. Such a
house is also assigned to the same agent under $PM(\succ)$ as $PM(\succ)$, because $\succ_j \Rightarrow_j$ for all $j \neq i$. Since $b \succ_i a$, agent $i$ is assigned $b$ under $PM(\succ)$. Thus, any agent with lower priority than $i$ faces the same remaining houses, and in particular, the agent assigned $a$ under $PM(\succ)$ is also assigned $a$ under $PM(\succ)$. Hence, $PM_{ja}(\succ) = PM_{ja}(\succ)$ for all $j \in N$. ◊

Example 10 (An envy-free and upper invariant mechanism that is not ordinally efficient.) Adjust the quotas of houses as $q'_{ia} \leq q_{ia}$ for all $a \in A$ so that $\sum_{a \in A} q'_{ia} = n$, where $n$ is the number of agents. Consider the mechanism that randomly assigns the houses with a uniform distribution according to the adjusted quotas, i.e., each agent receives each $a \in A$ with probability $\frac{q'_{ia}}{n}$. ◊

A Appendix

A.1 Proofs of Lemmas 1 and 2 and Corollary 1

Proof of Lemma 1. We prove this by induction on step $s$. First, we prove Part (a). For $s = 1$, if $i \not\in M(a, A^0)$, then $p^1_{ia} = p^0_{ia} \equiv 0$. Suppose it is true up to $s - 1$. Let $i \not\in M(a, A^{s-1})$. In words, agent $i$ does not want to eat house $a$ at step $s$. Then, she did not want to eat it at the previous step $s - 1 (i \not\in M(a, A^{s-2}))$. Thus,

$$p^s_{ia} = p^{s-1}_{ia} (\because i \not\in M(a, A^{s-1}))$$

$$= 0 (\because i \not\in M(a, A^{s-2}) and the induction hypothesis).$$

Next, we show Part (b). For $s = 1$, fix $a \in A^0$ and $i \in M(a, A^0)$. Then, $s_i(a) = 1$. Thus, since $y^0 \equiv 0$, we have $p^1_{ia} \equiv 0 + y^1 - y^0 = y^1 - y^{s_i(a)-1}$. Suppose it is true up to step $s - 1$. Fix $a \in A^{s-1}$ and $i \in M(a, A^{s-1})$ so that agent $i$ wants to eat house $a$ at step $s$. Depending on whether or not she wanted to eat it at the previous step $s - 1$, we consider two cases. First, consider the case where she wanted to eat it at step $s - 1$. Then, $s_i(a) \leq s - 1$, $a \in A^{s-2}$, and $i \in M(a, A^{s-2})$. Thus, it follows from the induction hypothesis that $p^{s-1}_{ia} = y^{s-1} - y^{s_i(a)-1}$. Hence, $p^s_{ia} \equiv p^{s-1}_{ia} + (y^s - y^{s-1}) = (y^{s-1} - y^{s_i(a)-1}) + (y^s - y^{s-1}) = y^s - y^{s_i(a)-1}$. On the other hand, consider the case where she did not want to eat $a$ at step $s - 1$. That is, $i \not\in M(a, A^{s-2})$. This implies $p^{s-1}_{ia} = 0$ from Part (a). Also, $s_i(a) = s$, because she started to eat $a$ at step $s$. Hence, $p^s_{ia} \equiv p^{s-1}_{ia} + (y^s - y^{s-1}) = y^s - y^{s-1} = y^s - y^{s_i(a)-1}$.

Proof of Corollary 1. Part (a) immediately follows from Lemma 1, and so we prove Part (b): Fix $i \in N$ and $a \in A^S$. In the last step $S$, $y^S = 1$. Since $a \in A^S = A^{S-1} \setminus B^S$, we have $y^S(a) = \infty$, and thus $M(a, A^{S-1}) = \emptyset$. It follows from Part (a) that $PS_{ia}(\succ) = 0$. ◊
Proof of Lemma 2. We prove this by induction on step \( s \). Let \( P^s = PS(\succ) \) be the random assignment obtained in the PS algorithm. For \( s = 1 \), fix \( a \in B^1 \) and \( i \in M(a, A^0) \). Then,

\[
F(\succ_{i}, a, P^{s}_{i}) = p^{s}_{ia} (\because i \in M(a, A^0) \text{ and } A^0 \equiv A)
\]

\[
= p^{1}_{ia} (\because \text{Corollary 1 with } a \in B^1 \text{ and } i \in M(a, A^0))
\]

\[
= y^{1} (\because i \in M(a, A^0)).
\]

Suppose it is true up to step \( s - 1 \). Fix \( a \in B^s \) and \( i \in M(a, A^{s-1}) \). That is, agent \( i \) wants to eat a disappearing house \( a \) at step \( s \). Depending on whether she has eaten house \( a \) from the beginning or she ate some other house \( a' \) at step \( s_{i}(a) - 1 \) that is the one before she started to eat \( a \), we consider two cases. Namely, either \( s_{i}(a) = 1 \) or \( \exists a' \in B^{s_{i}(a)-1} \text{ st } i \in M(a', A^{s_{i}(a)-2}) \). In the former case, \( a \) is the most preferred house in \( A \). Thus,

\[
F(\succ_{i}, a, P^{s}_{i}) = p^{s}_{ia} \quad (\because \text{Corollary 1 with } i \in M(a, A^{s-1}) \text{ and } a \in B^{s})
\]

\[
= y^{s} - y^{s_{i}(a)-1} (\because \text{Lemma 1 with } i \in M(a, A^{s-1}))
\]

\[
= y^{s} (\because y^{s_{i}(a)-1} = y^{0} \equiv 0).
\]

Consider the latter case. By the induction hypothesis, since \( a' \in B^{s_{i}(a)-1} \) and \( i \in M(a', A^{s_{i}(a)-2}) \),

\[
y^{s_{i}(a)-1} = F(\succ_{i}, a', P^{s}_{i}). \tag{3}
\]

Note \( a' \succ_{i} a \), because, at step \( s_{i}(a) - 1 \), both \( a \) and \( a' \) are available at step \( s_{i}(a) - 1 \) and agent \( i \) wants to eat \( a' \). Now, we prove

\[
\text{for all } a'' \in A \text{ with } a' \succ_{i} a'' \succ_{i} a, p^{s}_{ia''} = 0. \tag{4}
\]

Fix \( a'' \in A \) with \( a' \succ_{i} a'' \succ_{i} a \). First, \( a'' \) must disappear at step \( s'' \) before step \( s_{i}(a) \) \((s'' \leq s_{i}(a) - 1)\), otherwise agent \( i \) would want to eat \( a'' \) instead of \( a \) at step \( s_{i}(a) \). That is, \( a'' \in B^{s''} \) for some step \( s'' \leq s_{i}(a) - 1 \). Moreover, she did not want to eat \( a'' \) at step \( s'' \), because \( a' \) is available at step \( s_{i}(a) - 1 \) and thus was available at step \( s'' \leq s_{i}(a) - 1 \) with \( a' \succ_{i} a'' \). That is, \( i \notin M(a'', A^{s''-1}) \). Since \( a'' \in B^{s''} \) and \( i \notin M(a'', A^{s''-1}) \), Corollary 1 implies \( p^{s}_{ia''} = 0 \). Hence, (4) holds. Therefore,

\[
F(\succ_{i}, a, P^{s}_{i}) = F(\succ_{i}, a', P^{s}_{i}) + \sum_{a'' \in A \text{ s.t. } a' \succ_{i} a'' \succ_{i} a} p^{s}_{ia''} + p^{s}_{ia} (\because a' \succ_{i} a)
\]

\[
= y^{s_{i}(a)-1} + p^{s}_{ia} (\because \text{(3) and (4)})
\]

\[
= y^{s_{i}(a)-1} + p^{s}_{ia} (\because p^{s}_{ia} = p^{s}_{ia} \text{ by Corollary 1 with } a \in B^{s} \text{ and } i \in M(a, A^{s-1}))
\]

\[
= y^{s_{i}(a)-1} + (y^{s} - y^{s_{i}(a)-1}) (\because \text{Lemma 1 with } i \in M(a, A^{s-1}))
\]

\[
= y^{s}.
\]

\]
A.2 Proofs of the Claims in the Proof of Theorem 1

Part I. For each \( s = 1, \ldots, S \), we also prove the following claims, which will be used in proving Claims 5.1, 5.2, 5.3, and 5.4.

Claim A.1 For all \( a \in \bar{A}^{s-1} \) and \( i \in M(a, \bar{A}^{s-1}) \), \( \bar{y}^s_i(a) = F(\succ_i, a, \bar{P}) \).

Claim A.1 says that, at \( \bar{P} \), the individually demanded cumulative probability measured through \( \bar{y} \) for an agent who prefers \( a \) most among the remaining houses in any step \( s \) is equal to the probability of her being assigned in the upper contour set of \( a \). That is, we successfully index all cumulative probability of her upper contour set at \( a \) at \( \bar{P} \) through the indexing procedure.\(^{14}\)

Claim A.2 For all \( a \in \bar{A}^{s-1} \), if \( M(a, \bar{A}^{s-1}) \neq \emptyset \), then \( \bar{q}_a = \sum_{j \in N} \bar{p}_{ja} > 0 \). In particular, for all \( a \in \bar{B}^s \), \( \bar{q}_a > 0 \).

Claim A.2 results from Claim A.1 and non-wastefulness. This claim says that the sum of probabilities of any house that some agent wants to eat is always strictly positive.

Claim A.3 For all \( a \in \bar{B}^s \) and \( i \in M(a, \bar{A}^{s-1}) \), \( \bar{y}^s_i(a) = \frac{\bar{q}_a + \sum_{j \in M(a, \bar{A}^{s-1})} \bar{y}^{s-1}_j}{|M(a, \bar{A}^{s-1})|} \).

With Claim A.3, we have an explicit formula for \( \bar{y}^s_i(a) \). Note that the value of \( \bar{y}^s_i(a) \) is constant among agents in \( M(a, \bar{A}^{s-1}) \). Hence, its minimum, defined as \( \bar{y}^s \), is also equal to this constant.

Claim A.4 \( \bar{y}^{s-1} < \bar{y}^s \).

Claim A.4 says that the minimized demanded cumulative probability \( \bar{y}^s \) is strictly increasing in \( s \), i.e., in each step of the indexing procedure, we index some of each agent’s positive assignment probability at \( \bar{P} \). This is not always true for an arbitrary random assignment (see Example 4).

We prove all the claims together by induction on step \( s \) in the indexing procedure in the order A.1, A.2, 5.1, A.3, 5.2, A.4, 5.3, and 5.4:

**Step \( s = 1 \):**

**Proof of Claim A.1 for Step 1.** Fix \( a \in \bar{A}^0 \equiv A \). For each \( i \in M(a, \bar{A}^0) \), \( F(\succ_i, a, \bar{P}) = \bar{p}_{ia} = \bar{y}^1_i(a) \). \( \blacksquare \)

**Proof of Claim A.2 for Step 1.** Take any house \( a \in \bar{A}^0 \) with \( M(a, \bar{A}^0) \neq \emptyset \). Then, there exists \( i \in M(a, \bar{A}^0) \). For a contradiction, suppose \( \bar{q}_a = \sum_{j \in N} \bar{p}_{ja} = 0 \). Then, \( \bar{p}_{ia} = 0 \). Thus, since

\(^{14}\)See Lemma 2 for a similar claim for the simultaneous eating algorithm.

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\[ \sum_{b \in A} \bar{p}_{ib} = 1 \text{ and } i \in M(a, \bar{A}^0), \] 
there exists a house \( b \in A \) such that \( a \succ_i b \) and \( \bar{p}_{ib} > 0 \). It follows from non-wastefulness of \( \bar{P} \) at \( \succ \) that \( \bar{q}_a = q_a > 0 \), which is a contradiction.

The second statement follows from the fact that if \( a \in B^1 \) then there exists \( i \in M(a, \bar{A}^0) \) such that \( \bar{y}_i^1(a) = \bar{y}_i^1 \).

**Proof of Claim 5.1 for Step 1.** Fix \( a \in \bar{A}^0 \) and \( i \in M(a, \bar{A}^0) \). For a contradiction, suppose \( \bar{p}_{ia} = 0 \). Since \( \sum_{j \in N} \bar{p}_{ja} > 0 \) by Claim A.2, there is an agent \( j \in N \) such that \( \bar{p}_{ja} > 0 \). By ordinal fairness of \( \bar{P} \), \( 0 < \bar{p}_{ja} \leq F(\succ_j, a, \bar{P}_j) \leq F(\succ_i, a, \bar{P}_i) = \bar{p}_{ia} = 0 \). This is a contradiction. This concludes the proof of the first statement.

The second statement follows directly from the definition of \( \hat{B}^1 \).

**Proof of Claim A.3 for Step 1.** Fix \( a \in \hat{B}^1 \) and \( i \in M(a, \bar{A}^0) \). Since for all \( j \in N \), \( j \in M(a, \bar{A}^0) \Leftrightarrow \bar{p}_{ja} > 0 \) by Claim 5.1, for each agent \( j \in M(a, \bar{A}^0) \),

\[ F(\succ_j, a, \bar{P}_j) = F(\succ_j, a, \bar{P}_j) \quad (\because \text{ordinal fairness of } \bar{P} \text{ with } \bar{p}_{ia}, \bar{p}_{ja} > 0), \]

\[ \Rightarrow c := \bar{y}_i^1(a) = \bar{y}_i^1(a) \text{ where } c \text{ is constant } \quad (\because \text{Claim A.1}). \]

Also, because \( \bar{y}_j^1(a) = \bar{p}_{ja} = c \) for each \( j \in M(a, \bar{A}^0) \) and \( \sum_{j \in M(a, \bar{A}^0)} \bar{p}_{ja} = \bar{q}_a \), putting \( \bar{p}_{ja} \) in the former equation into the latter leads to \( \bar{y}_i^1(a) = c = \frac{\bar{q}_a}{|M(a, \bar{A}^0)|} \).

**Proof of Claim 5.2 for Step 1.** Let \( a \in \hat{B}^1 \). Take an agent \( i \in M(a, \bar{A}^0) \) such that \( \bar{y}_i^1(a) = \bar{y}_i^1 \). Then, \( \bar{q}_a > 0 \) by Claim A.2. For a contradiction, suppose \( a \not\in \hat{B}^1 \). Then, since \( \bar{q}_a > 0 \), \( \sum_{k \in M(a, \bar{A}^0)} \bar{p}_{ka} < \bar{q}_a \). Thus, there is \( j \not\in M(a, \bar{A}^0) \) with \( \bar{p}_{ja} > 0 \). Then, there is \( b \neq a \) with \( j \in M(b, \bar{A}^0) \). So, by Claim 5.1, \( \bar{p}_{ia} > 0 \) and \( \bar{p}_{jb} > 0 \). Therefore,

\[ F(\succ_i, a, \bar{P}_i) = F(\succ_j, a, \bar{P}_j) \quad (\because \text{ordinal fairness of } \bar{P}) \]

\[ \Rightarrow F(\succ_i, a, \bar{P}_i) \geq F(\succ_j, b, \bar{P}_j) + \bar{p}_{ja} \quad (\because b \succ_j a) \]

\[ \Rightarrow F(\succ_i, a, \bar{P}_i) > F(\succ_j, b, \bar{P}_j) \quad (\because \bar{p}_{ja} > 0) \]

\[ \Rightarrow \bar{y}_i^1(a) > \bar{y}_i^1(b) \quad (\because \text{Claim A.1}) \]

\[ \Rightarrow \bar{y}_i^1(a) > \bar{y}_i^1 \quad (\because \bar{y}_i^1(b) \geq \bar{y}_i^1 \text{ by definition of } \bar{y}_i^1) \]

which contradicts our assumption that \( \bar{y}_i^1 = \bar{y}_i^1(a) \).

This concludes the proof of \( \hat{B}^1 \subseteq \hat{B}^1 \).

This implies from the definition of \( \bar{y}_i^1 \) that \( \bar{y}_i^1 = \min\{\bar{y}_i^1(a) \mid a \in \hat{B}^1, i \in M(a, \bar{A}^0)\} \). Also, by Claim A.3, \( \bar{y}_i^1(a) \) is constant among all \( i \in M(a, \bar{A}^0) \) for some \( a \in \hat{B}^1 \).

**Proof of Claim A.4 for Step 1.** We have \( 0 = \bar{y}_0^1 < \bar{y}_1^1 \) by Claims A.3 and 5.2.

**Proof of Claim 5.3 for Step 1.** Since \( i \in M(a, \bar{A}^0) \), we have \( \bar{s}_i(a) = 1 \). Thus, \( \bar{y}_i^0 \equiv 0 \equiv \bar{y}_0^0 = \bar{y}_i^{\bar{s}_i(a) - 1} \).
**Step s:** Suppose Claims A.1 to 5.3 are true up to step \( s - 1 \) for some \( s > 1 \).

**Proof of Claim A.1 for Step s.** Fix \( a \in \tilde{A}^{s-1} \) and \( i \in M(a, \tilde{A}^{s-1}) \). At step \( s \), agent \( i \) demands house \( a \). Note that \( a \) did not disappear at step \( s - 1 \) \( (a \notin \tilde{B}^{s-1}) \), because house \( a \) is available at step \( s \). Depending on whether or not agent \( i \) demanded house \( a \) at step \( s - 1 \), we have two cases:

**Case 1:** Agent \( i \) demanded house \( a \) at step \( s - 1 \), i.e., \( i \in M(a, \tilde{A}^{s-2}) \). Then,

\[
\tilde{y}_i^s(a) = \tilde{y}_i^{s-1} + \tilde{p}_{ia} \quad (\because i \in M(a, \tilde{A}^{s-1}))
\]

\[
= \tilde{y}_i^{s-2} + \tilde{p}_{ia} \quad (\because \tilde{y}_i^{s-1} = \tilde{y}_i^{s-2} \text{ by definition with } a \notin \tilde{B}^{s-1}, i \in M(a, \tilde{A}^{s-2}))
\]

\[
= \tilde{y}_i^{s-1}(a) \quad (\because i \in M(a, \tilde{A}^{s-2}))
\]

\[
= F(\succ_i, a, \tilde{P}_i) \quad (\because \text{Claim A.1 for step } s - 1 \text{ with } i \in M(a, \tilde{A}^{s-2})).
\]

**Case 2:** Agent \( i \) did not demand house \( a \) at step \( s - 1 \), i.e., \( i \notin M(a, \tilde{A}^{s-2}) \).

Then, agent \( i \) demanded another house \( a' \neq a \) at step \( s - 1 \), i.e., \( i \in M(a', \tilde{A}^{s-2}) \). Note \( a' \succ_i a \). And \( a' \) must disappear at step \( s - 1 \), i.e., \( a' \in \tilde{B}^{s-1} \), otherwise she would demand \( a' \) instead of \( a \) at the next step \( s \). First, show

For all \( a'' \in \tilde{A} \) with \( a' \succ_i a'' \succ_i a \), \( \tilde{p}_{ia''} = 0 \). (5)

Fix a house \( a'' \in \tilde{A} \) with \( a' \succ_i a'' \succ_i a \). \( a'' \) must disappear at some step \( s'' \leq s - 1 \), otherwise agent \( i \) would demand \( a'' \) instead of \( a' \) at step \( s - 1 \). Moreover, she did not demand \( a'' \) at step \( s'' \), because \( a' \) is available at step \( s - 1 \) and thus was available at step \( s'' \leq s - 1 \). That is, we have \( i \notin M(a'', \tilde{A}^{s''-1}) \). Since \( a'' \in \tilde{B}^{s''} \), and \( \tilde{B}^{s''} \subseteq \tilde{B}^{s''} \) (\because Claim 5.2 for step \( s'' \leq s - 1 \)), and \( i \notin M(a'', \tilde{A}^{s''-1}) \), we have \( \tilde{p}_{ia''} = 0 \) by Claim 5.1 for step \( s'' \leq s - 1 \). Hence, (5) holds.

Now,

\[
F(\succ_i, a, \tilde{P}_i) = F(\succ_i, a', \tilde{P}_i) + \sum_{a'' \in \tilde{A}, a'' \succ_i a'' \succ_i a} \tilde{p}_{ia''} + \tilde{p}_{ia} \quad (\because a' \succ_i a)
\]

\[
= F(\succ_i, a', \tilde{P}_i) + \tilde{p}_{ia} \quad (\because (5))
\]

\[
= \tilde{y}_i^{s-1}(a') + \tilde{p}_{ia}
\]

(\because \( F(\succ_i, a', \tilde{P}_i) = \tilde{y}_i^{s-1}(a') \) by Claim A.1 for \( s - 1 \) with \( i \in M(a', \tilde{A}^{s-2}) \))

\[
= \tilde{y}_i^{s-1} + \tilde{p}_{ia} \quad (\because \tilde{y}_i^{s-1} = \tilde{y}_i^{s-1}(a') \text{ by definition with } i \in M(a', \tilde{A}^{s-2}) \text{ and } a' \in \tilde{B}^{s-1})
\]

\[
= \tilde{y}_i^s(a) \quad (\because i \in M(a, \tilde{A}^{s-1})).
\]

**Proof of Claim A.2 for Step s.** Take any house \( a \in \tilde{A}^{s-1} \) with \( M(a, \tilde{A}^{s-1}) \neq \emptyset \). Then, there is \( i \in M(a, \tilde{A}^{s-1}) \). For a contradiction, suppose \( \bar{q}_a = 0 \). Then, \( \bar{p}_{ia} = 0 \), and \( F(\succ_i, a, \tilde{P}_i) = 1 \)
by non-wastefulness. This implies that there exist \( s' < s \) and \( a' \in \dot{B}^{s'} \) such that \( \bar{p}_{ia'} > 0 \) and \( F(\succ_i, a', \dot{P}_i) = 1 \). Since \( \bar{p}_{ia'} > 0 \) and \( a \in \dot{B}^{s'} \) by Claim 5.2 for step \( s' \), we have \( i \in M(a', \dot{A}^{s'-1}) \) by Claim 5.1 for step \( s' \). Since \( i \in M(a', \dot{A}^{s'-1}) \) and \( a' \in B^{s'} \), it follows from Claim A.1 and 5.2 for step \( s' \) that \( \bar{y}^{s'} = \bar{g}_i^{s'}(a') = F(\succ_i, a', \dot{P}_i) \), and thus \( \bar{y}^{s'} = 1 \). Therefore, for all \( b \in \dot{A}^{s'-1} \) and \( j \in M(b, \dot{A}^{s'-1}) \), since \( \bar{g}_j^{s'}(b) \leq 1 \) and \( \bar{y}^{s'} = 1 \), it follows from the definition of \( \bar{y}^{s'} \) that \( \bar{g}_j^{s'}(b) = 1 \), and thus \( F(\succ_j, b, \dot{P}_j) = 1 \) by Claim A.1 for step \( s' \). Hence, \( P|_{\dot{A}^{s'}} = 0 \) and thus \( s' = S \), which contradicts the fact that \( s' < s \leq \dot{S} \). This concludes the proof of the first statement.

The second statement follows from the fact that if \( a \in \dot{B}^s \) then there exists \( i \in M(a, \dot{A}^{s-1}) \) such that \( \bar{g}_i^{s}(a) = \bar{y}^s \).

\boxed{\text{Proof of Claim 5.1 for Step } s.} \text{ Fix } a \in \dot{A}^{s-1} \text{ and } i \in M(a, \dot{A}^{s-1}). \text{ That is, agent } i \text{ demands house } a. \text{ If she demanded it at the previous step } s - 1 \text{ (i.e., } i \in M(a, \dot{A}^{s-2})) \text{, we have } \bar{p}_{ia} > 0 \text{ by Claim 5.1 for step } s - 1. \text{ Consider the other case where agent } i \text{ demanded another house } a' \neq a \text{ at the previous step } s - 1 \text{ (} i \in M(a', \dot{A}^{s-2}) \text{). Then, } a' \text{ must disappear at step } s - 1 \text{ (} a' \in \dot{B}^{s-1} \text{), otherwise she would have continued to consume } a \text{ at step } s. \text{ For a contradiction, suppose } \bar{p}_{ia} = 0. \text{ Then,}

\begin{align*}
F(\succ_i, a, \dot{P}_i) &= \bar{g}_i^{s}(a) \quad (\because \text{Claim A.1 for step } s \text{ with } i \in M(a, \dot{A}^{s-1})) \\
&= \bar{g}_i^{s-1} + \bar{p}_{ia} \quad (\because i \in M(a, \dot{A}^{s-1})) \\
&= \bar{g}_i^{s-1} \quad (\because \bar{p}_{ia} = 0) \\
&= \bar{y}^{s-1}_i(a') \quad (\because a' \in \dot{B}^{s-1}, i \in M(a', \dot{A}^{s-2})) \\
&= \bar{y}^{s-1} \quad (\because \text{Claim 5.2 for step } s - 1). \tag{6}
\end{align*}

Now, since \( \sum_{j \in N} \bar{p}_{ja} > 0 \) by Claim A.2, there is an agent \( j \in N \) with \( j \neq i \) such that \( \bar{p}_{ja} > 0 \). First, we have

\begin{align*}
F(\succ_j, a, \dot{P}_j) &\leq F(\succ_i, a, \dot{P}_i) \quad (\because \text{ordinal fairness of } \dot{P}) \\
&= \bar{y}^{s-1} \quad (\because (6)). \tag{7}
\end{align*}

Depending on whether or not agent \( j \) demanded some disappearing house at step \( s - 1 \), we consider two cases.

\text{Case 1: } j \notin \cup_{a'' \in B^{s-1}} M(a'', \dot{A}^{s-2}), \text{ i.e., agent } j \text{ did not demand any disappearing house at step } s - 1. \text{ Then, she demanded some non-disappearing house } b \text{ at step } s - 1, \text{ and thus continues to consume it}
at step s. That is, \( j \in M(b, \bar{A}^{s-1}) \), \( b \not\in \bar{B}^{s-1} \), and \( b \in \bar{A}^{s-1} \). Then,

\[
F(\succ_j, b, \bar{P}_j) = \bar{y}^s_j(b) \quad (\because \text{Claim A.1 for step } s \text{ with } j \in M(b, \bar{A}^{s-1}))
\]

\[
= \bar{y}^{s-1}_j + \bar{p}_{jb} \quad (\because j \in M(b, \bar{A}^{s-1}))
\]

\[
= \bar{y}^{s-2}_j + \bar{p}_{jb} \quad (\because \bar{y}^{s-1}_j = \bar{y}^{s-2}_j \text{ by definition with } b \not\in \bar{B}^{s-1}, j \in M(b, \bar{A}^{s-2}))
\]

\[
= \bar{y}^{s-1}_j(b) \quad (\because j \in M(b, A^{s-2}))
\]

\[
> \bar{y}^{s-1} \quad (\because b \not\in \bar{B}^{s-1}).
\]

(8)

If \( b = a \), then (8) implies \( F(\succ_j, a, \bar{P}_j) > \bar{y}^{s-1} \), which contradicts (7). On the other hand, consider the case where \( b \neq a \). Then, since \( j \in M(b, \bar{A}^{s-1}) \), we have \( b \succ_j a \). Then,

\[
F(\succ_j, a, \bar{P}_j) \geq F(\succ_j, b, \bar{P}_j) + \bar{p}_{ja} \quad (\because b \succ_j a)
\]

\[
> F(\succ_j, b, \bar{P}_j) \quad (\because \bar{p}_{ja} > 0)
\]

\[
> \bar{y}^{s-1} \quad (\because (8)),
\]

which contradicts (7).

Case 2: \( j \in M(a'', \bar{A}^{s-2}) \) for some \( a'' \in \bar{B}^{s-1} \), i.e., agent \( j \) demanded some disappearing house \( a'' \) in step \( s - 1 \).

Then,

\[
\bar{y}^{s-1}_j = \bar{y}^{s-1}_j(a'') \quad (\because a'' \in \bar{B}^{s-1} \text{ and } j \in M(a'', \bar{A}^{s-2}))
\]

\[
= \hat{y}^{s-1} \quad (\because \text{Claim 5.2 for step } s - 1).
\]

(9)

First, consider the case where agent \( j \) demands house \( a \) at step \( s \), i.e., \( j \in M(a, \bar{A}^{s-1}) \). Then,

\[
F(\succ_j, a, \bar{P}_j) = \hat{y}^s_j(a) \quad (\because \text{Claim A.1 for step } s \text{ with } j \in M(a, \bar{A}^{s-1}))
\]

\[
= \hat{y}^{s-1}_j + \bar{p}_{ja} \quad (\because j \in M(a, \bar{A}^{s-1}))
\]

\[
= \hat{y}^{s-1} + \bar{p}_{ja} \quad (\because (9))
\]

\[
> \hat{y}^{s-1} \quad (\because \bar{p}_{ja} > 0),
\]

which contradicts (7). On the other hand, consider the case where agent \( j \) demands some house
\[ b \neq a \text{ at step } s, \text{i.e., } j \not\in M(a, \bar{A}^{s-1}) \text{ but } j \in M(b, \bar{A}^{s-1}). \text{ Then,} \]

\[
\begin{align*}
\bar{y}^s - \cdot & = \bar{y}^s - (\cdot : (9)) \\
& \leq \bar{y}^s - + \bar{p}_{jb} \\
& = \bar{y}^s - (\cdot : j \in M(b, \bar{A}^{s-1})) \\
& = F(\succ j, b, \bar{P}_j) \quad (\cdot : \text{Claim A.1 for step } s \text{ with } j \in M(b, \bar{A}^{s-1})) \\
& < F(\succ j, b, \bar{P}_j) + \bar{p}_{ja} \quad (\cdot : \bar{p}_{ja} > 0) \\
& \leq F(\succ j, a, \bar{P}_j) \quad (\cdot : b \succ j a),
\end{align*}
\]

which contradicts (7).

In any case, we have a contradiction. Therefore, \( \bar{p}_{ia} > 0. \)

The second statement follows directly from the definition of \( \hat{B}^s. \) \[ \square \]

**Proof of Claim A.3 for Step s.** Fix \( a \in \hat{B}^s \) and \( i \in M(a, \bar{A}^{s-1}). \) Since for all \( j \in N, j \in M(a, \bar{A}^{s-1}) \Leftrightarrow \bar{p}_{ja} > 0 \) by Claim 5.1 for step \( s, \) for each \( j \in M(a, \bar{A}^{s-1}), \)

\[
F(\succ i, a, \bar{P}_i) = F(\succ j, a, \bar{P}_j) \quad (\cdot : \text{ordinal fairness of } \bar{P} \text{ with } \bar{p}_{ia}, \bar{p}_{ja} > 0)
\]

\[
\Rightarrow c := \bar{y}^s_i(a) = \bar{y}^s_i(a) \text{ where } c \text{ is constant} \quad (\cdot : \text{Claim A.1}).
\]

Also, because for all \( j \in M(a, \bar{A}^{s-1}), \bar{y}^s_j(a) = \bar{y}^s_j - + \bar{p}_{ja} = c \) and \( \sum_{j \in M(a, \bar{A}^{s-1})} \bar{p}_{ja} = \bar{q}_a, \) we obtain:

\[
\bar{y}^s_i(a) = c = \frac{\bar{q}_a + \sum_{j \in M(a, \bar{A}^{s-1})} \bar{y}^s_j}{|M(a, \bar{A}^{s-1})|}.
\]

\[ \square \]

**Proof of Claim 5.2 for Step s.** Let \( a \in \hat{B}^s. \) Take an agent \( i \in M(a, \bar{A}^{s-1}) \) such that \( \bar{y}^s_i(a) = \bar{y}^s. \) Then, \( \bar{q}_a > 0 \) by Claim A.2. For a contradiction, suppose \( a \not\in \hat{B}^s. \) Then, since \( \bar{q}_a > 0, \)

\[
\sum_{k \in M(a, \bar{A}^{s-1})} \bar{p}_{ka} < \bar{q}_a.
\]

Thus, there is \( j \not\in M(a, \bar{A}^{s-1}) \) with \( \bar{p}_{ja} > 0. \) Then, there is \( b \neq a \) with \( j \in M(b, \bar{A}^{s-1}). \) So, by Claim 5.1, \( \bar{p}_{ia} > 0 \) and \( \bar{p}_{jb} > 0. \) Therefore,

\[
\begin{align*}
F(\succ i, a, \bar{P}_i) &= F(\succ j, a, \bar{P}_j) \quad (\cdot : \text{ordinal fairness of } \bar{P}) \\
\Rightarrow F(\succ i, a, \bar{P}_i) &\geq F(\succ j, b, \bar{P}_j) + \bar{p}_{ja} \quad (\cdot : b \succ j a) \\
\Rightarrow F(\succ i, a, \bar{P}_i) &> F(\succ j, b, \bar{P}_j) \quad (\cdot : \bar{p}_{ja} > 0) \\
\Rightarrow \bar{y}^s_i(a) > \bar{y}^s_j(b) \quad (\cdot : \text{Claim A.1}) \\
\Rightarrow \bar{y}^s_i(a) > \bar{y}^s \quad (\cdot : \bar{y}^s_j(b) \geq \bar{y}^s \text{ by definition of } \bar{y}^s),
\end{align*}
\]

which contradicts our assumption that \( \bar{y}^s = \bar{y}^s_i(a). \)
This concludes the proof of $\bar{B}^s \subseteq \hat{B}^s$.

This implies from the definition of $\bar{y}^s$ that $\bar{y}^s = \min\{\bar{y}^s_i(a) | a \in \hat{B}^s, i \in M(a, \bar{A}^{s-1})\}$. Also, by Claim A.3 for step $s$, $\bar{y}^s_i(a)$ is constant among all $i \in M(a, \bar{A}^{s-1})$ for some $a \in \hat{B}^s$. ■

**Proof of Claim A.4 for Step $s$.** Take $a \in \hat{B}^s$ and $i \in M(a, \bar{A}^{s-1})$. We have $\bar{y}^s = \bar{y}^s_i(a)$ by Claim 5.2 for step $s$. That is, agent $i$ demands the disappearing house $a$ at step $s$. Depending on whether or not she demanded the house $a$ at the previous step $s - 1$, we consider two cases.

Case 1: Agent $i$ demanded the house $a$ at step $s - 1$, i.e., $i \in M(a, \bar{A}^{s-2})$.

Note that $a \notin \bar{B}^{s-1}$, i.e., $a$ is not a disappearing house at step $s - 1$, because it is available at step $s$. Then,

$$\bar{y}^s = \bar{y}^s_i(a)$$

$$= \hat{y}^s_i + p_{ia} \quad (\because i \in M(a, \bar{A}^{s-1}))$$

$$= \hat{y}^s_i - p_{ia} \quad (\because \hat{y}^s_i = \hat{y}^s_i \text{ by definition with } a \notin \bar{B}^{s-1} \text{ and } i \in M(a, \bar{A}^{s-2}))$$

$$= \bar{y}^s - p_{ia} \quad (\because i \in M(a, \bar{A}^{s-2}))$$

$$> \bar{y}^s \quad (\because a \notin \bar{B}^{s-1}).$$

Case 2: Agent $i$ did not demand the house $a$ but another house $a'$ at step $s - 1$.

Then, $a'$ must disappear at step $s - 1$ ($a' \in \bar{B}^{s-1}$), otherwise she would demand $a'$ at step $s$, as well. Note that $\bar{y}^{s-1} = \bar{y}^{s-1}_i(a')$ by Claim 5.2 for step $s - 1$. Thus,

$$\bar{y}^s = \bar{y}^s_i(a)$$

$$= \hat{y}^s_i + p_{ia} \quad (\because i \in M(a, \bar{A}^{s-1}))$$

$$= \hat{y}^s_i - p_{ia} \quad (\because \hat{y}^s_i = \hat{y}^s_i \text{ by definition with } a' \in \bar{B}^{s-1} \text{ and } i \in M(a', \bar{A}^{s-2}))$$

$$= \bar{y}^s - p_{ia} \quad (\because \text{Claim 5.2 for step } s - 1)$$

$$> \bar{y}^s \quad (\because p_{ia} > 0 \text{ by Claim 5.1 for step } s \text{ with } i \in M(a, \bar{A}^{s-1})).$$

■

**Proof of Claim 5.3 for Step $s$.** Suppose agent $i$ demands a house $a \in \bar{A}^{s-1}$ at step $s$, i.e., $i \in M(a, \bar{A}^{s-1})$. Depending on whether or not she demanded $a$ at the previous step $s - 1$, we consider two cases. First, consider the case where she demanded $a$ at step $s - 1$ ($i \in M(a, \bar{A}^{s-2})$). Then, since $a$ is available at step $s$, $a$ did not disappear at step $s - 1$ (i.e., $a \notin \bar{B}^{s-1}$). Thus,

$$\hat{y}^{s-1}_i = \hat{y}^{s-1}_i \quad (\because a \notin \bar{B}^{s-1} \text{ and } i \in M(a, \bar{A}^{s-2}))$$

$$= \bar{y}^{s}(a) \quad (\because \text{the induction hypothesis for step } s - 1 \text{ with } i \in M(a, \bar{A}^{s-2})).$$

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Consider the other case where she did not demand \( a \) but another house \( a' \neq a \) at step \( s - 1 \) \((i \in M(a', \bar{A}^{s-2}))\). Then, \( a' \) disappeared at step \( s - 1 \) \((\text{i.e., } a' \in \bar{B}^{s-1})\), otherwise she would demand \( a' \) instead of \( a \) at step \( s \). In addition, the step \( s \) is the first step in which she demands \( a \), i.e., \( \bar{s}_i(a) = s \). Thus,

\[
\bar{y}^{s-1}_i = \bar{y}^{s-1}(a') \ (\because i \in M(a', \bar{A}^{s-2}) \text{ and } a' \in \bar{B}^{s-1})
\]

\[
= \bar{y}^{s-1} \ (\because \text{Claim 5.2 for step } s - 1 \text{ with } a' \in \bar{B}^{s-1}, i \in M(a', \bar{A}^{s-2}))
\]

\[
= \bar{y}^{\bar{s}_i(a)-1} \ (\because \bar{s}_i(a) = s).
\]

\[\blacksquare\]

**Part II.**

**Proof of Claim 5.4.**

First, it follows from non-wastefulness of \( \bar{P} \) and Claim 5.1 that in the indexing procedure

\[
\text{for all } s = 1, \cdots, \bar{S} - 1 \text{ and } a \in \bar{B}^s, q_a = \bar{q}_a. \tag{10}
\]

Next, we show

\[
\bar{y}^{\bar{S}} = 1, \text{ and for all } s < \bar{S}, \bar{y}^s < 1. \tag{11}
\]

The second statement follows from the first one and Claim A.4, so we prove the first one. Take \( a \in \bar{B}^{\bar{S}} \) and \( i \in M(a, \bar{A}^{\bar{S}-1}) \). Then, it follows from Claim A.1 and 5.2 that \( \bar{y}^s = \bar{y}^{\bar{s}}(a) = F(\bar{s}_i, a, \bar{P}_i) \). Thus, we show \( F(\bar{s}_i, a, \bar{P}_i) = 1 \). To this end, it is sufficient to show that for all \( b \in A \) with \( a \succ_i b \), \( \bar{p}_{ib} = 0 \). Fix \( b \in A \) with \( a \succ_i b \). Then, either \( a \in B^{\bar{S}} \) or \( a \in \bar{A}^{\bar{S}} \). In the latter case, it follows from the stopping condition in the indexing procedure that \( \bar{p}_{ib} = 0 \). So, consider the former case. Since \( b \in B^{\bar{S}} \), we have \( b \in \hat{B}^{\bar{S}} \) by Claim 5.2. Thus, \( \bar{p}_{ib} = 0 \), because \( i \not\in M(b, \bar{A}^{\bar{S}-1}) \). This completes the proof of (11).

We prove by induction on step \( s \) that \( y^s = \bar{y}^s, B^s = \bar{B}^s \), and \( A^{s-1} = \bar{A}^{s-1} \) for each \( s = 1, \cdots, \min\{S, \bar{S}\} \). The variables \( y^s, B^s, A^{s-1}, S \) are those of the PS mechanism. Let \( s \geq 1 \). In the inductive step assume that for all \( t < s, y^t = \bar{y}^t, B^t = \bar{B}^t \), and \( A^{t-1} = \bar{A}^{t-1} \). First, by the induction hypothesis, \( A^{s-1} = A^{s-2} \setminus B^{s-1} = \bar{A}^{s-2} \setminus \bar{B}^{s-1} = \bar{A}^{s-1} \).

We know

\[
B^s = \text{arg min } \left\{ \min \left\{ 1, y^{s-1} + \frac{q_a - \sum_{i \in N} \bar{P}_{ia}^{s-1}}{|M(a, A^{s-1})|} \right\} \left| a \in A^{s-1} \text{ with } M(a, A^{s-1}) \neq \emptyset \right\} \right\} \ (\because \text{definition})
\]

\[
= \text{arg min } \left\{ \min \left\{ 1, y^{s-1} + \frac{q_a - \sum_{i \in M(a, A^{s-1})} \bar{P}_{ia}^{s-1}}{|M(a, A^{s-1})|} \right\} \left| a \in A^{s-1} \text{ with } M(a, A^{s-1}) \neq \emptyset \right\} \right\}
\]

where the minimized value is \( y^s \) \((\because \text{for all } i \not\in M(a, A^{s-1}), \bar{P}_{ia}^{s-1} = 0 \text{ by Lemma 1})\). \( \tag{12} \)
\( \bar{B}^s = \arg\min \left\{ \bar{y}_i^s(a) \mid a \in \hat{B}^s \text{ with } i \in M(a, \bar{A}^{s-1}) \right\} \quad (\because \text{Claim 5.2}) \\
= \arg\min \left\{ \bar{q}_a + \sum_{i \in M(a, \bar{A}^{s-1})} \bar{y}_i^{s-1} \middle| a \in \hat{B}^s \right\} \quad (\because \text{Claim A.3}) \\
= \arg\min \left\{ \bar{y}^{s-1} + \frac{\bar{q}_a - \sum_{i \in M(a, \bar{A}^{s-1})} \bar{p}_{ia}^{s-1}}{|M(a, \bar{A}^{s-1})|} a \in \hat{B}^s \right\} \text{ where the minimized value is } \bar{y}^s, \\
= \arg\min \left\{ \bar{y}^{s-1} + \frac{\bar{q}_a - \sum_{i \in M(a, \bar{A}^{s-1})} \bar{p}_{ia}^{s-1}}{|M(a, \bar{A}^{s-1})|} a \in \hat{B}^s \setminus \bar{B}^s \right\} \text{ if } s < \bar{S}, \quad (\because (11)). \quad (13) \\

In the above, \( \bar{p}_{ia}^{s-1} \) is defined as follows:

\[
\bar{p}_{ia}^{s-1} := \bar{y}^{s-1} - \bar{y}_i^{s-1}.
\]

Now, for each \( a \in \bar{A}^{s-1} \) and \( i \in M(a, \bar{A}^{s-1}) = M(a, \bar{A}^{s-1}) \),

\[
\bar{p}_{ia}^{s-1} = \bar{y}^{s-1} - \bar{y}_i^{s-1} = \bar{y}^{s-1} - \bar{y}^{i(a)-1} \quad (\because \text{Claim 5.3 with } i \in M(a, \bar{A}^{s-1})) \\
= y^{s-1} - y^{i(a)-1} \quad (\because \text{the induction hypothesis. Note } \bar{s}_i(a) = s_i(a)) \\
= \begin{cases} 
\bar{p}_{ia}^{s-1} & \text{if } i \in M(a, A^{s-2}) \quad (\because \text{Lemma 1}), \\
0 = \bar{p}_{ia}^{s-1} & \text{if } i \not\in M(a, A^{s-2}) \quad (\because \text{Lemma 1. Note } s_i(a) = s). 
\end{cases} \quad (14)
\]

Depending on whether or not the step \( s \) is the last step \( \bar{S} \), we consider two cases:

**Case 1:** \( s = \bar{S} \). First, we show

for all \( a \in \hat{B}^{\bar{S}} \), \( \bar{y}^{\bar{S}} = y^{\bar{S}}(a) = 1 \). \quad (15)

For all \( a \in \hat{B}^{\bar{S}} \),

\[
1 = \bar{y}^{\bar{S}} = \bar{y}^{\bar{S}-1} + \frac{\bar{q}_a - \sum_{i \in M(a, \bar{A}^{s-1})} \bar{p}_{ia}^{s-1}}{|M(a, \bar{A}^{s-1})|} \quad (\because (11) \text{ and } (13)) \\
\leq y^{\bar{S}-1} + \frac{\bar{q}_a - \sum_{i \in M(a, \bar{A}^{s-1})} \bar{p}_{ia}^{s-1}}{|M(a, \bar{A}^{s-1})|} \quad (\because \text{the induction hypothesis, } A^{\bar{S}-1} = \bar{A}^{\bar{S}-1}, \text{ and } \bar{q}_a \leq q_a). \\
\Rightarrow y^{\bar{S}}(a) = 1 \quad (\because (12) \text{ and } y^{\bar{S}} \leq 1).
\]

This completes the proof of (15).

To show \( B^{\bar{S}} = \bar{B}^{\bar{S}} \) and thus \( y^{\bar{S}} = \bar{y}^{\bar{S}} \), it is sufficient to prove that

for all \( a \in \bar{A}^{\bar{S}-1} \setminus \hat{B}^{\bar{S}} \), \( M(a, \bar{A}^{\bar{S}-1}) = \emptyset \), i.e., \( y^{\bar{S}} = \infty \).
For a contradiction, suppose there exist \( a \in \bar{A}^{s-1} \setminus \hat{B}^S \) and \( i \in M(a, \bar{A}^{s-1}) \). Then, \( a \not\in \bar{B}^S \) by Claim 5.2, and thus \( a \in \bar{A}^S \). On the other hand, it follows from Claim 5.1 that \( \bar{p}_{ia} > 0 \), which contradicts the stopping condition \( P|_{\bar{A}^S} = 0 \), in particular, \( \bar{p}_{ia} = 0 \).

Case 2: \( s < \bar{S} \). First, we show

\[
\text{for all } a \in \bar{A}^{s-1}, M(a, \bar{A}^{s-1}) \neq \emptyset \text{ implies } a \in \bar{B}^t \text{ for some } t = s, \ldots, \bar{S}.
\]

(16)

Fix \( a \in \bar{A}^{s-1} \) with \( M(a, \bar{A}^{s-1}) \neq \emptyset \). Then, \( \bar{q}_a > 0 \) by Claim A.2, and thus there is \( i \in N \) with \( \bar{p}_{ia} > 0 \). Moreover, since \( \bar{A}^{s-1} = \bigcup_{u=1}^{s-1} \bar{B}^u \) and \( \{\bar{B}^t\}_{t=1}^{\bar{S}} \) is a partition of \( A \setminus \bar{A}^S \), either \( a \in \bar{A}^S \) or \( a \in \bar{B}^t \) for some \( t = s, \ldots, \bar{S} \). If the former holds, the fact that \( \bar{p}_{ia} > 0 \) contradicts the stopping condition \( P|_{\bar{A}^S} = 0 \). Hence, the latter, i.e., (16), holds.

Together with the induction hypothesis, (12), (13), and (14) imply, for each \( a \in \hat{B}^s \setminus \bar{B}^S \) and \( i \in M(a, \bar{A}^{s-1}) \), since \( q_a = \bar{q}_a \) by (10) and (16),

\[
1 \geq \bar{y}^s_i(a) \equiv \bar{y}^{s-1} + \frac{\bar{q}_a - \sum_{i\in M(a,\bar{A}^{s-1})} \bar{p}_{ia}^{s-1}}{|M(a, \bar{A}^{s-1})|} = y^{s-1} + \frac{q_a - \sum_{i\in M(a, \bar{A}^{s-1})} p_{ia}^{s-1}}{|M(a, \bar{A}^{s-1})|} \equiv y^s(a),
\]

\[
\Rightarrow \quad \bar{B}^s = \arg \min \left\{ y^s(a) \middle| a \in \hat{B}^s \setminus \bar{B}^S \right\}.
\]

This indicates that, in both cases, the objective function is the same in the domain \( \hat{B}^s \setminus \bar{B}^S \). However, as \( \hat{B}^s \setminus \bar{B}^S \subseteq \{a \in A^{s-1} | M(a, A^{s-1}) \neq \emptyset \} \), \( B^s \) has a larger domain to minimize over than \( B^s \). Hence, to prove that \( B^s = \bar{B}^s \) and thus \( y^s = \bar{y}^s \), we need to show \( \bar{y}^s < \min \left\{ y^s(a) \middle| a \in (A^{s-1} \cup \bar{B}^S) \setminus \hat{B}^s, M(a, A^{s-1}) \neq \emptyset \right\} \). For a contradiction, suppose there is \( a \in (A^{s-1} \cup \bar{B}^S) \setminus \hat{B}^s \) with \( M(a, A^{s-1}) \neq \emptyset \) such that

\[
\bar{y}^s \geq y^s(a).
\]

(17)

Then, since \( a \not\in \hat{B}^s \), we have \( a \not\in \bar{B}^s \) by Claim 5.2. It follows from (16) that \( a \in \bar{B}^t \) for some \( t = s + 1, \ldots, \bar{S} \).

Next we show that \( \bar{s}_i(a) = s_i(a) \). We showed that \( \bar{A}^{s-1} = A^{s-1} \). By the inductive assumption we also know that for all \( s' < s, \bar{A}^{s'-1} = A^{s'-1} \). These two statements imply \( \bar{s}_i(a) = s_i(a) \).

Since for all \( a \in B^s \), by definition, \( y^s(a) \) is the smallest among all houses, without loss of generality
we choose $a \in B^s$ in Condition (17). Then,

$$
\bar{q}_a = \sum_{i \in M(a, A^s)} \bar{p}_{ia} \quad (\because \text{Since } a \in \bar{B}^t, \text{ we have } a \in \bar{B}^t \text{ by Claim 5.2})
$$

$$
= \sum_{i \in M(a, A^s)} \bar{p}_{ia} + \sum_{i \in M(a, A^s \setminus M(a, A^{s-1}))} \bar{p}_{ia} \quad (\because \emptyset \neq M(a, A^{s-1}) = M(a, \bar{A}^{s-1}) \subseteq M(a, \bar{A}^{t-1}))
$$

$$
\geq \sum_{i \in M(a, A^s)} \bar{p}_{ia} = \sum_{i \in M(a, A^{s-1})} (\bar{y}_i^t(a) - \bar{y}_i^{t-1})
$$

$$
= \sum_{i \in M(a, A^{s-1})} (\bar{y}_i^s - \bar{y}_i^{s-1}) \quad (\because \bar{y}_i^t = \bar{y}_i^s \text{ by Claim 5.2 with } a \in \bar{B}^t \text{ and } i \in M(a, \bar{A}^{s-1}) \subseteq M(a, \bar{A}^{t-1}))
$$

$$
> \sum_{i \in M(a, A^{s-1})} (\bar{y}_i^s - \bar{y}_i^{t-1}) \quad (\because \bar{y}_i^t > \bar{y}_i^s \text{ by Claim A.4 with } t > s)
$$

$$
= \sum_{i \in M(a, A^{s-1})} (\bar{y}_i^s - \bar{y}_i^{s_i(a)-1}) \quad (\because \bar{y}_i^{s_i(a)-1} = \bar{y}_i^{s_i(a)-1} \text{ by Claim 5.3 with } i \in M(a, \bar{A}^{s-1}) \subseteq M(a, \bar{A}^{t-1}))
$$

$$
\geq \sum_{i \in M(a, A^{s-1})} (y_i^s(a) - \bar{y}_i^{s_i(a)-1}) \quad (\because (17))
$$

$$
= \sum_{i \in M(a, A^{s-1})} (y_i^s(a) - y_i^{s_i(a)-1}) \quad (\because \bar{s}_i = s_i(a) \text{ and } y_i^{s_i(a)-1} = y_i^{s_i(a)-1} \text{ by the induction hypothesis})
$$

$$
= \sum_{i \in M(a, A^{s-1})} (y_i^s - y_i^{s_i(a)-1}) \quad (\because a \in B^s)
$$

$$
= \sum_{i \in M(a, A^{s-1})} p_{ia}^s \quad (\because \text{Lemma 1})
$$

$$
= q_a \quad (\because a \in B^s)
$$

$$
\geq \bar{q}_a \quad (\because \text{the definition}).
$$

This is a contradiction. Therefore, we proved that for each $s = 1, \ldots, \min\{S, \bar{S}\}$, $A^s = \bar{A}^{s-1}$, $B^s = \bar{B}^s$, and $y^s = \bar{y}^s$. Thus, $S = \bar{S}$. ■

### A.3 Proofs of the Claims in the Proof of Theorem 2

We introduce the following auxiliary concepts before we invoke upper invariance in our characterization:

**Definition 5 (Strongly Invariant Preferences:)**

- For an agent $i$, her preferences $\succ_i$ and $\succ_i'$ are **strongly upper invariant at house** $a$ if $\succ_i'$ is an upper invariant transformation of $\succ_i$ at $a$ and vice versa, that is, $U(\succ_i, a) = U(\succ_i', a)$ and $\succ_i | U(\succ_i, a) \Rightarrow \succ_i' | U(\succ_i, a)$.

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• Given a mechanism \( \phi \), a preference profile \( \succ \), a house \( b \), agent \( i \)'s preferences \( \succ_i \) and \( \succ'_i \) are strongly invariant by \( b \) if \( \phi_{ib}(\succ) = \phi_{ib}(\succ'_i, \succ_{\neg i}) = 0 \) and \( \succ_i \mid A \setminus \{b\} = \succ'_i \mid A \setminus \{b\} \).

The following lemma follows directly from the above two definitions. To prove Theorem 2, we use Lemma 3.

**Lemma 3** Let \( \phi \) be an upper invariant mechanism. Then, for all \( \succ, i \in N, \succ'_i \),

1. for all \( a \in A \), if \( \succ_i \) and \( \succ'_i \) are strongly upper invariant at \( a \), then \( \phi(\succ)|_{U(\succ_i, a)} = \phi(\succ'_i, \succ_{\neg i})|_{U(\succ_i, a)} \), and

2. if \( \succ_i \) and \( \succ'_i \) are strongly invariant by house \( a \), then \( \phi(\succ) = \phi(\succ'_i, \succ_{\neg i}) \).

Lemma 3 states that for an upper invariant mechanism, under a profile of given preferences for all agents but \( i \) and under a pair of strongly upper invariant preferences for \( i \) at a house \( a \), respectively, the induced random allocation vectors of each house in the upper contour sets of \( a \) for \( i \) are the same; moreover, if these preferences are also strongly invariant, then the whole respective random assignments are the same.

Next, we give four claims that hold for each step \( s = 1, \ldots, \bar{S} \) of the modified indexing procedure at \( \bar{P} \equiv \phi(\succ) \) under \( \succ \):

**Claim A.5** Consider any agent \( i \in N \). For each step \( s \), we have the following:

1. If \( i \in M(a^s, \bar{A}^s) \) and \( \bar{s}_i(a^s) = s \), then \( \succ^s_i = \left( \succ_{i}^{s-1} \mid U(\succ_{i}^{s-1}, a^s) \setminus a^s, a^s, \succ_i \mid \bar{A}^s \right) \).

2. If \( i \in M(a^s, \bar{A}^s) \) and \( \bar{s}_i(a^s) < s \), then \( \succ^s_i = \left( \succ_{i}^{s-1} \mid U(\succ_{i}^{s-1}, a^s) \setminus a^s, a^s, \bar{s}_i(a^s), a^s_i(a^s)+1, \ldots, a^s-1, \succ_i \mid \bar{A}^s \right) \).

3. If \( i \in M(b, \bar{A}^s) \) for some \( b \neq a^s \) and \( \bar{s}_i(b) = s \), then \( U(\succ_{i}^{s-1}, b) = U(\succ^s_i, b) = \{b, a^1, \ldots, a^s-1\} \) and \( \succ^s_i = \left( \succ_{i}^{s-1} \mid U(\succ_{i}^{s-1}, b) \setminus b, a^s, \succ_i \mid \bar{A}^s \right) \).

4. If \( i \in M(b, \bar{A}^s) \) for some \( b \neq a^s \) and \( \bar{s}_i(b) < s \), then \( U(\succ_{i}^{s-1}, b) = U(\succ^s_i, b) = \{b, a^1, \ldots, a^s-1\} \) and \( \succ^s_i = \left( \succ_{i}^{s-1} \mid U(\succ_{i}^{s-1}, b) \setminus b, a^s_i(b), a^s_i(b)+1, \ldots, a^s, \succ_i \mid \bar{A}^s \right) \).

To simplify notations, for any subset \( I \) of \( N \setminus M(a^s, \bar{A}^s) \), let

\[
\succ^s(I) := \left( \succ^s_{M(a^s, \bar{A}^s) \cup I}, \succ^s_{M(a^s, \bar{A}^s) \cup I} \right)
\]

and

\[
\bar{P}^s(I) := \phi(\succ^s(I)).
\]
Claim A.6  For all $I \subseteq N \setminus M(a^s, \bar{A}^{s-1})$,

1. $\bar{P}(I)|_{A \setminus \bar{A}^{s-1}} = \bar{P}|_{A \setminus \bar{A}^{s-1}}$, and

2. for all $i, j \in M(a^s, \bar{A}^{s-1})$, $F(\succ_i^s, a^s, \bar{P}(I)) = F(\succ_j^s, a^s, \bar{P}(I))$.

Claim A.6 says that if $I$ is an arbitrary set of agents who do not demand the disappearing house $a^s$ and if we change the preference profile for the agents who consume $a^s$ and the agents in $I$, the updated random assignments are the same for the houses that disappeared up to the previous period; moreover, the probabilities of upper contour sets of $a^s$ are equal for any two agents demanding $a^s$.

Claim A.7  For all $I \subseteq N \setminus M(a^s, \bar{A}^{s-1})$, $i \in M(a^s, \bar{A}^{s-1})$, $j \in N \setminus M(a^s, \bar{A}^{s-1})$ with $j \in M(b, \bar{A}^{s-1})$ for some $b \neq a^s$,

- if $j \in I$ and $\bar{p}_{ja^s}(I) > 0$, then $F(\succ_j^s, b, \bar{P}(I)) < F(\succ_i^s, a^s, \bar{P}(I))$,
- if $j \not\in I$ and $\bar{p}_{ja^s}(I) > 0$, then $F(\succ_j^{s-1}, b, \bar{P}(I)) < F(\succ_i^s, a^s, \bar{P}(I))$.

Claim A.7 is technical, but crucial. Ordinal efficiency is used directly only in the proof of this claim throughout the proof of Theorem 2.

Claim A.8  For all $I \subseteq N \setminus M(a^s, \bar{A}^{s-1})$,

1. $\bar{P}(I) = \bar{P}$. In particular, $\phi(\succ^s) = \bar{P}(N \setminus M(a^s, \bar{A}^{s-1})) = \bar{P}$.

2. for all $t = 1, \ldots, \bar{S}$ and $i \in M(a^t, \bar{A}^{t-1})$,

   (a) if $i \in M(a^s, \bar{A}^{s-1}) \cup I$, then $F(\succ_i^s, a^t, \bar{P}(I)) = F(\succ_i^s, a^s, \bar{P})$,
   (b) if $i \not\in M(a^s, \bar{A}^{s-1}) \cup I$, then $F(\succ_i^{s-1}, a^t, \bar{P}(I)) = F(\succ_i^s, a^s, \bar{P})$.

3. for all $i \in M(a^s, \bar{A}^{s-1})$, $F(\succ_i^s, a^s, \bar{P}(I)) = \bar{p}_i(I)$.  

4. $\sum_{i \in M(a^s, \bar{A}^{s-1})} \bar{p}_{ja^s}(I) = \bar{q}_a$. In particular, $a^s \in \hat{B}^s$, i.e., $\sum_{i \in M(a^s, \bar{A}^{s-1})} \bar{p}_{ia^s} = \bar{q}_a$.

Claim 6.2 follows from Claim A.8 by setting $I = N \setminus M(a^s, \bar{A}^{s-1})$.

We prove Claims A.5, A.6, A.7, and A.8 together in this order by induction on step $s$ of the modified indexing procedure for $P \equiv \phi(\succ)$ under $\succ$:

**Step $s = 1$:**

---

\(^{15}\)Recall that $\hat{y}^s$ is defined in the indexing procedure as the minimum of $\{\hat{y}^s_i(\cdot)\}$ at step $s$. 

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Proof of Claim A.5 for Step 1.

First, consider the case where \( i \in M(a^1, \tilde{A}^0) \). Then, \( \bar{s}_i(a^1) = 1 \). Thus, Part (2) for step 1 is vacuously true. For Part (1) for step 1, we have the desired result as follows:

\[
\succ^1_i = \succ^0_i \equiv \succ_i \quad (\because i \in M(a^1, \tilde{A}^0)) \\
= (a^1, \succ_i |_{A^0 \setminus a^1}) \quad (\because i \in M(a^1, \tilde{A}^0) \text{ and } \tilde{A}^0 \equiv A) \\
= (a^1, \succ_i |_{\bar{A}^1}) \quad (\because \bar{A}^1 = \tilde{A}^0 \setminus a^1).
\]

Next, consider the case where \( i \not\in M(a^1, \tilde{A}^0) \) but \( i \in M(b, \tilde{A}^0) \) for some \( b \in \tilde{A}^0 \). Then, \( \bar{s}_i(b) = 1 \). Thus, Part (4) for step 1 is vacuously true. We prove Part (3) for step 1: Since \( \bar{s}_i(b) = 1 \), we have \( \succ^0_i \equiv \succ_i \equiv (b, \succ_i |_{A^0 \setminus b}) \). Now, \( \succ^1_i \) upgrades \( a^1 \) just below \( b \) in \( \succ^0_i \), because \( i \in M(b, \tilde{A}^0) \) and \( \bar{s}_i(b) = 1 \). We have the desired equality as follows:

\[
\succ^1_i = (b, a^1, \succ_i |_{A^0 \setminus b}) \quad (\because \tilde{A}^1 = \tilde{A}^0 \setminus a^1).
\]

Moreover, it follows from the above expressions that \( U(\succ^0_i, b) = U(\succ^1_i, b) = \{b\} \).

Proof of Claim A.6 for Step 1. The proof is the same as that of Claim A.1 for step 1.

Proof of Claim A.7 for Step 1. Take any set \( I \subseteq N \setminus M(a^1, \tilde{A}^0) \), and let \( \tilde{P}' := \tilde{P}^1(I) \). Fix two agents \( i, j \in M(a^1, \tilde{A}^0) = M(a^1, A) \). Note that \( \succ^1_i = \succ^0_i \equiv \succ_i \) and \( \succ^1_j = \succ^0_j \equiv \succ_j \). Then,

\[
F(\succ^1_i, a^1, \tilde{P}_i') \geq F(\succ^1_i, a^1, \tilde{P}_j') \quad (\because \text{envy-freeness}) \\
= F(\succ^1_i, a^1, \tilde{P}_j') \quad (\because i \in M(a^1, A), \succ^1_i = \succ_i \text{ and } a^1 \text{ is the top choice in } \succ^1_i) \\
= F(\succ^1_j, a^1, \tilde{P}_j') \quad (\because j \in M(a^1, A), \succ^1_j = \succ_j \text{ and } a^1 \text{ is the top choice in } \succ^1_j)
\]

The opposite inequality is obtained by the symmetric argument. Hence, we have the desired equality.

Proof of Claim A.8 for Step 1. Take any set \( I \subseteq N \setminus M(a^1, \tilde{A}^0) \), any agent \( i \in M(a^1, \tilde{A}^0) \), and any agent \( j \not\in M(a^1, \tilde{A}^0) \) so that \( j \in M(b, \tilde{A}^0) \) for some house \( b \neq a^1 \). We prove the case \( j \in I \). Let \( \tilde{P}' := \tilde{P}^1(I) \). Suppose \( \tilde{p}'_{ja} > 0 \). We first prove it is sufficient to show \( \tilde{p}'_{jb} < \tilde{p}'_{ia} \). Suppose it is true. Note \( \tilde{A}^0 = A \). Since \( \succ^1_i = \succ^1_j \), \( F(\succ^1_i, a^1, \tilde{P}_i') = \tilde{p}'_{ia} \). Since \( j \in I \), \( \succ^1_j \) upgrades \( a^1 \) just below \( b \) and has \( b \) as the top choice. Thus, \( F(\succ^1_j, b, \tilde{P}_j') = \tilde{p}'_{jb} < \tilde{p}'_{ia} = F(\succ^1_i, a^1, \tilde{P}_i') \) as \( a^1 \) is the top choice of \( i \) and \( b \) is the top choice of \( j \).

We consider the preference \( \succ''_i \) that upgrades \( b \) just below \( a^1 \) in \( \succ^i_i \). Consider the random assignment \( \tilde{P}'' := \phi(\succ''_i) \) where \( \succ'' := (\succ'', \succ^1_i, \succ^1_i |_{M(a^1, \tilde{A}^0) \setminus i}^0, \succ^1_i |_{M(a^1, \tilde{A}^0) \setminus i}) \).

43
First, since $\succ_i''$ and $\succ_i^1$ are strongly upper invariant at $a^1$, Lemma 3 implies through upper invariance of $\phi$

$$\bar{p}''_{ia} = \bar{p}'_{ia} \text{ and } \bar{p}''_{ja} = \bar{p}'_{ja}. \quad (18)$$

Second, since $\bar{p}'_{ja} > 0$ (by our assumption) and $b \succ_j^1 a^1$, this implies

$$\bar{p}'_{ib} = 0. \quad (19)$$

Otherwise, $i$ could give a small probability from $b$ to $j$, and in return, she could get an equal probability of getting $a^1$ from $b$, and both agents would be strictly better off (under stochastic dominance) with respect to $\bar{P}'$; however, this would contradict ordinal efficiency of $\phi$ at $\bar{P}' \equiv \phi(\succ (I))$.

Third, note that $\bar{p}''_{ja} = \bar{p}'_{ja} > 0$ by (18), and $b \succ_j^1 a^1$. This implies

$$\bar{p}''_{jb} = 0. \quad (20)$$

Otherwise, $i$ could give a small probability from $b$ to $j$, and in return, she could get an equal probability of getting $a^1$ from $b$, and both agents would be strictly better off (under stochastic dominance) with respect to $\bar{P}''$; however, this would contradict ordinal efficiency of $\phi$ at $\bar{P}'' \equiv \phi(\succ'')$.

Fourth, notice from (19) and (20) that $\succ''_i$ and $\succ_i^1$ are strongly invariant by $b$. It follows from Lemma 3 that $\bar{P}'' = \bar{P}'$, in particular,

$$\bar{p}'_{jb} = \bar{p}''_{jb}. \quad (21)$$

Finally, since $\bar{P}''$ is envy-free at $\succ''$, we have the desired inequality:

$$F(\succ''_i, b, \bar{P}'') \geq F(\succ''_i, b, \bar{P}''') \quad (\because \text{envy-freeness of } \bar{P}'' \text{ at } b \text{ for } i)$$

$$\Leftrightarrow \quad \bar{p}''_{ia} + \bar{p}''_{ib} \geq \bar{p}''_{ja} + \bar{p}''_{jb}$$

$$\Rightarrow \quad \bar{p}''_{ia} \geq \bar{p}''_{ja} + \bar{p}''_{jb} \quad (\because (18), (20), \text{ and (21)})$$

$$\Rightarrow \quad \bar{p}''_{ia} > \bar{p}''_{jb} \quad (\because \bar{p}'_{ja} > 0).$$

The proof of the other case, $j \notin I$, is the same as the above case, after replacing $\succ_j^1$ to $\succ_j^0$. ■

**Proof of Claim A.8 for Step 1.** We prove the claim for step 1 of the indexing procedure by induction on $|I|$. Suppose this claim is true for any proper subsets of $I$ (this statement is vacuously true for the initial step, $I = \emptyset$). Pick any agent $k \in I$ so that $k \notin M(a^1, \bar{A}^0)$. Then, $k \in M(b, \bar{A}^0)$ for some house $b \neq a^1$. Let $\bar{P}' = \bar{P}^1(I \setminus k)$ and $\bar{P}'' = \bar{P}^1(I)$. Then, the induction hypothesis implies

$$\bar{P}' = \bar{P}. \quad (22)$$

Proof of **Part (1):** Note that $\succ_k^1$ upgrades $a^1$ just below $b$ in $\succ_k^0$.

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We first prove that to show $\bar{P} = \bar{P}''$, it is sufficient to prove $\bar{p}_{ka1}'' = 0$: Suppose $\bar{p}_{ka1}'' = 0$. Now, since $k \not\in M(a^1, A^0)$, we have $\bar{p}_{ka1}' = 0$ by the induction hypothesis of Part (4) for $I \setminus k$. Hence, $\succ^1_k$ and $\succ^0_k$ are strongly invariant by $a^1$ and thus $P'' = P'$ by Lemma 3, and $P' = \bar{P}$ by (22). Therefore, $\bar{P} = \bar{P}''$.

We next prove that $\bar{p}_{ka1}'' = 0$: For a contradiction, suppose $\bar{p}_{ka1}'' > 0$. Pick any agent $i \in M(a^1, A^0)$.

First, we show

$$F(\succ^1_i, a^1, \bar{P}_i'') = \bar{p}_{ia1}'' < \bar{y}_i.$$  \hspace{1cm} (23)

Note that for all $j \in M(a^1, A^0)$, $F(\succ^1_j, a^1, \bar{P}_j'') = \bar{p}_{ja1}''$. Since $\bar{p}_{ja1}'' > 0$ by our assumption, $\sum_{j \in M(a^1, A^0)} \bar{p}_{ja1}'' = \bar{q}_a$. Also, by Claim A.6 for step 1, $\bar{p}_{ja1}'' = \bar{p}_{ja1}'$. Thus,

$$F(\succ^1_i, a^1, \bar{P}_i'') = \bar{p}_{ia1}'' < \frac{\bar{q}_a}{|M(a^1, A^0)|}. \hspace{1cm} (24)$$

Now, from the induction hypothesis for $I \setminus k$, $\sum_{j \in M(a^1, A^0)} \bar{p}_{ja1}'' = \bar{q}_a$ and for all $j \in M(a^1, A^0)$, $F(\succ^1_j, a^1, P_j') = \bar{p}_{ja1}' = \bar{y}_i$. Hence, $\bar{y}_i = \bar{q}_a/|M(a^1, A^0)|$, which implies (23) by (24).

Next, we show

$$F(\succ^0_k, b, P_k'') = F(\succ_k, b, \bar{P}_k). \hspace{1cm} (25)$$

Since $\succ^0_k$ upgrades $a^1$ just below $b$, $\succ^1_k$ and $\succ^0_k$ are strongly upper invariant at $b$. Thus, $F(\succ^0_k, b, \bar{P}_k'') = F(\succ^0_k, b, P_k')$. Also, we have $F(\succ^0_k, b, \bar{P}_k) = F(\succ_k, b, \bar{P}_k)$ because we have $\succ^0_k \Rightarrow \succ_k$ and $\bar{P}' = \bar{P}$ by (22). Thus, we have (25).

Therefore,

$$F(\succ^1_k, b, \bar{P}''_k) < F(\succ^1_i, a^1, \bar{P}''_i) \hspace{1cm} (\because \text{Claim A.7 for step 1})$$

$$\Leftrightarrow F(\succ_k, b, \bar{P}_k) < \bar{y}_i \hspace{1cm} (\because (23) \text{ and } (25))$$

$$\Leftrightarrow \bar{y}_{ik}(b) < \bar{y}_i \hspace{1cm} (\because \text{Claim 6.1 for step 1}).$$

This contradicts the definition of $\bar{y}_i$. Hence, Part (1) holds.

Proof of Part (2): Note from the proof of Part (1), $\succ^1_k$ and $\succ^0_k$ are strongly invariant by $a^1$ and thus $\bar{P}'' = \bar{P}'$, and $\bar{P}' = \bar{P}$. With these, Lemma 3 immediately implies Part (2).
Proof of Part (3): Take any agent \( i \in M(a^1, \bar{A}^0) \). Note from the proof of Part (1), \( \succ^1_k \) and \( \succ^0_k \) are strongly invariant by \( a^1 \).

\[
F(\succ^1_i, a^1, \bar{P}''_i) = F(\succ^1_i, a^1, \bar{P}'_i) \quad (\because \text{Lemma 3})
\]

\[
= \bar{y}^1 \quad (\because \text{induction hypothesis for } I \setminus k).
\]

Proof of Part (4): From the proof of Part (1), we know \( \bar{P}'' = \bar{P}' \). Also, we have \( \sum_{j \in M(a^1, \bar{A}^0)} \bar{P}''_{ja^1} = \bar{q}_a \) by the induction hypothesis for \( I \setminus k \). Thus, we have \( \sum_{j \in M(a^1, \bar{A}^0)} \bar{P}''_{ja^1} = \bar{q}_a. \) ☐

**Step** \( s \), for \( s > 1 \): Suppose Claims A.5, 6.1, A.6, A.7, and A.8 are true up to \( s - 1 \).

**Proof of Claim A.5 for Step** \( s \). Fix agent \( i \in N \).

Proof of Part (1): Suppose \( i \in M(a^s, \bar{A}^{s-1}) \) and \( \bar{s}_i(a^s) = s \). Note first that \( \succ_i^s = \succ_i^{s-1} \) because \( i \in M(a^s, \bar{A}^{s-1}) \). At step \( s - 1 \), since \( \bar{s}_i(a^s) = s \), she demanded house \( a^{s-1} \), i.e., \( i \in M(a^{s-1}, \bar{A}^{s-2}) \).

Otherwise, she demanded some house \( c \neq a^{s-1}, a^s \), and both \( c \) and \( a^s \) are available at step \( s \), she would continue to demand \( c \). This contradicts \( i \in M(a^s, \bar{A}^{s-1}) \).

If \( \bar{s}_i(a^{s-1}) = s - 1 \), then Part (1) for step \( s - 1 \) implies

\[
\succ_i^{s-1} = \left( \succ_i^{s-2} \mid U(\succ_i^{s-2, a^{s-1}}) \setminus a^{s-1}, a^{s-1}, i \mid \bar{A}^{s-1} \right)
\]

\[
= \left( \succ_i^{s-2} \mid U(\succ_i^{s-2, a^{s-1}}) \setminus a^{s-1}, a^{s-1}, a^s, i \mid \bar{A}^{s-1} \setminus a^s \right) \quad (\because i \in M(a^s, \bar{A}^{s-1}))
\]

Since \( \succ_i^s = \succ_i^{s-1} \) and \( \bar{A}^s = \bar{A}^{s-1} \setminus a^s \), we have \( \succ_i^s = \left( \succ_i^{s-1} \mid U(\succ_i^{s-1, a^s}) \setminus a^s, a^s, i \mid \bar{A} \right) \).

On the other hand, if \( \bar{s}_i(a^{s-1}) < s - 1 \), then Part (2) for step \( s - 1 \) implies

\[
\succ_i^{s-1} = \left( \succ_i^{s-2} \mid U(\succ_i^{s-2, a^{s-1}}) \setminus a^{s-1}, a^{s-1}, a_i^s(a^{s-1}), \ldots, a^{s-2}, i \mid \bar{A}^{s-1} \right)
\]

\[
= \left( \succ_i^{s-2} \mid U(\succ_i^{s-2, a^{s-1}}) \setminus a^{s-1}, a^{s-1}, a_i^s(a^{s-1}), \ldots, a^{s-2}, a^s, i \mid \bar{A}^{s-1} \setminus a^s \right) \quad (\because i \in M(a^s, \bar{A}^{s-1}))
\]

Since \( \succ_i^s = \succ_i^{s-1} \) and \( \bar{A}^s = \bar{A}^{s-1} \setminus a^s \), we have \( \succ_i^s = \left( \succ_i^{s-1} \mid U(\succ_i^{s-1, a^s}) \setminus a^s, a^s, i \mid \bar{A} \right) \).

Proof of Part (2): Suppose \( i \in M(a^s, \bar{A}^{s-1}) \) and \( \bar{s}_i(a^s) < s \). Note that \( \succ_i^s = \succ_i^{s-1} \) because \( i \in M(a^s, \bar{A}^{s-1}) \), and \( i \in M(a^s, \bar{A}^{s-2}) \) because \( \bar{s}_i(a^s) < s \). If \( \bar{s}_i(a^s) = s - 1 \), then Part (3) for step \( s - 1 \) implies

\[
\succ_i^{s-1} = \left( \succ_i^{s-2} \mid U(\succ_i^{s-2, a^s}) \setminus a^s, a^{s-1}, i \mid \bar{A}^{s-1} \setminus a^s \right)
\]
Since $\succ_i^s = \succ_i^{s-1}$ and $\bar{A}^s = \bar{A}^{s-1}\backslash a^s$, and $\bar{s}_i(a^s) = s-1$, we have the desired equality $\succ_i^s = (\succ_i^{s-1} | U(\succ_i^{s-1}, a^s) \backslash a^s, a^{s-1} \succ_i | \bar{A}^{s-1})$.

On the other hand, if $\bar{s}_i(a^s) < s-1$, then Part (4) for step $s-1$ implies

$$\succ_i^{s-1} = (\succ_i^{s-2} | U(\succ_i^{s-2}, a^s) \backslash a^s, a^s, a^{s_i(a^s)}, \ldots, a^{s-1}, \succ_i | \bar{A}^{s-1})$$

Since $\succ_i^s = \succ_i^{s-1}$ and $\bar{A}^s = \bar{A}^{s-1}\backslash a^s$, we have the desired equality:

$$\succ_i^s = (\succ_i^{s-1} | U(\succ_i^{s-1}, a^s) \backslash a^s, a^s, a^{s_i(a^s)}, \ldots, a^{s-1}, \succ_i | \bar{A}^s).$$

Proof of Part (3): Suppose $i \in M(b, \bar{A}^{s-1})$ for some $b \neq a^s$, and $\bar{s}_i(b) = s$. At step $s-1$, since $\bar{s}_i(b) = a^s$, she demanded house $a^{s-1}$, i.e., $i \in M(a^{s-1}, \bar{A}^{s-2})$. Otherwise, she demanded some house $c \neq a^{s-1}, b$, and both $c$ and $b$ are available at step $s$, she would continue to demand $c$. This contradicts $i \in M(b, \bar{A}^{s-1})$.

Suppose $\bar{s}_i(a^{s-1}) = s-1$. Then, Part (1) for step $s-1$ implies

$$\succ_i^{s-1} = (\succ_i^{s-2} | U(\succ_i^{s-2}, a^{s-1}) \backslash a^{s-1}, a^{s-1}, \succ_i | \bar{A}^{s-1})$$

$$= (\succ_i^{s-2} | U(\succ_i^{s-2}, a^{s-1}) \backslash a^{s-1}, a^{s-1}, b, \succ_i | \bar{A}^{s-1})$$

and then

$$U(\succ_i^{s-1}, b) = \{b, a^s, \ldots, a^{s-1}\}.$$

Thus, since $\succ_i^s$ upgrades $a^s$ just below $b$ in $\succ_i^{s-1}$, we have the desired result:

$$\succ_i^s = (\succ_i^{s-1} | U(\succ_i^{s-1}, b) \backslash b, a^s, \succ_i | \bar{A}^{s-1})$$

$$= (\succ_i^{s-1} | U(\succ_i^{s-1}, b) \backslash b, a^s, \succ_i | \bar{A}^{s-1} \backslash \{b, a^s\})$$

$$U(\succ_i^s, b) = \{b, a^s, \ldots, a^{s-1}\}.$$

On the other hand, suppose $\bar{s}_i(a^{s-1}) < s-1$. Then, Part (2) for step $s-1$ implies

$$\succ_i^{s-1} = (\succ_i^{s-2} | U(\succ_i^{s-2}, a^{s-1}) \backslash a^{s-1}, a^{s-1}, a^{s_i(a^{s-1})}, \ldots, a^{s-2}, \succ_i | \bar{A}^{s-1})$$

$$= (\succ_i^{s-2} | U(\succ_i^{s-2}, a^{s-1}) \backslash a^{s-1}, a^{s-1}, a^{s_i(a^{s-1})}, \ldots, a^{s-2}, b, \succ_i | \bar{A}^{s-1})$$

and then

$$U(\succ_i^{s-1}, b) = \{b, a^s, \ldots, a^{s-1}\}.$$
Thus, since $\vDash_i^s$ upgrades $a^s$ just below $b$ in $\vDash_i^{s-1}$, we have the desired result.

\[
\vDash_i^s \, = \, \left( \vDash_i^{s-1} \mid U(\vDash_i^{s-1}, b), b, a^s, \vDash_i \mid \bar{A}_{s-1} \setminus \{b, a^s\} \right) \\
\, = \, \left( \vDash_i^{s-1} \mid U(\vDash_i^{s-1}, b), b, a^s, \vDash_i \mid \bar{A}_{s-1} \setminus \{b, a^s\} \right) \quad (\because \bar{A}^s \setminus b = \bar{A}^{s-1} \setminus \{b, a^s\}), \\
U(\vDash_i^s, b) \, = \, \{b, a^1, \ldots, a^{s-1}\}.
\]

Proof of **Part (4):** Suppose $i \in M(b, \bar{A}^{s-1})$ for some $b \neq b^s$, and $\bar{s}_i(b) < s$. Note that $i \in M(b, \bar{A}^{s-2})$ because $\bar{s}_i(b) < s$. Suppose $\bar{s}_i(b) = s - 1$. Then, Part (3) for step $s - 1$ implies

\[
\vDash_i^{s-1} = \left( \vDash_i^{s-2} \mid U(\vDash_i^{s-2}, b), b, a^{s-1}, \vDash_i \mid \bar{A}^{s-1} \setminus b \right),
\]

and then

\[
U(\vDash_i^{s-1}, b) \subseteq \{b, a^1, \ldots, a^{s-1}\}.
\]

Since $\vDash_i^s$ upgrades $a^s$ just below $a^{s-1}$ in $\vDash_i^{s-1}$, we have the desired result:

\[
\vDash_i^s \, = \, \left( \vDash_i^{s-1} \mid U(\vDash_i^{s-1}, b), b, a^{s-1}, a^s, \vDash_i \mid \bar{A}^{s-1} \setminus \{b, a^s\} \right) \\
\, = \, \left( \vDash_i^{s-1} \mid U(\vDash_i^{s-1}, b), b, a^{s-1}, a^s, \vDash_i \mid \bar{A}^{s-1} \setminus \{b, a^s\} \right) \quad (\because \bar{s}_i(b) = s - 1, \bar{A}^s \setminus b = \bar{A}^{s-1} \setminus \{b, a^s\}), \\
U(\vDash_i^s, b) \, = \, U(\vDash_i^{s-1}, b) \subseteq \{b, a^1, \ldots, a^{s-1}\}.
\]

On the other hand, suppose $\bar{s}_i(b) < s - 1$. Then, Part (4) for step $s - 1$ implies

\[
\vDash_i^{s-1} = \left( \vDash_i^{s-2} \mid U(\vDash_i^{s-2}, b), b, a^{s-1}, \ldots, a^s, \vDash_i \mid \bar{A}^{s-1} \setminus b \right),
\]

and then

\[
U(\vDash_i^{s-1}, b) \subseteq \{b, a^1, \ldots, a^{s-1}\}.
\]

Thus, since $\vDash_i^s$ upgrades $a^s$ just below $a^{s-1}$ in $\vDash_i^{s-1}$, we have the desired result:

\[
\vDash_i^s \, = \, \left( \vDash_i^{s-1} \mid U(\vDash_i^{s-1}, b), b, a^{s-1}, a^s, \vDash_i \mid \bar{A}^{s-1} \setminus \{b, a^s\} \right) \\
\, = \, \left( \vDash_i^{s-1} \mid U(\vDash_i^{s-1}, b), b, a^{s-1}, a^s, \vDash_i \mid \bar{A}^{s-1} \setminus \{b, a^s\} \right) \quad (\because \bar{A}^s \setminus b = \bar{A}^{s-1} \setminus \{b, a^s\}), \\
U(\vDash_i^s, b) \, = \, U(\vDash_i^{s-1}, b) \subseteq \{b, a^1, \ldots, a^{s-1}\}.
\]

**Proof of Claim 6.1 for Step s.** This can be easily verified by using the same idea as Claim A.1 for step $s$. ■
Proof of Claim A.6 for Step $s$. We prove the claim by induction on $|I|$. Let $I \subseteq N \setminus M(a^s, \tilde{A}^{s-1})$. As the inductive hypothesis, assume that the claim holds for all proper subsets of $I$ (the inductive hypothesis is vacuously true for the initial step $I = \emptyset$). Consider $I$:

Pick an agent $k \in I$ such that $k \not\in M(a^s, \tilde{A}^{s-1})$. Then, $k \in M(b, \tilde{A}^{s-1})$ for some house $b \neq a^s$. Let $\bar{P}' = \bar{P}^s(I)$.

Proof of Part (1): If $\bar{s}_k(b) = s$, then it follows from Part (3) of Claim A.5 for step $s$ that $\succ_k^s$ and $\succ_k^{s-1}$ are strongly upper invariant at $b$ and $U(\succ_k^s, b) = b \cup (A \setminus \tilde{A}^{s-1})$. On the other hand, if $\bar{s}_k(b) < s$, then it follows from Part (4) of Claim A.5 for step $s$ that $\succ_k^s$ and $\succ_k^{s-1}$ are strongly upper invariant at $a^{s-1}$ and $U(\succ_k^s, a^{s-1}) = b \cup (A \setminus \tilde{A}^{s-1})$. Thus, in either case, Lemma 3 implies that

\[
\bar{P}'_{|A \setminus \tilde{A}^{s-1}} = \bar{P}^s(I)_{|A \setminus \tilde{A}^{s-1}} = \bar{P}^s(I \setminus k)_{|A \setminus \tilde{A}^{s-1}} = \bar{P}_{|A \setminus \tilde{A}^{s-1}} \quad \text{(: induction hypothesis for step $s - 1$)}
\]

Hence, Part (1) holds.

Proof of Part (2): Fix any two agents $i, j \in M(a^s, \tilde{A}^{s-1})$. Consider first the case where $\bar{s}_i(a^s) = s$. Then, the first two parts of Claim A.5 for step $s$ imply $U(\succ_i^s, a^s) = \{a^1, \ldots, a^s\}$ and $U(\succ_j^s, a^s) \subseteq \{a^1, \ldots, a^s\}$. Thus,

\[
U(\succ_j^s, a^s) \subseteq U(\succ_i^s, a^s).
\] (26)

Hence, we have the desired inequality as follows:

\[
F(\succ_i^s, a^s, \bar{P}'_{\bar{s}_i}) \geq F(\succ_i^s, a^s, \bar{P}'_{\bar{s}_j}) \quad \text{(: envy-freeness of $\bar{P}'$ at $a^s$)}
\]
\[
\geq F(\succ_j^s, a^s, \bar{P}'_{\bar{s}_j}) \quad \text{(: (26))}.
\]

On the other hand, consider the case where $\bar{s}_i(a^s) < s$. First, note that, because $U(\succ_i^s, a^{s-1}) = \{a^1, \ldots, a^s\}$ and $U(\succ_j^s, a^s) \subseteq \{a^1, \ldots, a^s\}$ by Claim A.5. Thus,

\[
U(\succ_j^s, a^s) \subseteq U(\succ_i^s, a^{s-1}).
\] (27)

First, we will prove that

\[
\text{for all } b \in U(\succ_i^s, a^{s-1}) \setminus U(\succ_i^s, a^s), \quad \bar{p}_{ib} = 0.
\] (28)

To show this, notice from Part (2) of Claim A.5 for step $s$ that $U(\succ_i^s, a^s) \subseteq U(\succ_i^s, a^{s-1}) = \{a^1, \ldots, a^s\}$ and $U(\succ_i^s, a^{s-1}) \setminus U(\succ_i^s, a^s) = \{a^{\bar{s}_i(a^s)}, a^{\bar{s}_i(a^s)+1}, \ldots, a^{s-1}\} \subseteq A \setminus \tilde{A}^{s-1}$. Fix such a house $b \in U(\succ_i^s$
, $a^{s-1}) \setminus U(\succ_i^s, a^s)$. Then, $b = a^t \in A \setminus \bar{A}^{s-1}$ for some $t$ with $\bar{s}_i(a^s) \leq t \leq s-1$. Part (1) (of the claim for step $s$) implies that $\bar{p}_{ib} = \bar{p}_{ib}$. Also, $\bar{p}_{ib} = 0$ by Part (4) of Claim A.8 for the induction hypothesis for step $t \leq s-1$, because $i \notin M(a^s, \bar{A}^{t-1})$ but $i \in M(a^s, \bar{A}^{t-1})$. Thus, (28) holds.

Thus, we have the desired inequality as follows:

\[
F(\succ_i^s, a^s, \bar{P}_i^t) \equiv \sum_{b \in U(\succ_i^s, a^s)} \bar{p}_{ib} \\
= \sum_{b \in U(\succ_i^s, a^s)} \bar{p}_{ib} + \sum_{b \in U(\succ_i^s, a^{s-1}) \setminus U(\succ_i^s, a^s)} \bar{p}_{ib} \quad (\because \text{the second term is zero by (28)}) \\
= \sum_{b \in U(\succ_i^s, a^{s-1})} \bar{p}_{ib} \equiv F(\succ_i^s, a^{s-1}, \bar{P}_i^t) \\
\geq F(\succ_i^s, a^{s-1}, \bar{P}_i^t) \quad (\because \text{envy-freeness of } \bar{P}^t \text{ at } a^{s-1}) \\
\geq F(\succ_i^s, a^s, \bar{P}_i^t) \quad (\because (27)).
\]

The opposite inequality is obtained by the symmetric argument. Hence, we have the desired result.

\textbf{Proof of Claim A.7 for Step } s. \text{ Take any set } I \subseteq N \setminus M(a^s, \bar{A}^{s-1}), \text{ any agent } i \in M(a^s, \bar{A}^{s-1}), \text{ and any agent } j \in N \setminus M(a^s, \bar{A}^{s-1}) \text{ with } j \in M(b, \bar{A}^{s-1}) \text{ for some } b \neq a^s. \text{ Let } \bar{P}^t := \bar{P}^t(I). \text{ Suppose } \bar{p}_{ja^s} > 0.

Define

\[
a^s = \begin{cases} 
a^s & \text{if } \bar{s}_i(a^s) = s \\
a^{s-1} & \text{if } \bar{s}_i(a^s) < s.
\end{cases}
\]

We consider the preference $\succ_i''$ that upgrades $b$ just below $a^s$ in $\succ_i^s$. Notice that $\succ_i^s$ takes the form reported in Part (1) or Part (2) of Claim A.5 for step $s$ depending on $\bar{s}_i(a^s) = s$ or $\bar{s}_i(a^s) < s$, respectively. Consider the random assignment $\bar{P}'' := \phi(\succ'')$ by the mechanism when only agent $i$ changes her preference to $\succ_i''$ where $\succ'' = \left(\succ_i'' \setminus \succ_i'(M(a^s, \bar{A}^{s-1}) \cup I) \setminus \succ_i''(M(a^s, \bar{A}^{s-1}) \cup I)\right)$:

- Since $\succ_i''$ upgrades $b$ just below $a^s$ in $\succ_i^s$, $\succ_i''$ and $\succ_i^s$ are strongly upper invariant at $a^s$ and $U(\succ_i^s, a^s) = \{a^1, \ldots, a^s\}$ by Part (1) or (2) of Claim A.5. Thus, by upper invariance of $\phi$, Lemma 3 implies

\[
\bar{p}_{ja^s}'' = \bar{p}_{ja^s} > 0. \tag{29}
\]

- We have $\bar{p}_{ja^s} > 0$ by our assumption, and $b \succ_i^s a^s$ because $j \in M(b, \bar{A}^{s-1})$. This implies

\[
\bar{p}_{ib} = 0. \quad \tag{30}
\]

50
Otherwise, agent $i$ could exchange a small probability of $b$ with agent $j$ to receive a small probability of $a^s$ (note $a^s \succ_i^s b$ because $i \in M(a^s, \bar{A}^{t-1})$ and Claim A.5), and both $i$ and $j$ would be strictly better off with respect to $\bar{P}'$ (under stochastic dominance), while the other agents would be indifferent. This would contradict the ordinal efficiency of $\phi$ at $\bar{P}' \equiv \phi(\succ^s (I))$.

- We have $\bar{p}_{ja}'' > 0$ by (29), and $b \succ_j^s a^s$. This implies

$$\bar{p}_{ib}'' = 0.$$  \hspace{1cm} (31)

Otherwise, agent $i$ could exchange a small probability of $b$ with agent $j$ to receive a small probability of $a^s$ (note $a^s \succ_i^s b$ because $i \in M(a^s, \bar{A}^{t-1})$ and Claim A.5), and both $i$ and $j$ would be strictly better off with respect to $\bar{P}''$ (under stochastic dominance), while the other agents would be indifferent. This would contradict the ordinal efficiency of $\phi$ at $\bar{P}'' \equiv \phi(\succ'')$.

- Notice from (30), (31) that $\succ_i^s$ and $\succ_i''$ are strongly invariant by $b$. Thus, by Lemma 3,

$$\bar{P}' = \bar{P}''.$$  \hspace{1cm} (32)

- It follows from Parts (1) and (2) of Claim A.5 that $U(\succ_i^s, a^s) = \{a^1, \ldots, a^s\}$. Since $\succ_i''$ upgrades $b$ just below $a^s$ in $\succ_i^s$, we have

$$U(\succ_i'', a^s) = U(\succ_i^s, a^s) = \{a^1, \ldots, a^s\},$$

$$U(\succ_i'', b) = U(\succ_i''', a^s) \cup \{b\} = \{b, a^1, \ldots, a^s\}.$$  \hspace{1cm} (33)

Moreover, it follows from Parts (3) and (4) of Claim A.5 that $U(\succ_j^{s-1}, b) = U(\succ_j^s, b) \subset \{b, a^1, \ldots, a^{s-1}\}$. Thus,

$$U(\succ_j^{s-1}, b) \cup \{a^s\} = U(\succ_j^s, b) \cup \{a^s\} \subset U(\succ_i''', b).$$  \hspace{1cm} (34)

- We will show that

$$\text{for all } c \in U(\succ_i^s, a^s) \setminus U(\succ_i^s, a^s), \quad \bar{p}_{ic}'' = 0$$  \hspace{1cm} (35)

as follows:

If $\bar{s}_i(a^s) = s$, then this is vacuously true as $a^* = a^s$. Thus, suppose that $\bar{s}_i(a^s) < s$. Notice from Part (2) of Claim A.5 that $U(\succ_i^s, a^s) \subset U(\succ_i^s, a^s)$ and $U(\succ_i^s, a^s) \setminus U(\succ_i^s, a^s) = \{a^{\bar{s}_i(a^s)}, a^{\bar{s}_i(a^s)} + 1, \ldots, a^{s-1}\}$. Fix any house $c$ in the set above. Then, $c = a^t$ for some $t$ with $\bar{s}_i(a^s) \leq t \leq s - 1$. Then, it follows from Part (1) of Claim A.6 for step $s$ that $\bar{p}_{ic}' = \bar{p}_{tc}$. Also, $\bar{p}_{ic} = 0$ by inductive hypothesis of Part (4) for Claim A.8 for step $t$, because $i \notin M(a^t, \bar{A}^{t-1})$ but $i \in M(a^s, \bar{A}^{t-1})$. Thus, (35) holds.
• Finally, if \( j \in I \), we obtain the desired inequality as follows:

\[
F(\succ_i^s, a^s, \bar{P}_i') \equiv \sum_{c \in U(\succ_i^s, a^s)} \bar{p}_{ic}' = \sum_{c \in U(\succ_i^s, a^s)} \bar{p}_{ic}' + \sum_{c \in U(\succ_i^s, a^s) \setminus U(\succ_i^s, a^s)} \bar{p}_{ic}' \quad (\because \text{the second term is zero by (35)})
\]

\[
= \sum_{c \in U(\succ_i^s, a^s)} \bar{p}_{ic}'
\]

\[
= \sum_{c \in U(\succ_i^s, a^s)} \bar{p}_{ic}' \equiv F(\succ_i'', a^s, \bar{P}_i') \quad (\because U(\succ_i'', a^s) = U(\succ_i^s, a^s) \text{ by (33)})
\]

\[
= F(\succ_i'', a^s, \bar{P}_i'') \quad (\because \bar{P}_i'' = \bar{P}' \text{ by (32)})
\]

\[
= F(\succ_i'', a^s, \bar{P}_i'') + \bar{p}_{i}'' \quad (\because \bar{p}_{i}'' = 0 \text{ by (31)})
\]

\[
= F(\succ_i'', b, \bar{P}_i'') \quad (\because \succ_i'' \text{ ranks } b \text{ just below } a^s)
\]

\[
\geq F(\succ_i'', b, \bar{P}_i'') \equiv \sum_{c \in U(\succ_i'', b)} \bar{p}_{jc}'' \quad (\because \text{envy-freeness of } \bar{P}_i'' \text{ for } i \text{ at } b)
\]

\[
\geq \sum_{c \in U(\succ_i'', b)} \bar{p}_{jc}'' + \bar{p}_{ja}'' \quad (\because U(\succ_j'', b) \cup \{a^s\} \subseteq U(\succ_i'', b) \text{ by (34)})
\]

\[
> \sum_{c \in U(\succ_i'', b)} \bar{p}_{jc}'' \equiv F(\succ_j'', b, \bar{P}_j'') \quad (\because \bar{p}_{ja}'' > 0 \text{ by (29)})
\]

\[
= F(\succ_j'', b, \bar{P}_j') \quad (\because \bar{P}_j'' = \bar{P}' \text{ by (32)})
\]

The case with \( j \not\in I \) has the same proof as above, after replacing \( \succ_{j-1}^s \) with \( \succ_{j}^s \).

---

**Proof of Claim A.8 for Step s.** We prove the claim for step \( s \) by induction on \(|I|\). Let \( I \subseteq N \setminus M(a^s, \bar{A}^{s-1}) \). In the inductive step, assume that for all proper subsets of \( I \) the claim is true at step S. Consider \( I \):

Pick agent \( k \in I \) so that \( k \not\in M(a^s, \bar{A}^{s-1}) \). Then, \( k \in M(b, \bar{A}^{s-1}) \) for some house \( b \neq a^s \). Let \( \bar{P}' := \bar{P}^s(I \setminus k) \) and \( \bar{P}'' := \bar{P}^s(I) \). Then, the induction hypothesis for \( I \setminus k \) implies that

\[
\bar{P}' = \bar{P}.
\]

(36)

Proof of Part (1): Note that whether or not \( s_k(b) = s, \succ_k^s \) upgrades \( a^s \) just below some house in \( \succ_{k-1}^s \).

• We first prove that to show \( \bar{P} = \bar{P}'' \), it is sufficient to prove \( \bar{p}_{ks}'' = 0 \): Suppose \( \bar{p}_{ks}'' = 0 \). Now, since \( k \not\in M(a^s, \bar{A}^{s-1}) \), we have \( \bar{p}_{ks}'' = 0 \) by the induction hypothesis of Part (4) for \( I \setminus k \). Hence, \( \succ_k^s \) and \( \succ_{k-1}^s \) are strongly invariant by \( a^s \) and thus \( \bar{P}'' = \bar{P}' \) by Lemma 3, and \( \bar{P}' = \bar{P} \) by (36). Therefore, \( \bar{P} = \bar{P}'' \).
• We show $p''_{ka^s} = 0$: For a contradiction, suppose $p''_{ka^s} > 0$. Pick any agent $i \in M(a^s, \bar{A}^{s-1})$.

* First, we show

$$F(\succ^s_i, a^s, \bar{P}'') < \bar{y}^s : \quad (37)$$

Since $p''_{ka^s} > 0$ by our assumption, we have $\sum_{j \in M(a^s, \bar{A}^{s-1})} p''_{ja^s} < \bar{q}_a$, which is equivalent to

$$\sum_{j \in M(a^s, \bar{A}^{s-1})} \left[ F(\succ^s_j, a^s, \bar{P}'') - \sum_{b \in U(\succ^s_j, a^s) \setminus a^s} \bar{p}'_{jb} \right] < \bar{q}_a. \quad (38)$$

Since $\sum_{j \in M(a^s, \bar{A}^{s-1})} \bar{p}'_{ja^s} = \bar{q}_a$ by the induction hypothesis of Part (4) for $I \setminus k$, we have

$$\Leftrightarrow \sum_{j \in M(a^s, \bar{A}^{s-1})} \left[ \bar{y}^s - \sum_{b \in U(\succ^s_j, a^s) \setminus a^s} \bar{p}'_{jb} \right] = \bar{q}_a. \quad (39)$$

In both equations (38) and (39), the second term is equal, because we have $\bar{P}''_{|A \setminus \bar{A}^{s-1}} = \bar{P}'_{|A \setminus \bar{A}^{s-1}} = \bar{P}''_{|A \setminus \bar{A}^{s-1}}$ by Part (1) of Claim A.6 for step $s$, and $U(\succ^s_j, a^s) \setminus a^s \subseteq A \setminus \bar{A}^{s-1}$ by Claim A.5 for step $s$. Also, we have for all $j \in M(a^s, \bar{A}^{s-1}), F(\succ^s_j, a^s, \bar{P}''_{j}) = F(\succ^s_j, a^s, \bar{P}''_{j})$ by Part (2) of Claim A.6. Hence, both equations (38) and (39) imply (37).

* Next, we show

$$F(\succ^s_k, b, \bar{P}''_k) = F(\succ^s_k, b, \bar{P}_k) : \quad (40)$$

Whether or not $s_k(b) = s, \succ^s_k$ upgrades $a^s$ just below some house. Looking at Claim A.5 for step $s, \succ^s_k$ and $\succ^{s-1}_k$ are strongly upper invariant at $b$. Thus, upper invariance of $\phi$ implies that $F(\succ^s_k, b, \bar{P}''_k) = F(\succ^{s-1}_k, b, \bar{P}_k)$ by Lemma 3. Also, we have $F(\succ^{s-1}_k, b, \bar{P}_k) = F(\succ^s_k, b, \bar{P}_k)$ from the induction hypothesis of Part (2) for $I \setminus k$. Thus, we have (40).

* Therefore, we have

$$F(\succ^s_k, b, \bar{P}'') < F(\succ^s_i, a^s, \bar{P}'') \quad (\therefore \text{Claim A.7 for step } s)$$

$$\Rightarrow F(\succ^s_k, b, \bar{P}_k) < \bar{y}^s \quad (\therefore (37) \text{ and (40)})$$

$$\Leftrightarrow \bar{y}^s_k(b) < \bar{y}^s. \quad (\therefore \text{Claim 6.1 for step } s)$$

This contradicts the definition of $\bar{y}^s$. Therefore, Part (1) holds.
Proof of Part (2): Note from the proof of Part (1) that $\succ^s_k$ and $\succ^{s-1}_k$ are strongly invariant by $a^s$ and thus $\bar{P}'' = \bar{P}'$, and $\bar{P}' = \bar{P}$. With these, Lemma 3 immediately implies Part (2).

Proof of Part (3): Take any agent $i \in M(a^s, A^{s-1})$. Note from the proof of Part (1) that $\succ^s_k$ and $\succ^{s-1}_k$ are strongly invariant by $a^s$. 

$$F(\succ_i^s, a^s, \bar{P}'') = F(\succ_i^s, a^s, \bar{P}') \quad (\because \text{Lemma 3})$$

$$= \bar{y}^s \quad (\because \text{induction hypothesis-Part (3) for } I \setminus k).$$

Proof of Part (4): From the proof of Part (1), we know $\bar{P}'' = \bar{P}'$. Also, we have $\sum_{j \in M(a^s, A^{s-1})} \bar{p}'_{ja^s} = \bar{q}_a$ from the induction hypothesis of Part (4) for $I \setminus k$. Thus, $\sum_{j \in M(a^s, A^{s-1})} \bar{p}''_{ja^s} = \bar{q}_a$. □

References


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23–38.


1433.


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A.4 Online Appendix: Proof of Proposition 3

We show that the mechanism in Example 5 is ordinally efficient, envy-free, and weakly strategy-proof.

First, note that $P^*$ is ordinally efficient at $\succ^*$, and the PS mechanism is ordinally efficient (Bo
gomolnaia and Moulin, 2001). Thus, the mechanism $\phi$ is ordinally efficient.

Second, we show that the mechanism $\phi$ is envy-free. Since the PS mechanism is envy-free (Bogomolnaia and Moulin, 2001), we need to show that $P^*$ is envy-free at $\succ^*$. Let $sd(\succ_i)$ be the stochastic dominance relation induced by preference $\succ_i$. For example, we need to show, for agent 1, $P^*_1^{sd(\succ^-_i)}P^*_1^{sd(\succ^-_i)}$.
and $P_1^*sd(\succ^*_1)P_5^*$. We have similar conditions for agents 4 and 5. To this end, we use the following table:

<table>
<thead>
<tr>
<th>$\succ_i^*$</th>
<th>a</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_i^*$</td>
<td>220</td>
<td>430</td>
<td>615</td>
<td>720</td>
<td>720</td>
</tr>
<tr>
<td>$P_4^*$</td>
<td>0</td>
<td>75</td>
<td>240</td>
<td>360</td>
<td>720</td>
</tr>
<tr>
<td>$P_5^*$</td>
<td>60</td>
<td>75</td>
<td>75</td>
<td>360</td>
<td>720</td>
</tr>
</tbody>
</table>

The first row indicates the houses in order of the preference $\succ^*_1$ of agent 1. The second row calculates $F(\succ^*_1, a', P_i^*)$ for each corresponding house $a'$ in the first row. The third row calculates $F(\succ^*_1, a', P_4^*)$ for each corresponding house $a'$ in the first row. Similarly defined is the fourth row. To have $P_i^*sd(\succ^*_1)P_4^*$, we need to compare the row of $P_i^*$ with that of $P_4^*$. That is, for each column, the number in the second row must be greater than or equal to the number in the third row. The above table actually shows $P_1^*sd(\succ^*_1)P_4^*$ and $P_1^*sd(\succ^*_1)P_5^*$. Similarly, we have tables for agents 4 and 5:

<table>
<thead>
<tr>
<th>$\succ_i^*$</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_4^*$</td>
<td>360</td>
<td>435</td>
<td>600</td>
<td>720</td>
<td>720</td>
</tr>
<tr>
<td>$P_1^*$</td>
<td>0</td>
<td>210</td>
<td>395</td>
<td>500</td>
<td>720</td>
</tr>
<tr>
<td>$P_5^*$</td>
<td>360</td>
<td>375</td>
<td>375</td>
<td>660</td>
<td>720</td>
</tr>
</tbody>
</table>

Looking at the above tables, we conclude that $P^*$ is envy-free at $\succ^*$.

Finally, we show that the mechanism $\phi$ is weakly strategy-proof. For notational simplicity, we use $P = \phi(\succ_i, \succ^*_i)$ for a preference $\succ_i \neq \succ^*_i$ of an agent $i$. Recall $P^* = \phi(\succ^*)$. First, because $\phi$ consists of the PS mechanism that is weakly strategy-proof (Bogomolnaia and Moulin, 2001), we need to show for all $i \in N, \succ_i$, if $P_i \neq P_i^*$, then

1. it is not possible that $P_i sd(\succ^*_i) P_i^*$, and

2. it is not possible that $P_i^* sd(\succ^*_i) P_i$.

By symmetry, we show this for $i = 1, 4, 5$. Before checking the above two conditions, we introduce two kinds of tables. For example, consider the case where $i = 1$ and her preference is $\succ_1 = (e, b, a, c, c) \neq \succ^*_1$. We denote $P = \phi(\succ_1, \succ^*_1) \equiv PS(\succ_1, \succ^*_1)$, and recall $P^* \equiv \phi(\succ^*_1, \succ^*_1)$. To examine the first condition in the above, we use the following table.

<table>
<thead>
<tr>
<th>$\succ^*_1$</th>
<th>a</th>
<th>c</th>
<th>b</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1^*$</td>
<td>$F(\succ^<em>_1, a, P_1^</em>)$</td>
<td>$F(\succ^<em>_1, c, P_1^</em>)$</td>
<td>$F(\succ^<em>_1, b, P_1^</em>)$</td>
<td>$F(\succ^<em>_1, d, P_1^</em>)$</td>
<td>$F(\succ^<em>_1, e, P_1^</em>)$</td>
</tr>
<tr>
<td>$P_1$</td>
<td>$F(\succ^*_1, a, P_1)$</td>
<td>$F(\succ^*_1, c, P_1)$</td>
<td>$F(\succ^*_1, b, P_1)$</td>
<td>$F(\succ^*_1, d, P_1)$</td>
<td>$F(\succ^*_1, e, P_1)$</td>
</tr>
</tbody>
</table>
Here the first row indicates the houses in order of preference $\succ^*_1$. To verify the first condition (i.e., we cannot have $P_1sd(\succ^*_1)P_1^*$, it suffices to have that, at some column, the number in the second row is strictly greater than the one in the third row. Thus, we will list houses until the column with this condition.

Similarly, to examine the second condition in the above, we use the following table.

<table>
<thead>
<tr>
<th>$\succ_1$</th>
<th>e</th>
<th>$\succ_1$</th>
<th>b</th>
<th>$\succ_1$</th>
<th>a</th>
<th>$\succ_1$</th>
<th>c</th>
<th>$\succ_1$</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>$F(\succ_1, e, P_1)$</td>
<td>$F(\succ_1, b, P_1)$</td>
<td>$F(\succ_1, a, P_1)$</td>
<td>$F(\succ_1, c, P_1)$</td>
<td>$F(\succ_1, d, P_1)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P_1^*$</td>
<td>$F(\succ_1, e, P_1^*)$</td>
<td>$F(\succ_1, b, P_1^*)$</td>
<td>$F(\succ_1, a, P_1^*)$</td>
<td>$F(\succ_1, c, P_1^*)$</td>
<td>$F(\succ_1, d, P_1^*)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here the first row indicates the houses in order of preference $\succ_1$. To verify the condition 2, it suffices to have that, at some column, the number in the second row is strictly greater than the one in the third row. Thus, we will list houses until the column with this condition.

Now we start checking each case.

First, consider agent 1. Take any preference $\succ_1$.

Case 1-1: $\succ_1 = (a, c, d, b, e)$ or $(a, c, b, d, e)$.
Then, $P_1 = PS(\succ^*)$ (Recall $P_1 = PS(\succ_1, \succ^*_{-1})$) and thus $P_1 = \frac{1}{720}(240, 0, 192, 180, 108)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_1$</th>
<th>a</th>
<th>$\succ^*_1$</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1^*$</td>
<td>220</td>
<td>$P_1^*$</td>
<td>220</td>
</tr>
<tr>
<td>$P_1$</td>
<td>240</td>
<td>$P_1^*$</td>
<td>220</td>
</tr>
</tbody>
</table>

Case 1-2: $\succ_1 = (a, c, b, e, d)$ or $(a, c, e, \cdots)$.
Then, $P_1 = \frac{1}{720}(240, 0, 192, 0, 288)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_1$</th>
<th>a</th>
<th>$\succ^*_1$</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1^*$</td>
<td>220</td>
<td>$P_1^*$</td>
<td>220</td>
</tr>
<tr>
<td>$P_1$</td>
<td>240</td>
<td>$P_1^*$</td>
<td>220</td>
</tr>
</tbody>
</table>

Case 1-3: $\succ_1 = (a, b, c, \cdots)$.
Then, $P_1 = \frac{1}{720}(240, 80, 112, \cdots)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_1$</th>
<th>a</th>
<th>$\succ^*_1$</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1^*$</td>
<td>220</td>
<td>$P_1^*$</td>
<td>220</td>
</tr>
<tr>
<td>$P_1$</td>
<td>240</td>
<td>$P_1^*$</td>
<td>220</td>
</tr>
</tbody>
</table>

Case 1-4: $\succ_1 = (a, b, d, \cdots)$ or $(a, b, e, \cdots)$.
Then, $P_1 = \frac{1}{720}(240, 80, 0, \cdots)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_1$</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1^*$</td>
<td>220</td>
</tr>
<tr>
<td>$P_1$</td>
<td>60</td>
</tr>
</tbody>
</table>

Now we start checking each case.

First, consider agent 1. Take any preference $\succ_1$.

Case 1-1: $\succ_1 = (a, c, d, b, e)$ or $(a, c, b, d, e)$.
Then, $P_1 = PS(\succ^*)$ (Recall $P_1 = PS(\succ_1, \succ^*_{-1})$) and thus $P_1 = \frac{1}{720}(240, 0, 192, 180, 108)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_1$</th>
<th>a</th>
<th>$\succ^*_1$</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1^*$</td>
<td>220</td>
<td>$P_1^*$</td>
<td>220</td>
</tr>
<tr>
<td>$P_1$</td>
<td>240</td>
<td>$P_1^*$</td>
<td>220</td>
</tr>
</tbody>
</table>

Case 1-5: $\succ_1 = (a, d, \cdots)$ or $(a, e, \cdots)$.
Then, $P_1 = \frac{1}{720}(240, 0, 0, \cdots)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_1$</th>
<th>a</th>
<th>$\succ^*_1$</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1^*$</td>
<td>220</td>
<td>$P_1^*$</td>
<td>220</td>
</tr>
<tr>
<td>$P_1$</td>
<td>240</td>
<td>$P_1^*$</td>
<td>220</td>
</tr>
</tbody>
</table>

Case 1-6: $\succ_1 = (b, \cdots)$.
Obviously, $p_{1b} = 1/3 = 240/720$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_1$</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1^*$</td>
<td>220</td>
</tr>
<tr>
<td>$P_1$</td>
<td>60</td>
</tr>
</tbody>
</table>

Note that $p_{1a}$ is the largest if $\succ_1 = (b, a, \cdots)$. Suppose $\succ_1 = (b, a, \cdots)$. Then, $P_1 = \frac{1}{720}(60, 240, \cdots)$. Hence, the other table is

<table>
<thead>
<tr>
<th>$\succ^*_1$</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1^*$</td>
<td>220</td>
</tr>
<tr>
<td>$P_1$</td>
<td>60</td>
</tr>
</tbody>
</table>
Thus, for any preference $\succ_1$, we have the desired result.

Case 1-7: $\succ_1 = (c, \cdots)$. Then, $P_1 = \frac{1}{720}(0,0,432,\cdots)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_1$</th>
<th>$a$</th>
<th>$\succ_1$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^*_1$</td>
<td>220</td>
<td>$P_1$</td>
<td>432</td>
</tr>
<tr>
<td>$P_1$</td>
<td>60</td>
<td>$P^*_1$</td>
<td>210</td>
</tr>
</tbody>
</table>

Case 1-8: $\succ_1 = (d, \cdots)$. Then, $P_1 = \frac{1}{720}(0,0,0,585,135)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_1$</th>
<th>$a$</th>
<th>$\succ_1$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^*_1$</td>
<td>220</td>
<td>$P_1$</td>
<td>585</td>
</tr>
<tr>
<td>$P_1$</td>
<td>0</td>
<td>$P^*_1$</td>
<td>185</td>
</tr>
</tbody>
</table>

Case 1-9: $\succ_1 = (e, \cdots)$. Then, $P_1 = \frac{1}{720}(0,0,0,90,630)$.

<table>
<thead>
<tr>
<th>$\succ^*_1$</th>
<th>$a$</th>
<th>$\succ_1$</th>
<th>$e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^*_1$</td>
<td>220</td>
<td>$P_1$</td>
<td>630</td>
</tr>
<tr>
<td>$P_1$</td>
<td>0</td>
<td>$P^*_1$</td>
<td>105</td>
</tr>
</tbody>
</table>

Next, consider agent 4. Take any preference $\succ_4$ ($\neq \succ^*_4$). We denote $P = \phi(\succ_4, \succ^*_4) \equiv PS(\succ_4, \succ^*_4)$, and recall $P^* = \phi(\succ^*_4, \succ^*_4)$.

Case 4-1: $\succ_4 = (b,c,d,a,e)$, $(b,c,a,d,e)$, or $(b,a,c,d,e)$.

Then, $P = PS(\succ^*)$. Thus,

<table>
<thead>
<tr>
<th>$\succ^*_4$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^*_4$</td>
<td>360</td>
<td>435</td>
</tr>
<tr>
<td>$P_4$</td>
<td>360</td>
<td>432</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\succ_4$</th>
<th>$b$</th>
<th>$\ (a)$</th>
<th>$c$</th>
<th>$\ (a)$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_4$</td>
<td>360</td>
<td>$(360)$</td>
<td>432</td>
<td>$(432)$</td>
<td>612</td>
</tr>
<tr>
<td>$P^*_4$</td>
<td>360</td>
<td>$(360)$</td>
<td>435</td>
<td>$(435)$</td>
<td>600</td>
</tr>
</tbody>
</table>

Case 4-2: $\succ_4 = (b,c,a,e,d)$, $(b,c,e,\cdots)$, or $(b,a,c,e,d)$.

Then, $P_4 = \frac{1}{720}(0,360,72,0,288)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_4$</th>
<th>$b$</th>
<th>$\succ_4$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^*_4$</td>
<td>360</td>
<td>$P_4$</td>
<td>360</td>
</tr>
<tr>
<td>$P_4$</td>
<td>360</td>
<td>$P^*_4$</td>
<td>75</td>
</tr>
</tbody>
</table>

Thus, for any preference, we have the desired result.

Case 4-3: $\succ_4 = (b,a,d,\cdots)$ or $(b,d,\cdots)$. Then, $P_4 = \frac{1}{720}(0,360,0,247.5,112.5)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_4$</th>
<th>$b$</th>
<th>$\ (a)$</th>
<th>$c$</th>
<th>$\ (a)$</th>
<th>$e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^*_4$</td>
<td>360</td>
<td>$(360)$</td>
<td>432</td>
<td>$(432)$</td>
<td>720</td>
</tr>
<tr>
<td>$P_4$</td>
<td>360</td>
<td>$(360)$</td>
<td>435</td>
<td>$(435)$</td>
<td>555</td>
</tr>
</tbody>
</table>

Case 4-4: $\succ_4 = (b,a,e,\cdots)$ or $(b,e,\cdots)$. Then, $P_4 = \frac{1}{720}(0,360,0,0,360)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_4$</th>
<th>$b$</th>
<th>$\succ_4$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^*_4$</td>
<td>360</td>
<td>$P_4$</td>
<td>360</td>
</tr>
<tr>
<td>$P_4$</td>
<td>360</td>
<td>$P^*_4$</td>
<td>525</td>
</tr>
</tbody>
</table>

Case 4-5: $\succ_4 = (a,\cdots)$. Obviously, $p_{4a} = 1/4 = 180/720$. Thus,

<table>
<thead>
<tr>
<th>$\succ_4$</th>
<th>$\succ^*_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_4$</td>
<td>180</td>
</tr>
<tr>
<td>$P^*_4$</td>
<td>0</td>
</tr>
</tbody>
</table>

Note that $p_{4b}$ is the largest if $\succ_4 = (a,b,\cdots)$. Suppose $\succ_4 = (a,b,\cdots)$. Then, $P_4 = \frac{1}{720}(180,270,0,\cdots)$. And the other table is

<table>
<thead>
<tr>
<th>$\succ_4$</th>
<th>$\succ^*_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_4$</td>
<td>360</td>
</tr>
<tr>
<td>$P^*_4$</td>
<td>270</td>
</tr>
</tbody>
</table>

Thus, for any preference, we have the desired result.

Case 4-6: $\succ_4 = (c,\cdots)$. We can calculate as $p_{4c} = 1/2 = 360/720$. Thus,

<table>
<thead>
<tr>
<th>$\succ_4$</th>
<th>$\succ^*_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_4$</td>
<td>360</td>
</tr>
<tr>
<td>$P^*_4$</td>
<td>75</td>
</tr>
</tbody>
</table>
Note that $p_{4b}$ is the largest if $\succ_4 = (c, b, \cdots)$. Suppose $\succ_4 = (c, b, \cdots)$. Then, $P_4 = \frac{1}{720}(0, 180, 360, \cdots)$. And the other table is

<table>
<thead>
<tr>
<th>$\succ^*$</th>
<th>b</th>
<th>$\succ$</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_4$</td>
<td>360</td>
<td>$P_4$</td>
<td>720</td>
</tr>
<tr>
<td>$P_4^*$</td>
<td>0</td>
<td>$P_4^*$</td>
<td>120</td>
</tr>
</tbody>
</table>

Thus, for any preference, we have the desired result.

Case 5-1: $\succ_5 = (b, a, d, \cdots)$. Then, $P_5 = \frac{1}{720}(0, 360, 0, 216, 144)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_5$</th>
<th>b</th>
<th>a</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_5$</td>
<td>360</td>
<td>360</td>
<td>432</td>
<td>576</td>
</tr>
<tr>
<td>$P_5^*$</td>
<td>360</td>
<td>420</td>
<td>435</td>
<td>435</td>
</tr>
</tbody>
</table>

Case 5-2: $\succ_5 = (b, a, d, \cdots)$. Then, $P_5 = \frac{1}{720}(0, 360, 0, 216, 144)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_5$</th>
<th>b</th>
<th>a</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_5$</td>
<td>360</td>
<td>360</td>
<td>576</td>
</tr>
<tr>
<td>$P_5^*$</td>
<td>360</td>
<td>420</td>
<td>420</td>
</tr>
</tbody>
</table>

Case 5-3: $\succ_5 = (b, a, e, \cdots)$. Then, $P_5 = \frac{1}{720}(0, 360, 0, 360)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_5$</th>
<th>b</th>
<th>a</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_5$</td>
<td>360</td>
<td>360</td>
<td>720</td>
</tr>
<tr>
<td>$P_5^*$</td>
<td>360</td>
<td>420</td>
<td>705</td>
</tr>
</tbody>
</table>

Case 5-4: $\succ_5 = (b, c, \cdots)$. Then, $P_5 = \frac{1}{720}(0, 360, 72, \cdots)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_5$</th>
<th>b</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_5$</td>
<td>360</td>
<td>420</td>
</tr>
<tr>
<td>$P_5^*$</td>
<td>360</td>
<td>360</td>
</tr>
</tbody>
</table>

Case 5-5: $\succ_5 = (b, d, \cdots)$. $P$ coincides with the one in Case 5-2, i.e., $P_5 = \frac{1}{720}(0, 360, 0, 216, 144)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_5$</th>
<th>b</th>
<th>a</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_5$</td>
<td>360</td>
<td>420</td>
<td>576</td>
</tr>
<tr>
<td>$P_5^*$</td>
<td>360</td>
<td>360</td>
<td>360</td>
</tr>
</tbody>
</table>

Case 5-6: $\succ_5 = (b, e, \cdots)$. $P$ coincides with the one in Case 5-3, i.e., $P_5 = \frac{1}{720}(0, 360, 0, 360)$. Hence,

<table>
<thead>
<tr>
<th>$\succ^*_5$</th>
<th>b</th>
<th>a</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_5$</td>
<td>360</td>
<td>420</td>
<td>720</td>
</tr>
<tr>
<td>$P_5^*$</td>
<td>360</td>
<td>360</td>
<td>645</td>
</tr>
</tbody>
</table>

Case 5-7: $\succ_5 = (a, \cdots)$. Obviously, $p_{5a} = 1/4 = 180/720$. Thus,
<table>
<thead>
<tr>
<th>$\succ_5$</th>
<th>$a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_5$</td>
<td>180</td>
</tr>
<tr>
<td>$P_5^*$</td>
<td>60</td>
</tr>
</tbody>
</table>

Note that $p_{5b}$ is the largest if $\succ_5 = (a, b, \cdots)$. Suppose $\succ_5 = (a, b, \cdots)$. Then, $P_5 = \frac{1}{720}(180, 270, 0, \cdots)$.

<table>
<thead>
<tr>
<th>$\succ^*_5$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_5^*$</td>
<td>360</td>
</tr>
<tr>
<td>$P_5$</td>
<td>270</td>
</tr>
</tbody>
</table>

Thus, for any preference, we have the desired result.

Case 5-9: $\succ_5 = (d, \cdots)$.

Obviously, $p_{5d} \geq 1/3 = 240/720$. Thus, $P_5 = \frac{1}{720}(0, 90, 540, 90)$.

<table>
<thead>
<tr>
<th>$\succ^*_5$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_5^*$</td>
<td>360</td>
</tr>
<tr>
<td>$P_5$</td>
<td>90</td>
</tr>
</tbody>
</table>

Thus, for any preference, we have the desired result.

We next show that mechanism $\phi$ is not upper invariant: Consider the preference of agent 1, $\succ'_1 = (a, c, d, b, e)$. Then, $\phi(\succ'_1, \succ^*_{1-1}) = PS(\succ'_1, \succ^*_{1-1}) = PS(\succ^*)$. In particular, $\phi_{ib}(\succ'_1, \succ^*_{1-1}) = 0$. Thus, $\succ'_1 = (a, c, d, e, b)$ is an upper invariant transformation of $\succ'_1$ at $e$ under $\phi(\succ'_1, \succ^*_{1-1})$. However, $\phi_{je}(\succ'_1, \succ^*_{1-1}) \neq \phi_{je}(\succ'_1, \succ^*_{1-1})$ for all $j = 1, 2, \cdots, 5$. Hence, $\phi$ is not upper invariant.