# Inferring Beliefs from Actions*

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**Abstract**

This paper considers single-agent decision making under uncertainty and addresses the question to which degree the private information of an agent is revealed through his optimal action. We show that the agents optimal action reveals his posterior distribution for all but a negligible subset (formally, a meager subset) of the space of continuous utility functions if and only if the set of actions contains no isolated points. If the action set is uncountable (not necessarily perfect), then there exists a continuous utility function such that actions reveal beliefs. On the basis of the single-agent belief revelation result we establish information aggregation results in the framework of repeated interaction in social networks and in the sequential social learning model.

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# 1 Introduction

The focus of this paper lies on observational learning. That is, the ability to make inferences regarding the private information of an agent based on the actions he takes. The big picture question we address is to what extend does an agents action reveal his private information. This is a central question in economic theory. The literatures on social learning[1], knowledge and consensus[2], and the belief elicitation (scoring rule) literature in statistics[3] and economic theory[4] crucially evolve around this point.

In a single-agent decision problem, the agent is endowed with a probability distribution, or belief, on the state space. This belief might be based upon updating a prior distribution conditioning on private information the agent observes. The private information might be given by a realization of a state-dependent signal or represented by a partition of the state space. The agent maximizes his expected utility conditional on his posterior distribution.

The objective of this paper is to gain a deeper understanding of the phenomenon of actions revealing beliefs. We consider a single agent who faces a decision under uncertainty. The state space is compact metrizable. The agent has a private probability distribution on the state space and selects an action out of a compact metrizable set. His utility depends on the action and the realized state of the world and we assume that the utility function is continuous.

Rather than considering a specific utility function we consider the space of continuous utility functions. The central question the paper addresses is the following: how much structure needs to be imposed on this general environment such that the expected utility maximizing action reveals the agents belief, that is his posterior distribution. For the general state and action space considered, it is

[1]See Banerjee [7], Bikhchandani, Hirshleifer and Welch [9], Smith and Sorensen [37], Gale and Kariv [19].

[2]See Geanakoplos and Polemarchakis [20], Parikh and Krasucki [33], Krasucki [26]

[3]For example, Brier [11], de Finetti [14], Gneiting and Raftery [21], Good [22] and Savage [36].

[4]See Karni [24].

unclear whether continuous utility functions satisfying belief revelation exist. It is clear however that under a discrete action space belief revelation cannot occur as the space of distributions is uncountable. Therefore, a necessary condition is that the action space is uncountable. Our first result, Theorem 1, establishes a necessary and sufficient condition on the action space such that belief revelation is a common property of the utility functions. To be precise, Theorem 1 establishes that generic continuous utility functions have the property that optimal actions reveal beliefs if and only if the set of actions is perfect (has no isolated points).[5]

To say that actions reveal beliefs is somewhat counter-intuitive. The contribution of the strict proper scoring rule literature in statistics[6] is to show that one can construct payoff functions such that risk-neutral agents indeed reveal their posterior belief through their action. The action space of such proper scoring rules equals the space of probability measures and the functions itself have a very particular structure. In contrast, our theorem states the following result: If the action set is sufficiently "dense", then all continuous utility functions, but for a negligible set, have the property that optimal actions reveal private beliefs.

One direct implication of Theorem 1 is the following: any continuous function that fails belief revelation can be arbitrarily approximated by a sequence of continuous functions which satisfy belief revelation. This gives rise to an alternative interpretation of the implication of the result. Based on the existing literature, belief revelation has to be considered as a rather rare phenomenon and hence failure of belief revelation as common. However, the theorem implies that failure of belief revelation is not robust as every neighborhood of a function failing the property contains functions that satisfy belief revelation.

Our second result concerning belief revelation in the space of continuous func-

[5]Genericity refers to the standard notion of topological genericity, i.e. residual set, the complement of a meager set. Meagerness is the standard notion for negligible sets in topological spaces and is widely used in economic theory. For examples see Chen and Xiong [12], Dekel, Fudenberg and Morris [16], Dekel and Feinberg [15], Ely and Peski [18], and Reny and Perry [34].

[6]For example, Brier [11], de Finetti [14], Gneiting and Raftery [21], Good [22], and Savage [36].

tions, Theorem 2, establishes that an uncountable action space is sufficient for the existence of a continuous utility function that satisfies belief revelation.

Economic models typically work with smooth functions. The set of smooth functions is a negligible subset of the space of continuous utility functions and as such our main theorem does not allow for any inference regarding the belief revelation properties of smooth functions. However, we establish the following additional result concerning approximate belief revelation: for any smooth function and for every $\epsilon, \delta$ there exists a smooth function within distance $\delta$ that reveals beliefs up to precision $\epsilon$ (Theorem 3).

The generic belief revelation result is derived in a single-agent setting and does not directly imply results in multi-agent settings. However, it serves as a powerful tool with which one can derive learning results in multi-agent settings, in particular the framework of repeated interaction in social networks and the sequential social learning model.

## 1.1 Implications for Repeated Interaction in Social Networks

The theoretical literature on repeated interaction of privately informed agents in social networks is mostly concerned with the long run properties of aggregate behavior. That is, under which conditions does rational learning lead to consensus, indifference or disagreement among the actions chosen.[7]

We consider the following model. A finite set of agents are organized in a strongly connected social network. The agents share a common prior over the set of states of the world and each agent has private information represented by a countable partition of the state space. The timing of the repeated interaction game is as follows. The state of the world is realized and agents observe their corresponding private partition cell. Thereafter, in each of countable periods all agents simultaneously select an action. The stage utility of an agent depends only on the stage action and the realized state of the world. The stream of stage

[7]See Gale and Kariv [19], Rosenberg, Solan and Vieille [35], and Mueller-Frank [29].

utilities is discounted. We allow agents to be fully strategic.

The existing literature establishes that asymptotically a local indifference result holds, that is one's neighbors limit action is optimal conditional on one's own limit information. However, the literature has so far neglected the information aggregation properties of the equilibria in the general model.

We consider two different concepts of information aggregation. Asymptotic information aggregation holds if the limit action of each agent maximizes his expected utility conditional on the pooled private information of all agents. A stronger concept is that of perfect information aggregation which is satisfied if agents learn along the shortest path. That is, if the period $t$ action of every agent maximizes his expected utility conditional on the pooled private information of all agents within distance of $t-1$.

We consider the space of continuous utility functions and allow the utility functions to differ across agents. In a repeated environment strategic considerations play a role as each agent faces a trade-off in every round of interaction between selecting a stage utility maximizing action or trying to optimize future information gains. The strategic aspect is the main complication when going from a single-agent to a multi-agent repeated interaction setting.

We provide two results. First, we show that for generic utility function tuples asymptotic information aggregation holds in any Perfect Bayesian equilibrium if the action set is perfect (Theorem 4). This result is established using our main theorem together with a result from Rosenberg, Solan and Vieille [35] which states that limit actions are best replies to limit beliefs in any Perfect Bayesian equilibrium. Second, we use Theorem 1 to show that for generic utility function tuples there exists a Perfect Bayesian equilibrium that satisfies perfect information aggregation if the set of actions is perfect (Theorem 5).

To the best of our knowledge, our results are the first to establish perfect and/or asymptotic information aggregation for general games of repeated interaction in social networks. Only the special case of quadratic loss utility functions has been analyzed so far: Mueller-Frank [29] shows that perfect information aggregation holds for generic priors in any strongly connected network.[8] We show that information aggregation is perfect for all but a negligible set of utility functions in any strongly connected network.

## 1.2 Implications for Sequential Social Learning

In the sequential social learning model going back to Banerjee [7] and Bikhchandani, Hirshleifer and Welch [9], a countable set of fully rational agents make a one-time, irreversible choice in a predetermined sequence. The utility of each agent depends only on his action and the (unknown) realized state of the world. Every agent receives a conditionally independent, identically distributed and informative private signal, and observes the history of choices of his predecessors.

So far, the literature on sequential social learning has mostly focused on a binary state space, finite action (in most cases binary) setting. The learning benchmark typically applied is that of asymptotic learning, in the sense of the optimal action being chosen asymptotically with probability one. Smith and Sorensen [37] show that asymptotic learning occurs if and only if private signals are unbounded.[9] However, from an economic perspective not only the asymptotic properties of the equilibrium of a game are relevant but also properties of the equilibrium path.

We consider a concept of learning along the equilibrium path: $n$-perfect learning is satisfied if, with probability one, the action agent $n$ selects equals the expected utility maximizing action conditioning on the realized signals of agent $n$ and all his predecessors. Learning is perfect, if learning is $n$-perfect for all agents $n$.[10] Hence, in any setting where an infinite sequence of signals allows an observer to learn the realized state of the world, perfect learning is a considerably stronger concept than asymptotic learning.

Our final result, Theorem 6, shows that if the set of actions is compact metrizable and perfect, and private signals are independent conditional on the realized

[8] In a strongly connected network there exists a directed path among every pair of agents.

[9] Under unbounded signals their informativeness is unbounded.

[10] Lee [27] also considers the concept of perfect learning in a setting with a strict proper scoring rule utility function.

state, then perfect learning holds generically in the space of continuous utility functions. In other words, Theorem 6 states that under conditional independent signals and a perfect action set all but a negligible set of continuous utility functions lead to perfect learning as the only equilibrium outcome. This theorem provides a broad perspective of the sequential social learning literature. In a general environment the properties of the action not the signal space determine the possibility of perfect learning. Failure of perfect learning, and hence the occurrence of informational cascades, is a negligible phenomenon if the action set is sufficiently "dense".

The rest of the paper is organized as follows. Section 2 introduces the single-agent model and our general results on single-agent belief revelation. Section 3 considers approximate belief revelation in the space of smooth functions. Section 4 analyzes the framework of repeated interaction in social networks and provides our results on asymptotic and perfect information aggregation in social networks. Section 5 considers the implications of the single-agent result to the sequential social learning model and presents the result of generic perfect learning. Section 6 concludes. All proofs are relegated to the appendix.

# 2 Perfect Revelation in a Single-Agent Setting

Let us start by defining a single-agent model of decision making under uncertainty.

**Definition 1.** A single-agent model of decision making under uncertainty is described by the following objects:

- $\Omega$- a compact metrizable set of states of the world.
- $A$- a compact metrizable action space.
- $u : \Omega \times A \to \mathbb{R}$- a continuous utility function.
- $\mu \in \Delta(\Omega)$- a probability distribution over the state space $\Omega$ that represents the agent's belief.

The model above describes a general framework of uncertainty that is common in many fields of economic theory. The agent has a belief that is represented by a probability distribution $\mu$ over the set of states of the world. Based on the belief $\mu$ the agent chooses an action $a \in A$.

Our formulation captures the standard approaches of modeling private information. For example, the belief $\mu$ might represent the posterior distribution on the state space conditional on realized private information. The private information might be a realized partition cell of the agent, as in the literature on knowledge and consensus, or we can think of it as a signal generated according to a certain probability distribution over a signal space conditional on the realized state of the world, which, for example, is the common formulation in the sequential social learning literature.

For any belief $\mu \in \Delta(\Omega)$ and a utility function $u : \Omega \times A \to \mathbb{R}$ we let $\mathrm{br}_u(\mu)$ be the set of actions that maximizes the agent's expected utility given $\mu$. That is,

$$\mathrm{br}_u(\mu) = \left\{ a \in A : a \in \arg\max_{b \in A} \int_\Omega u(\omega, b)\mu(\mathrm{d}\omega) \right\}.$$

We say that the agent is *rational* if he chooses an action $a \in \mathrm{br}_u(\mu)$. A rational agent always maximizes his expected utility with respect to his belief. We shall assume throughout that the agent is rational.

Consider an outside observer who is ignorant in regards to the way that the agent makes his decisions or forms his beliefs. That is, the observer has no knowledge of the agent's underlying informational structure that gives rise to his belief. In particular, we do not impose a common prior on the state space. The outside observer knows the state space, the agent's action space and his utility function. We shall assume that the observer sees the action that the agent takes and that he considers the agent to be rational. That is, after observing the action of the agent the observer interprets it as a best-reply action that maximizes the agent's expected payoff. The objective of the outside observer is to predict the agent's belief based upon the action the agent selects.

A *prediction function* assigns a conditional probability measure on the state space to each action in $A$.

**Definition 2.** Let $B \subset A$ be the set of actions which are a best-reply to some belief $\mu \in \Delta(\Omega)$. That is,

$$B = \{a \in A : \exists\ \mu \in \Delta(\Omega) \text{ s.t., } a \in \mathrm{br}_u(\mu)\}.$$

A *prediction function* is a mapping $Q : A \to \Delta(\Omega)$ such that $a \in \mathrm{br}_u(Q(a))$ for every $a \in B$.

Conditional on observing a best-reply action $a$ the observer predicts a belief $Q(a) = \mu \in \Delta(\Omega)$ such that $a$ is optimal with respect to $\mu$. In general, for a given action $a'$ there might be an uncountable set of beliefs, such that $a'$ maximizes the expected utility conditional on each of the beliefs.

The objective of this paper is to gain a deeper understanding of the phenomenon of actions revealing beliefs. Our main goal is to establish underlying conditions on the environment of decision making under uncertainty such that the observer can perfectly infer, or predict, the agents posterior distribution from his action.

**Definition 3.** The payoff function $u : \Omega \times A \to \mathbb{R}$ satisfies *perfect revelation* if for any belief $\mu \in \Delta(\Omega)$ with a corresponding optimal action $a \in \mathrm{br}_u(\mu)$ we have $Q(a) = \mu$ for every prediction function $Q : A \to \Delta(\Omega)$.

That is, perfect revelation occurs whenever any best-reply action of the agent fully reveals his belief over the state of the world.

**Example 1.** To gain intuition consider the setting where the set of states of the world is binary $\Omega = \{0, 1\}$. The action set of the player is the unit interval $[0, 1]$ and his payoff is the squared loss utility function, $u(a, \omega) = -(a - \omega)^2$. Given any belief $\mu \in \Delta(\{0, 1\})$ note that the optimal action of the player is the action $a$ that is equal to $\mu(\{1\})$, his probability of the true state being state 1. As a result there exists a unique prediction function for the outside observer such that $Q(a) = \mu_a$ where $\mu_a(1) = a$. Hence in this particular case the observer gets to learn the belief of the agent over the true state of the world. Set instead $\Omega = [0, 1]$, and assume that the agents posterior distribution is $\nu \in \Delta([0, 1])$. In that case the optimal action of the agent is $a = \int_{[0,1]} x\nu(\mathrm{d}x)$. Hence the action of the agent only reveals the mean of the agent's distribution on $[0,1]$. In this case there exists no prediction function that predicts the belief of the agent. $\square$

Example 1 demonstrates that in some settings it is possible to carefully design the utility function such that actions fully reveal beliefs. And, that in general it is not clear under which conditions perfect revelation is possible and how common it is for a utility function to satisfy the perfect revelation property.

**Example 2.** The role of this example is to further highlight the difficulty of inferring beliefs from actions. Let the set of states of the world be given by the unit square $[0,1]^2 \subset \mathbb{R}^2$ and the set of actions by the unit interval. Consider the space of continuous real-valued utility functions defined on $[0,1] \times [0,1]^2$. Suppose that the agent knows the true state of the world. In this case the belief of the agent is simply $1_\omega$, the probability measure that assigns probability 1 to the realized state $\omega \in [0,1]^2$. This belief space is homeomorphic to the unit square $[0,1]^2$. Hence in particular for perfect revelation to be satisfied for a given utility function, it needs to generate an argmax which is injective from the square to the interval. While both have the same cardinality, in general injections from higher to lower dimensional spaces require careful construction. This suggests that a sufficient condition for perfect revelation might be hard to achieve. $\square$

In general we have the following observation.

**Observation 1.** *Consider a utility function $u$. In order for perfect revelation to hold it is necessary and sufficient that $\mathrm{br}_u(\mu) \cap \mathrm{br}_u(\nu) = \varnothing$ for every two distinct beliefs $\mu, \nu \in \Delta(\Omega)$.*

The sufficiency of the above conditions is clear. If every two different beliefs are associated with a disjoint set of best-reply actions then the set of beliefs forms a partition over the set of optimal actions. Hence the observer gets to learn the agents posterior distribution from his action. To see the necessity, assume that a given action $a$ lies in $\mathrm{br}_u(\mu) \cap \mathrm{br}_u(\nu)$ for two distinct $\mu, \nu \in \Delta(\Omega)$. In that case the action $a$ does not reveal enough information for the observer in order to determine precisely the belief of the agent. Hence perfect revelation fails.

The two above examples and the observation suggest that perfect revelation might be a too strong condition to be satisfied. In contrast, we shall show that if the action space is uncountable then there exists a continuous utility function that satisfies perfect revelation. Moreover, failure of perfect revelation is a negligible phenomenon under only mild conditions on the action space.

The action space $A$ is *perfect* if it contains no isolated point. A point $a \in A$ is *isolated* if it is an open set. Since $A$ is a metrizable space, a point $a$ is isolated if there exists an open ball around $a$ that comprises $a$ uniquely.[11] This property provides a certain "denseness" condition on the action space.

Consider the space $C(\Omega \times A)$ of all continuous utility functions endowed with the sup-norm topology. A generic property is one that holds everywhere but on a negligible set. In a topological space, as is $C(\Omega \times A)$, a property holds generically if it holds on a *residual set*, the complement of a *meager* set, which can be represented by a countable union of nowhere dense sets (sets whose closure has an empty interior).[12] Meagerness is the standard notion for negligible sets in topological spaces and is widely used in economic theory.[13]

Our first main result provides a necessary and sufficient condition for perfect revelation to hold as a generic property.

**Theorem 1.** *Consider a compact metrizable set of actions $A$. Perfect revelation is satisfied generically in the space of continuous utility functions $C(\Omega \times A)$ if and only if the set of actions $A$ is perfect.*

In regards to example 2 our main theorem demonstrates that it is not only possible to achieve perfect revelation in general environments, but for a perfect

[11]In the real line for example perfect sets goes beyond just intervals and union of intervals, the cantor set for example is a perfect set.

[12]To use the concept of a meager set as a notion of smallness makes only sense in a Baire space where not both a set and its complement can be meager. As $C(\Omega \times A)$ endowed with the sup-norm is a Banach space, by the Baire category theorem $C(\Omega \times A)$ is a Baire space.

[13]See for example, Chen and Xiong [12], Dekel, Fudenberg and Morris [16], Dekel and Feinberg [15], Ely and Peski [18], and Reny and Perry [34].

action space perfect revelation is satisfied for all but a negligible set of continuous utility functions. In order to get some perspective on the result, let us discuss it in context of the scoring rule literature.[14] A strict proper scoring rule is a payoff function under which the unique expected payoff maximizing action is equal to the decision makers probability of some uncertain event, or more generally his distribution on the state space. Strict proper scoring rules are used to entice agents to reveal their true probabilistic beliefs and they play an important role in probability assessment. The literature has characterized conditions under which a scoring rule is strictly proper, and moreover shown that strict proper scoring rules can take different functional forms, for example quadratic and logarithmic.

However, based on the existing literature on scoring rules these are considered as something very specific, carefully designed and by no means as a common, or even generic property.[15] Our result however, establishes that "almost all" continuous payoff functions satisfy perfect revelation if the set of actions satisfies a "denseness" property. In relation to the scoring rule literature it is important to emphasize the following distinction. Under strict proper scoring rules the optimal action is in fact equal to the belief which makes the inferences straight-forward. In contrast, perfect revelation only says that the utility function induces an injective mapping from beliefs to actions.

Note that meagerness is the standard notion for a negligible set in topological spaces but not the only such notion.[16] Sidestepping the different notions of genericity in general spaces, there is an interesting direct implication of Theorem 1. As the space of continuous functions on $\Omega \times A$ is a Baire space a residual set is dense. This implies the following corollary. Recall that the sup-norm distance between two continuous functions $u, v \in C(\Omega \times A)$ is equal to

$$\|u - v\|_\infty = \max_{(a,\omega)\in\Omega\times A} |u(\omega, a) - v(\omega, a)| .$$

[14]For an overview of the main points of the scoring rule literature see Gneiting and Raftery [21].

[15]For example, see Gneiting and Raftery [21].

[16]An alternative notion is shyness, see Anderson and Zame [4].

**Corollary 1.** *Consider a perfect compact metrizable set of actions $A$. For every continuous utility function $v \in C(\Omega \times A)$, and $\epsilon > 0$ there exists a continuous utility function $u \in C(\Omega \times A)$ that satisfies perfect revelation such that $\|u - v\|_\infty < \epsilon$.*

The corollary states that any continuous utility function which fails perfect revelation can be arbitrarily approximated by continuous functions satisfying perfect revelation. Therefore, failure of perfect revelation is not a robust phenomenon.

The first theorem provides necessary and sufficient conditions for perfect revelation to hold as a generic property. However, in context of the scoring rule literature, it seems important to understand under which conditions there exists a continuous utility function that satisfies perfect revelation. Our second main result provides a sufficient condition for such existence.

**Theorem 2.** *If the action space $A$ is compact metrizable and uncountable, then there exists a continuous utility function $u : A \times \Omega \to \mathbb{R}$ such that perfect revelation is satisfied.*

One might summarize the first two theorems informally as follows. Richness of the action space in a strong sense, containing no isolated points, is necessary and sufficient for perfect revelation to be satisfied for generic continuous utility functions. Richness of the action space in a weaker sense, it being uncountable, is sufficient for the existence of a utility function that satisfies perfect revelation.[17]

In the following subsection we provide an outline of the proof of the theorems.

## 2.1 Outline of the Proof of the Main results

The sufficiency part of Theorem 1 readily follows from the following proposition and Observation 1.

**Proposition 1.** *There exists a generic set of utility functions $\mathcal{U} \subset C(A \times \Omega)$ such that for every $u \in \mathcal{U}$ and for every two different beliefs $\mu, \nu \in \Delta(\Omega)$ it holds that $\mathrm{br}_u(\mu) \cap \mathrm{br}_u(\nu) = \varnothing$.*

[17] A perfect set in a metrizable space is uncountable.

The proof of the proposition consists of two parts. In the first part, we consider a pair of disjoint convex closed subsets $U, V \subset \Delta(\Omega)$ of the space of distributions over the set of states of the world. We show that given that the action space is perfect, the following statement holds for a generic set of payoff functions: There is no action $a \in A$ that is both a best reply for a belief that lies in $U$ and a different belief that lies in $V$. That is, the best reply actions generically separate between these two sets.

Given the above, we provide a countable collection of disjoint pairs $\{(U_n, V_n)\}_n \subset \Delta(\Omega) \times \Delta(\Omega)$ that separate each pair of distinct beliefs $\mu, \nu \in \Delta(\Omega)$. That is for every two distinct $\mu, \nu \in \Delta(\Omega)$ there exists $n$ such that $\mu \in U_n$ and $\nu \in V_n$. If we apply the previous statement to each and every pair in this collection, and take the intersection of the corresponding sets we get a generic set $\mathcal{U}$ of utilities for which $\mathrm{br}_u : \Delta(\Omega) \to A$ induces a partition for every element $u \in \mathcal{U}$.

For the necessity part we show that for every action set containing at least one isolated point $b$, there exists an open set of utility functions for which $b$ is the only optimal action given every belief $\mu \in \Delta(\Omega)$.

The proof of Theorem 2 relies on the Cantor Bendixson theorem which states that every uncountable compact metric space can be decomposed uniquely as a disjoint union of a countable set and a perfect set. We then use Theorem 1 to construct a continuous utility function on the perfect subset of the action space that satisfies the perfect revelation property. We extend it to a continuous function over the whole product space of actions and states such that any action outside of the perfect set is strictly dominated by some action in the perfect set. It then follows that this extension has the perfect revelation property.

# 3 Approximate Revelation for Smooth Functions

An important class of function in economics is the class of smooth functions. Although this class is a negligible (meager) subset in the space of all continuous functions, typically functions used in economic models are in fact smooth. While our result sheds new light on the commonness of perfect revelation in the space of continuous functions, the implication to smooth functions is not clear. In this section we study *approximate belief revelation* properties of smooth functions.

Let $\Omega \subset \mathbb{R}^m$ be a compact set of states of the world, and let $A \subset \mathbb{R}^n$ be a compact and perfect set of actions. For any metric $d$ over $\Delta(\Omega)$ that induces the weak* topology let $\mathrm{B}_\epsilon(\mu)$, be the closed ball around $\mu$ with a radius $\epsilon$. Choose any such metric $d$ with the additional property that every closed ball with respect to $d$ is a convex set (the Prokhorov metric is an example of such a metric, see Billingsley [10]). Our definition of $\epsilon$-belief revelation is the following.

**Definition 4.** A function $u : \Omega \times A \to \mathbb{R}$ satisfies $\epsilon$-belief revelation if for every $\mu \in \Delta(\Omega)$ and for every prediction function $Q : A \to \Delta(\Omega)$ it holds that,

$$d(\mu, Q(a)) \leq \epsilon,$$

for every $a \in \mathrm{br}_u(\mu)$.

$\epsilon$-belief revelation is satisfied if and only if for every $\mu \in \Delta(\Omega)$ and for every $a \in \mathrm{br}_u(\mu)$ we have

$$\{\nu : a \in \mathrm{br}_u(\nu)\} \subset \mathrm{B}_\epsilon(\mu).$$

This in turn implies that any rational action taken by the agent reveals his belief up to a precision of $\epsilon$.

Call a function $u : \Omega \times A \to \mathbb{R}$ smooth if it has a smooth extension to an open set in $\mathbb{R}^m \times \mathbb{R}^n$ that contains $\Omega \times A$.

**Definition 5.** A smooth function $u : \Omega \times A \to \mathbb{R}$ satisfies *approximate belief revelation* if for every $\epsilon, \delta > 0$ there exists another smooth function $v$ such that $\|v - u\|_\infty \leq \delta$ and $v$ satisfies $\epsilon$-belief revelation.

We have the following belief revelation result for smooth functions.

**Theorem 3.** *Every smooth function $u : \Omega \times A \to \mathbb{R}$ satisfies approximate belief revelation.*

Smooth functions may or may not satisfy belief revelation. While belief revelation might not be a generic property of smooth functions, we establish that any smooth function can be approximated by smooth functions that satisfy $\epsilon$-belief revelation, for any $\epsilon > 0$.

# 4 Repeated Interaction in Networks

Social networks are an important factor for the diffusion of information, opinions and behavior. This role of social networks as a conduit of information, behavior and opinions has been documented empirically and experimentally.[18] The theoretical literature analyzing observational learning under repeated interaction in social networks mainly focuses on the following framework. A finite set of rational agents are embedded in a social structure. Each agent initially receives private information. Thereafter, in each of countable rounds every agent selects an action. All agents share the same utility function with the utility depending only the state of the world and the action (no payoff externalities). Gale and Kariv [19], Mueller-Frank [29] and Rosenberg, Solan and Vieille [35] analyze versions of this framework differing mainly in their degree of generality and assumptions on myopic versus strategic behavior of agents (see also Mossel, Sly and Tamuz [28]). However, a common result holds: in the long run, or once learning ends, a local indifference property is satisfied. That is, if two neighbors select different actions infinitely often, then they are indifferent among their choices.

Mueller-Frank [29] considers a special case where the stage utility maximizing action of each agent equals the posterior probability[19] of some uncertain event thereby building upon Aumann [6], and Geanakoplos and Polemarchakis [20]. He establishes that for generic common priors information aggregation is perfect and occurs along the shortest path in any strongly connected network.[20] Beyond this special case the literature on repeated interaction in networks focused on the distribution of aggregate and local behavior in the long run as opposed to information aggregation. Our goal in this subsection is to derive information aggregation results in a general network interaction game.

Let us consider the following model. A finite set of agents $V$ is organized in a

[18] For examples see Munshi [31], Banerjee, Chandrasekhar, Duflo and Jackson [8] and Choi, Gale and Kariv [13].

[19] This corresponds to squared loss stage utility functions.

[20] A network is strongly connected if from any given agent every other agent can be reached by a directed path.

strongly connected network $G = (V, E)$. The agents face uncertainty represented by a probability space $(\Omega, \mathcal{F}, \lambda)$ where $\Omega$ is compact metrizable and $\lambda$ is a common prior probability measure. Keeping with the tradition of the literature on knowledge and consensus, we represent the private information of agents via partitions. That is, each agent $i$ is endowed with a countable partition $\mathcal{P}_i$ of the state space $\Omega$. The set of partitions of all players $\{\mathcal{P}_i\}_{i \in V}$ is commonly known. We assume that the sigma algebra $\mathcal{F}$ is generated by the join of partitions of all agents, $\bigvee_{j \in V} \mathcal{P}_j$.[21] Time is discrete.

At the beginning of time, $t = 0$, one state $\omega$ is realized and each agent observes the realized cell of his partitions. Thereafter, in each of countable rounds of interaction all players simultaneously select an action out of a compact metrizable set of actions $A$. The network determines the observability of actions: agents observe the history of actions of their neighbors. The stage utility of each agent $i$ is determined by a (across agents possibly distinct) continuous utility function $u_i : A \times \Omega \to \mathbb{R}$. The stage-$t$ utility of an agent depends only on his stage action $a_i^t$ and the realized state of the world. The total utility of an agent is given by the discounted sum of stage utilities with discount factor $\delta \in (0, 1)$.

Let $h_i^t$ denote the history of actions (including his own) that agent $i$ observed prior to selecting his action in round $t$. The information set of agent $i$ in round $t$ is given by her realized partition cell $\mathcal{P}_i(\omega)$ and the history $h_i^t$ of actions of her neighbors $j \in N_i$,

$$I_i^t = \left\{\mathcal{P}_i(\omega), h_i^t\right\}.$$

The set of all possible information sets of agent $i$ is denoted by $\mathcal{I}_i$ and ranges over all possible partition cells, histories $h_i^t$ and stages $t \in \mathbb{N}$. A strategy of agent $i$ is a measurable function that assigns a mixed action to each possible information set, $\sigma_i : \mathcal{I}_i \to \Delta(A)$. The strategy profile $\langle \sigma_i \rangle_{i \in V}$ is denoted by $\sigma$. Let $\mathbf{a}^t$ denote the vector of actions chosen in period $t$. Given a strategy profile $\sigma$, the sequence of action vectors $\langle \mathbf{a}^t \rangle_{t \in \mathbb{N}}$ is a stochastic process with probability measure $\Pr_\sigma$. We focus on the Perfect Bayesian equilibria of the network interaction game

[21] The join of a set of partitions is their coarsest common refinement.

therefore allowing for fully strategic agents. This approach differs from much of the literature on repeated interaction which restricts agents to act myopically in the sense of maximizing the expected stage utility in each round. Myopia is typically imposed due to the additional complications that are associated with strategic behavior.[22]

When considering the information aggregation properties of the network interaction game we focus on the following concepts.

**Definition 6.** A strategy profile $\sigma$ satisfies *asymptotic information aggregation*, if for all agent $i \in V$

$$\Pr_\sigma\left(\lim_{t\to\infty} E\left[u_i(a_i^t,\omega)|I_i^t\right] = \max_{a\in A} E[u_i(a,\omega)|\bigvee_{j\in V}\mathcal{P}_j]\right) = 1.$$

Under asymptotic information aggregation the expected payoffs of all agents converge to the pooled information expected utility with probability one.

Let $d_G(i,j)$ denote the length of the shortest directed path from $i$ to $j$ in network $G$. The optimality benchmark for learning in the network interaction game is denoted as perfect information aggregation.

**Definition 7.** A strategy profile $\sigma$ satisfies *perfect information aggregation*, if for all agents $i \in V$ and rounds $t \in \mathbb{N}$

$$\Pr_\sigma\left(\sigma_i(\mathcal{P}_i(\omega), h_i^t) \in \arg\max_{a\in A} E[u_i(a,\omega)| \bigvee_{j:d_G(i,j)\leq t-1} \mathcal{P}_j]\right) = 1.$$

Under perfect information aggregation each agent learns the private information of all other agents along the shortest path. That is, at the beginning of stage $t$ agent $i$ knows the realized partition cell of each agent within network distance of $t-1$. Since we consider only finite networks, all agents select the action that maximizes their respective expected utility conditional on the pooled private information of all agents from round $t = \mathrm{diam}(G)+1$ onward, where the diameter, $\mathrm{diam}(G)$, equals the greatest distance between any two agents in network $G$.

[22]See for example Geanakoplos and Polemarchakis [20], Parikh and Krasucki [33] and Gale and Kariv [20]. Rosenberg, Solan and Vieille [35] on the other hand allow for strategic behavior in a general repeated interaction environment.

The existing literature on information aggregation in social networks focuses on squared loss utility functions, see for example Acemoglu, Bimpikis and Ozdaglar [1], and Mueller-Frank [29, 30]. Instead, we take a more general approach. We consider the product space of continuous utility functions $C(A \times \Omega)^V$ endowed with the product topology. A typical element $\mathbf{u} \in C(A \times \Omega)^V$ consists of the tuple $\langle u_i \rangle_{i \in V}$ of utility functions across all agents. We assume that $\mathbf{u}$ is common knowledge.

**Theorem 4.** *Let the partition of each agent be countable and the network strongly connected. If the set of actions $A$ is perfect, then there exists a generic subset of $C(A \times \Omega)^V$ such that for each of its utility function tuples $\mathbf{u}$ asymptotic information aggregation is satisfied in every Perfect Bayesian equilibrium.*

It is important to note that the network interaction game is very different from the sequential social learning setting to be considered in the next section. This is due to the fact that agents interact repeatedly. Whenever the discount factor is positive agents face a trade-off in each stage between maximizing their stage utility and maximizing the future information gain. In the sequential social learning setting such strategic considerations play no role as each agent acts only once. Our Theorem 4 establishes that the strategic component does not affect long run information aggregation. For all but a negligible set of utility function tuples, each agent, in any strongly connected network, converges to the optimal action conditional on the pooled private information of all agents if the set of actions has no isolated points. Theorem 4 complements the existing asymptotic local indifference result established in the literature by providing sufficient conditions for long run actions to perfectly aggregate the dispersed private information. The proof of Theorem 4 makes use of a result by Rosenberg, Solan and Vieille [35] which states that the limit action of an agent is optimal conditional on his limit information in any Perfect Bayesian equilibrium, together with Theorem 1. See the appendix for details.

While it is important to know that in the long run information perfectly spreads in the network, Theorem 4 says nothing about the equilibrium path. A natural question is under which conditions perfect information aggregation is achievable.

**Theorem 5.** *Let the partition of each agent be countable and the network strongly connected. If the set of actions $A$ is perfect, then there exists a generic subset of $C(A \times \Omega)^V$ such that for each of its utility function tuples $\mathbf{u}$ there exists a Perfect Bayesian equilibrium that satisfies perfect information aggregation.*

The theorem states that for any strongly connected network there exists an equilibrium in which each agent, within finite time, selects an action which is optimal conditional on the pooled private information of all agents, if the set of actions has no isolated points. This is the case despite the fact that the history of actions is not common knowledge in incomplete networks. In other words, Theorem 5 shows that even under lack of common knowledge of the history Bayesian learning leaves nothing to be desired. Agents learn along the shortest path for all but a negligible set of utility function tuples.

In context of the literature on rational learning in networks there are two features of our results that are worthwhile to mention. First, ours are the first results on properties of the evolution of actions in social networks that do not rely on agents being homogeneous in the sense of sharing the same utility function. Second, our results are the first to establish information aggregation properties in a general environment of repeated interaction in social networks. Theorem 5 establishes that the perfect information aggregation result of Mueller-Frank [29] is not an artefact of squared loss utility functions applied there but in fact holds for generic utility functions if the set of actions is perfect.

The proof of Theorem 5 is omitted. The result follows from the fact that in a partitional information setting posterior distributions reveal information sets. That is, there is an injective mapping from the set of possible information sets to posterior distributions. By Theorem 1 there is an injective mapping from posterior distributions to actions. Jointly they imply that the stage utility maximizing action reveals the private information of the agent. Finally, selecting this myopically optimal action is a Perfect Bayesian equilibrium because the myopic action maximizes the expected stage utility and learning cannot be improved upon by deviating.

## 5 Sequential Social Learning

In this section we consider information aggregation in the sequential social learning model which was first introduced by Banerjee [7] and Bikhchandani, Hirshleifer and Welch [9]. The existing literature focuses on the following standard framework: binary states, binary actions (equalling the states), and conditionally independent, identically distributed signals. The standard utility function considered has the property that a utility of one is achieved if the action matches the realized state and zero otherwise. A fundamental result in this setting due to Smith and Sorensen [37] is that asymptotic learning, the correct action being chosen with probability one asymptotically, occurs if and only if private signals are unbounded.[23]

We analyze the following general model. A countable infinite set of agents denoted by $N$ faces uncertainty regarding the realized state of the world. Let the state space $\Omega$ be compact metrizable and $\lambda \in \Delta(\Omega)$ be the common prior distribution on the state space. The timing of the game is the following. In round $t = 0$ the state of the world is drawn according to $\lambda$. In every round exactly one agent selects an irreversible action $a \in A$, where $A$ is a compact metrizable space. The utility of every agent $n$ depends only on his action and the state of the world, $u_n : A \times \Omega \to \mathbb{R}$. The utility functions are assumed to be continuous and, as typical in the literature, identical for all agents, $u_n = u$ for all $n$.

Agents do not know the realized state but each agent $n$ observes a private signal $s_n$ belonging to a measurable signal space $S$. The signal $s_n$ is drawn according to the following state-dependent distribution,

$$F_n : \Omega \to \Delta(S).$$

Signals are conditionally independent across agents. In addition to the signal, each agent $n$ observes the history of actions $h_n$ of all his predecessors. A strategy $\sigma_n$ of agent $n$ is a measurable mapping that assigns an action to each possible

[23] Under unbounded signals their informativeness is unbounded.

information set, $\sigma_n : A^{n-1} \times S \to A$. As common in the literature we solve the game for its pure strategy Perfect Bayesian equilibria, the set of strategy profiles $\langle \sigma_n \rangle_{n \in N}$ such that each $\sigma_n$ maximizes the expected utility of agent $n$ given the strategies of all other agents.

Our analysis departs from the existing literature in two ways. First, we consider a more general model. Second, our focus lies on a stronger concept of learning. While it is important to understand under which conditions asymptotic learning occurs, the properties of the equilibrium path are of importance as well, mainly for two reasons. First, from an individual agents perspective asymptotic outcomes are irrelevant. Agent $n$ cannot aspire to perfectly know the state of the world as only a finite set of signals are realized, those of his predecessors and himself, when agent $n$ selects his action. From an informational perspective, the optimal action of agent $n$ equals the action that maximizes his expected utility conditioning on the set of realized signals. Second, the expected welfare varies with the speed of convergence to the correct action.[24]

The focus of this section lies on the information aggregation properties of the equilibrium path in the general sequential social learning model. In particular, we are interested in $n$-perfect and perfect learning.

**Definition 8.** A strategy profile $\sigma$ satisfies *n-perfect learning*, if

$$\Pr{}_{\sigma}\left(\sigma_n(h_n, s_n) \in \arg\max_{a \in A} E\left[u(a,\omega)\,|s_1,...,s_n\right]\right) = 1.$$

The strategy profile $\sigma$ satisfies *perfect learning*, if $\sigma$ satisfies $n$-perfect learning for all $n \in N$.

In the sequential social learning model, the signal realizations are private. Informally, in a perfect learning equilibrium each agent acts as if he had observed the signals of his predecessors rather than their actions. Note that if the infinite sequence of signals reveals the true state with probability one, then perfect learning is a substantially stronger concept than asymptotic learning.

[24] For more details on the advantages of the concept of perfect rather than asymptotic learning see the earlier working paper "Generic Outcomes of Observational Learning", pages 7-8.

Lee [27] addresses the question of perfect learning in a specific setting. He considers the sequential social learning setting with finite states, binary signals, real valued actions and a quadratic loss utility function that induces agents to reveal their signal through their optimal action. Lee [27] shows that asymptotic learning occurs if and only if the action space contains a closed interval whose endpoints depend on the state dependent signal distributions.

Our objective in regards to the sequential social learning setting is to identify sufficient conditions on the utility functions and environment such that perfect learning occurs. Theorem 1 plays an important role towards establishing the following result.

**Theorem 6.** *If the action space is perfect and private signals conditional independent across agents, then there exists a generic subset of $C(A \times \Omega)$ for which any Perfect Bayesian equilibrium satisfies perfect learning.*

Theorem 6 provides a broad perspective of the sequential social learning literature. In a general environment, i.e. class of continuous utility functions and only minimal conditions on the state, signal and action space, the properties of the action not the signal space determine the possibility of perfect learning. Failure of perfect learning is a negligible phenomenon if the action set is sufficiently rich. Note that an equilibrium satisfying perfect learning is both individually optimal for each agent as well as socially optimal from a planners perspective. This contrasts the equilibrium properties in the standard framework where both individual optimality and social efficiency fail to be satisfied.[25] The main difference to Lee [27] can be summarized with the following sentence. While he considers a specific utility function and setting, and shows that perfect learning holds if the action set contains a closed interval, we show that in general environments a general richness property of the action space is sufficient such that perfect learning occurs for generic utility functions.

Our result complements the existing literature. The standard binary state, binary action setting is an important special case and has many interesting appli-

[25]See Smith and Sorensen [38] for the failure of social efficiency in the standard framework.

cations. The contribution of Theorem 6 is to point out that in settings where a rich action space is a better description of reality than a discrete or binary one, perfect learning is the only generic equilibrium outcome.

We take a two step approach towards proving Theorem 6. First, consider a given utility function and the correspondence from posterior distributions on the state space to expected utility maximizing actions. Note that any given agent $n$ uses the information available to him in order to update his posterior probability distribution on the state space $\Omega$. He then selects an expected utility maximizing action for the given posterior distribution. Formally, given a profile of strategies $\sigma$ let $f_n^\sigma(h_n, s_n) = \Pr_\sigma(\mathrm{d}\omega \,|h_n, s_n)$ be the conditional distribution of player $n$ over the state space given the history of actions he observed $h_n \in A^{n-1}$, and his private signal $s_n \in S$.[26]

A strategy profile $\sigma$ is an equilibrium if for each player $n$:

$$\sigma_n(h_n, s_n) \in \arg\max_{a \in A} \int_\Omega u(a, \omega) \mathrm{d} f_n^\sigma(h_n, s).$$

The above shows that observing actions allows for inferences on private signals through inferences on posterior distributions on the state space. The following proposition establishes a sufficient condition for perfect learning.

**Proposition 2.** *Let the private signals be conditionally independent across agents. If* $\Pr_\sigma(\mathrm{d}\omega \,|h_n, \sigma_n(h_n, s)) = f_n^\sigma(h_n, s)$ *holds almost surely for all* $n$*, then*

$$f_{n+1}^\sigma(h_{n+1}, s_{n+1}) = \Pr(\mathrm{d}\omega \,|s_1, ..., s_{n+1}) \tag{1}$$

*almost surely for all* $n$*. Hence, in particular, perfect learning holds.*

The proposition tells us that for perfect learning it is enough for the action to reveal the agent's posterior distribution over the state space as opposed to the action revealing the agent's private signal. The sufficiency of actions revealing

[26]Note that the conditional posterior distribution $f_n^\sigma(h_n, s_n)$ is well defined by Dudley [17] Theorem 10.2.2.

beliefs crucially depends on the assumption of conditional independent signals.[27] For the proof of Theorem 6 see the appendix.

# 6 Conclusion

To what extend does an agents action reveal his private information? An agent might have no incentive to communicate his private information to an outside observer. When facing a decision problem under uncertainty where his choice impacts his utility, however, he will act optimally given his private information. After all, actions speak louder than words. In such a decision problem under uncertainty the private information of an agent is relevant only in as much as it generates a probability distribution on the state space. This paper is concerned with the question under which conditions on a general environment an agents optimal action serves as a sufficient statistic for his private belief. In other words, under which conditions do actions reveal beliefs?

Our main result, Theorem 1, is quite counter-intuitive seen in context of the scoring rule literature. We show that if and only if the action set contains no isolated points, then all but a negligible set of continuous utility functions have the desired property: actions reveal private beliefs. This directly implies that any continuous function failing belief revelation can be arbitrarily approximated by continuous functions satisfying belief revelation (Corollary 1). Theorem 2 establishes that if the set of actions is uncountable (not necessarily perfect) then there exists a continuous utility function such that actions reveal beliefs. We then consider belief revelation in the space of smooth functions and show that any smooth function satisfies approximate belief revelation.

Our single-agent belief revelation result has direct implication for two prominent multi-agent settings of observational learning. For observational learning under repeated interaction in social networks let the private information of agents

[27]For an example providing intuition for the importance of the conditional independence assumption see the earlier working paper "Generic Outcomes of Observational Learning", pages 9-10.

be represented by countable partitions and allow the utility functions to differ across agents. We provide the following two results. First, if the set of actions is perfect and the network is strongly connected, then there exists a generic set of utility function tuples such that asymptotic information aggregation is satisfied in any Perfect Bayesian equilibrium. Second, if the set of actions contains no isolated points and the network is strongly connected, then there exists a generic set of continuous utility function tuples such that there exists a Perfect Bayesian equilibrium that satisfies perfect information aggregation. That is, at the beginning of round $t$ each agent knows the private information of all agents within network distance $t-1$.

We consider a generalized version of the sequential social learning setting and a stronger learning concept than the standard one of asymptotic learning. Under perfect learning each agent acts as if he observed the private signals of his predecessors rather than their actions. We show that if signals are conditionally independent across agents and the set of actions contains no isolated points, then for generic continuous utility functions every Perfect Bayesian equilibrium satisfies perfect learning.

The main conclusion of our paper is that under the Bayesian paradigm observational learning leads to perfect aggregation of all for utility purposes relevant dispersed private information, if the set of actions is sufficiently "dense".

# A Appendix

## A.1 Proof of Theorem 1

For a belief $\mu \in \Delta(\Omega)$ and action $a \in A$ let $u(\mu, a)$ be the agents expected payoff if his belief is represented by $\mu$ and he plays action $a$. That is,

$$u(\mu, a) := \int_\Omega u(\omega, a)\mu(\mathrm{d}\omega).$$

For any belief $\mu \in \Delta(\Omega)$ recall that $\text{br}_u(\mu) \subset A$ is the set of best-reply actions to the belief $\mu$ and for the utility $u$. That is,

$$\text{br}_u(\mu) = \{a \in A : u(\mu, a) \geq u(\mu, b) \ \forall b \in A\}.$$

For a closed $V \subset \Delta(\Omega)$ and a utility function $u \in C(\Omega \times A)$ similarly let $\text{br}_u(V)$ be the set of actions which are best reply to some belief $\mu \in V$.

#### A.1.1 Proof of the Proposition 1

Let $K : C(\Omega \times A) \times \Delta(\Omega) \twoheadrightarrow A$ be a set valued map defined as follows:

$$K(u, \mu) = \text{br}_u(\mu).$$

We endow $\Delta(\Omega)$ with the weak* topology.

**Lemma 1.** *$K$ has a closed graph.*

*Proof.* Let $\{(u_n, \mu_n, a_n)\}_n \subset C(\Omega \times A) \times \Delta(\Omega) \times A$ be a sequence that converges to $(u, \mu, a) \in C(\Omega \times A) \times \Delta(\Omega) \times A$ such that $a_n \in K(u_n, \mu_n) = \text{br}_{u_n}(\mu_n)$. We need to show that $a \in K(u, \mu) = \text{br}_u(\mu)$. We shall show first that $u_n(a_n, \mu_n) \to_{n \to \infty} u(a, \mu)$. Note first that:

$$\begin{aligned}
& \lim_n |u_n(\mu_n, a_n) - u(\mu, a)| \\
\leq \ & \limsup_n |u_n(\mu_n, a_n) - u(\mu_n, a_n)| && (2) \\
+ \ & \limsup_n |u(\mu_n, a_n) - u(\mu_n, a)| && (3) \\
+ \ & \limsup_n |u_n(\mu_n, a) - u(\mu, a)| && (4)
\end{aligned}$$

By definition $u_n \to u$ according to the sup norm. That is for every $\epsilon$ there exists $N$ such that for all $n > N$,

$$\sup_{a \in A, \omega \in \Omega} |u(\omega, a) - u_n(\omega, a)| < \epsilon.$$

Hence the expression in equation 2 converges to 0.

Since $u$ is a continuous function on a compact set it is uniformly continuous. In particular if we let $d_A$ be a metric of $A$ then for every $\epsilon$ there exists a $\delta$ such that if $d_A(a, a') < \delta$ then $|u(\omega, a) - u(\omega, a')| < \epsilon$ for every $\omega \in \Omega$. Hence the expression in equation 3 converges to 0.

Note that for a fixed $a \in A$ the function $u(a, \cdot) : \Omega \to \mathbb{R}$ is continuous. Therefore the expression in equation 4 converges to 0 by definition of the weak* topology. Hence $\lim_n u_n(\mu_n, a_n) = u(\mu, a)$. Therefore for every action $b \in A$ we have,

$$\begin{aligned} u(\mu, b) &= \lim_n u_n(\mu_n, b) \qquad (5) \\ &\leq \lim_n u_n(\mu_n, a_n) \qquad (6) \\ &= u(\mu, a). \qquad (7) \end{aligned}$$

Equation 5 follows from the preceding calculation. Equation 6 follows since

$$a_n \in \mathrm{br}_{u_n}(\mu_n).$$

And equation 7 again follows from the preceding calculation. Therefore by definition $a \in \mathrm{br}_u(\mu)$. □

**Lemma 2.** *Let $V, W \subset \Delta(\Omega)$ be two convex closed disjoint subsets. The set of utility functions $E \subset C(A \times \Omega)$ such that $\mathrm{br}_u(V) \cap \mathrm{br}_u(W) \neq \varnothing$ is closed and nowhere dense.*

*Proof.* We shall show first that $E$ is a closed set. Let $\{u_n\}_{n=1}^{\infty} \subset E$ be a sequence of function within $E$ that converges to the function $u$, we show that $u \in E$. Since for every $n$ we have $u_n \in E$, by definition we can find an action $a_n$ and probability distributions $\mu_n \in V$ and $\nu_n \in W$ such that $a_n \in \mathrm{br}_{u_n}(\mu_n)$ and $a_n \in \mathrm{br}_{u_n}(\nu_n)$. Since both the action set $A$ and $\Delta(\Omega)$ are compact sets by taking subsequences we can assume that $a_n \to a$, $\mu_n \to \mu$ and $\nu_n \to \nu$. Since both $V$ and $W$ are closed sets one has $\mu \in V$ and $\nu \in W$. As a consequence of Lemma 1 we deduce that $a \in \mathrm{br}_u(\mu)$ and $a \in \mathrm{br}_u(\nu)$. Hence $u \in E$, and $E$ is a closed set.

We shall show next that $E$ is nowhere dense, that is every $u \in E$ can be approximated by a function outside $E$. Let $\epsilon > 0$. We shall establish that there exists a function $v \in E^c$ such that $\|u - v\| < \epsilon$. From the fact that $V$ and $W$ are convex, weakly compact, and disjoint we have, by the Hahn Banach separation theorem, a weakly continuous linear functional that strictly separates $V$ and $W$. That is, there exists a weakly continuous linear functional $\mathrm{g}$ over $\Delta(\Omega)$ such that for every $\mu \in V$ and $\nu \in W$,

$$\mathrm{g}(\mu) > r > -r > \mathrm{g}(\nu)$$

for some $0 < r$.

By Theorem 5.93 in Aliprantis and Border, there exists a continuous function $g : \Omega \to \mathbb{R}$ such that for every $\mu \in \Delta(\Omega)$,

$$\mathrm{g}(\mu) = \int g(\omega) \mathrm{d}\mu(\omega).$$

By dividing $g$ with a constant if necessary one can assume that $\|g\|_\infty < \epsilon$ and that there exists $0 < c < 1$ such that for every $\mu \in V$ and $\nu \in W$,

$$\int_\Omega g(\omega) \mathrm{d}\mu(\omega) > c \cdot \epsilon > -c \cdot \epsilon > \int_\Omega g(\omega) \mathrm{d}\nu(\omega). \tag{8}$$

From compactness and perfectness of $A$ there exist disjoint finite sets $A_1 = \{a_1, \dots, a_n\}$ and $A_2 = \{a_{n+1}, \dots, a_{n+m}\}$ such that for every $\mu \in V$ there exists $a \in A_1$ such that

$$u(\mu, a) + c \cdot \epsilon > \max_{b \in A} u(\mu, b). \tag{9}$$

And similarly for $W$ with respect to $A_2$. Let $\gamma$ be small enough such that, for every $a \in A_1 \cup A_2$, $\mathrm{B}_{2\gamma}(a) \cap (A_1 \cup A_2) = a$. For each $a \in A_1 \cup A_2$ let $v_a : A \to [0, 1]$ be a continuous function such that $v_a(a) = 1$ and $v_a(b) = 0$ for $b \notin \mathrm{B}_\gamma(a)$. The existence of such function is guaranteed by Urysohn's lemma.

Let

$$v(\omega, a) = u(\omega, a) + g(\omega)\Big(\sum_{k=1}^{n} v_{a_k}(a) - \sum_{k=1}^{m} v_{a_{n+k}}(a)\Big).$$

Since for $i \neq j$ the functions $v_{a_i}$ and $v_{a_j}$ have a disjoint support and since $\|g\|_\infty < \epsilon$, one has that $\|u - v\|_\infty < \epsilon$. Moreover, one can deduce from equations (8) and (9) that for every $\mu \in V$ there exists $a \in A_1$ such that,

$$v(\mu, a) > u(\mu, a) + c\epsilon > \max_{b \in A} u(\mu, b).$$

On the other hand for every $b \notin (A_1)^\gamma$ it holds that,[28]

$$v(\mu, b) \leq u(\mu, b) \leq \max_{b \in A} u(\mu, a).$$

As a result we have that, for every $\mu \in V$,

$$\text{br}_v(\mu) \subset (A_1)^\gamma.$$

Similarly for every $\nu \in W$,

$$\text{argmax}_a v(a, \nu) \subset (A_2)^\gamma.$$

Since by construction $(A_1)^\gamma$ and $(A_2)^\gamma$ are disjoint sets we get that $v \in E^c$. This completes the proof of the lemma. $\square$

*Proof of Proposition 1.* Since the space $\Delta(\Omega)$ is a compact convex set in a locally convex vector space, there exists a countable base, $\{W_n\}_n$,[29] of closed convex sets for $\Delta(\Omega)$. Let $C$ be a collection of all pairs of sets from the base $\{W_n\}_n$ which has an empty intersection. That is,

$$C = \{(U, V) : U = W_m \text{ for some } m,\ V = W_l \text{ for some } l,\ \text{and } W_m \cap W_l = \emptyset\}.$$

We note that $C$ is a countable collection of sets. Therefore, we can write

$$C = \{(U_k, V_k)\}_k.$$

We let $E_k$ be the set of all utilities $u \in C(\Omega \times A)$ for which $\text{br}_u(U_k) \cap \text{br}_u(V_k) \neq \emptyset$. By Lemma 2 the set $E_k$ is nowhere dense. We set $\mathcal{U} = C(\Omega \times A) \setminus \{\cup_k E_k\}$. $\mathcal{U}$ is therefore a generic set.

Let $\mu, \nu \in \Delta(\Omega)$ be two distinct probability measures. We note that since $\{W_n\}_n$ is a base there exists $(U_k, V_k) \in C$ such that $\mu \in U_k$ and $\nu \in V_k$. Since $u \notin E_k$ by definition $\text{br}_u(U_k) \cap \text{br}_u(V_k) = \varnothing$. Hence in particular $\text{br}_u(\mu) \cap \text{br}_u(\nu) = \varnothing$. That completes the proof of the proposition. $\square$

[28] Recall that $(A_1)^\gamma = \{a \in A : \exists a_i \in A_1 \text{ s.t. } d_A(a, a_i) \leq \gamma\}$.

[29] To be more precise each $W_n$ corresponds to the closure of an open convex base element.

### A.1.2 Failure of Perfect Revelation

In order to conclude the proof Theorem 1 we have left to show that perfect revelation fails when the action space of the agent has an isolated point. We shall show that if $b \in A$ is an isolated point of $A$ and $\Omega$ contains at least two elements then there exists a continuous utility function $u : \Omega \times A \to \mathbb{R}$ such that perfect revelation fails for every continuous utility function $v$ for which

$$\|u - v\|_\infty = \max_{\omega \in \Omega, a \in A} |u(\omega, a) - v(\omega, a)| < \frac{1}{2}.$$

Define $u$ as follows:

$$u(\omega, a) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases} \tag{10}$$

$u$ is a continuous function since $b$ is an isolated point. Moreover, $\mathrm{br}_u(\mu) = b$ for every $\mu \in \Delta(\Omega)$. Let $v$ be such that $\|u - v\|_\infty < \frac{1}{2}$, it must hold that for every $a \neq b$ and $\omega \in \Omega$ that

$$v(\omega, a) < \frac{1}{2} < v(\omega, b).$$

Hence $\mathrm{br}_v(\mu) = b$ for every $\mu \in \Delta(\Omega)$. Therefore in the presence of an isolated point perfect revelation fails in a strong sense as the observer cannot infer anything about the agent underlying belief for an open set of utility functions. $\square$

## A.2 Proof of Theorem 2

Let $A$ be an uncountable set. By the Cantor Bendixson Theorem (Theorem 6.4 in Kechris [25]) $A$ can be uniquely decomposed as $A = K \cup B$ where $K$ is a compact perfect set and $B$ is a countable set. Let $d_A$ be a metric on $A$ that induces its topology. For every $r > 0$ we let $K^r$ be the set of points that lie within a distance of at most $r$ from the set $K$. For every interval $[c, f]$ such that $0 < c < f$ let $K^{[c,f]}$ be the set of points that lie in a distance of at least $c$ but no more than $f$ from the set $K$.

**Lemma 3.** *For every $0 < r_2 < r_1$ there exists $r$ such that $r_2 < r < r_1$ and $\delta > 0$ such that,*

$$K^{[r-\delta, r+\delta]} = \varnothing. \tag{11}$$

*Proof of Lemma 3.* Assume by contradiction that the lemma does not hold for some $0 < r_2 < r_1$. Then $K^{[r-\delta,r+\delta]} \neq \varnothing$ for every $r_2 < r < r_1$ and for every $0 < \delta \leq r$. The set $K^{[r-\delta,r+\delta]}$ is a closed set and therefore compact. Hence the intersection $\cap_{0<\delta\leq r} K^{[r-\delta,r+\delta]}$ is non empty and comprises of all points that lie in a distance of *precisely* $r$ from the set $K$. In particular for every $r_2 < r < r_1$ we can find $a_r \notin K$ that lies in a distance $r$ from $K$. That is a contradiction to the fact that $B$ is a countable set. $\square$

By Theorem 1 one can find a utility function $v : \Omega \times K \to \mathbb{R}$ that satisfies the perfect revelation property. We shall show that one can extend $v$ to a function $u : \Omega \times A \to \mathbb{R}$ such that every action $b \in A \setminus K$ is strictly dominated by some action $a \in K$. It readily follows that such $u$ has the desired properties since for every $\mu \in \Delta(\Omega)$ one has that $\mathrm{br}_u(\mu) = \mathrm{br}_v(\mu)$.

By the Tietze extension theorem one can find a continuous function,

$$g : \Omega \times A \to \mathbb{R}$$

that coincide with $v$ on $\Omega \times K$. Since the function $g$ is uniformly continuous (as a continuous function on a compact domain) for each $n \geq 1$ there exists $\beta_n$ such that if $a, b \in A$ and $d_A(a, b) < \beta_n$ then $\forall \omega \in \Omega$ $|g(\omega, a) - g(\omega, b)| < \frac{1}{n}$. One can assume that $\{\beta_n\}_n$ is a strictly decreasing sequence that converges to 0. By Lemma 3 we can find a strictly decreasing sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers that converges to zero such that each $r_n$ satisfies the property listed in equation (11) and $\beta_{n+1} < r_n < \beta_n$. Let $M > \sup_{\omega,a} |g(\omega, a)|$. Define

$$u(\omega, a) = \begin{cases} -M & \text{if } a \notin K^{r_1} \\ g(\omega, a) - \frac{1}{n} & \text{if } a \in K^{[r_{n+1}, r_n]} \\ g(\omega, a) & \text{if } a \in K. \end{cases}$$

Note that $u$ is well defined by the definition of $r_n$ and equation 11. In addition for all $\omega \in \Omega$, and $a \in K$, $u(a, \omega) = g(\omega, a) = v(\omega, a)$. To see that $u$ is continuous let $(\omega_k, a_k) \to (\omega, a)$ be a convergent sequence. Assume that $a \in K^{[r_{n+1}, r_n]}$ for some $n > 0$. By definition of $r_{n+1}$ and $r_n$ one can find a small enough $\delta > 0$ such that

$b \in K^{[r_{n+1}, r_n]}$ if and only if $b \in K^{[r_{n+1}+\delta, r_n-\delta]}$. Hence there must exist $k_0$ such that for every $k > k_0$, we have $a_k \in K^{[r_{n+1}, r_n]}$. In that case by the continuity of $g$,

$$\lim_k u(\omega_k, a_k) = \lim_k g(\omega_k, a_k) - \frac{1}{n} = g(\omega, a) - \frac{1}{n} = u(\omega, a).$$

Similar argument applies when $a \notin K^{r_1}$. If $a \in K$ then $\lim_k u(\omega_k, a_k) = u(\omega, a)$ by definition of $u$ and the continuity of $g$.

The only property left to be shown is that every $b \notin K$ is strictly dominated by some $a \in K$. For $b \notin K_{r_1}$ this is clear. Let $b \in K^{[r_{n+1}, r_n]}$. By definition there exists $a \in K$ such that $d_A(a, b) \leq r_n$. Since $r_n < \beta_n$ we have by definition of $\beta_n$ that for all $\omega \in \Omega$, $|g(\omega, a) - g(\omega, b)| < \frac{1}{n}$. Hence by definition of $u$,

$$\forall \omega \in \Omega,\ u(\omega, a) = g(\omega, a) > g(\omega, b) - \frac{1}{n} = u(\omega, b).$$

In particular we have that $u(\mu, a) > u(\mu, b)$ for every $\mu \in \Delta(\Omega)$, therefore $b$ can never be a best-reply action. $\square$

## A.3 Proof of Theorem 3

The proof of the theorem is in fact a corollary of the proof of our main theorem.

*Proof of Theorem 3.* Fix $\epsilon$ and $\delta$. By Theorem 1 there exists a function

$$w : \Omega \times A \to \mathbb{R},$$

that satisfies perfect revelation and lies within $\delta$ of the function $u$. Let $\mathcal{B}$ be a cover of $\Delta(\Omega)$ with balls of radius $\frac{\epsilon}{4}$. By compactness of $\Delta(\Omega)$ we can assume that $\mathcal{B}$ is finite. Let $C$ be the set of all disjoint pairs from $\mathcal{B}$. That is,

$$C = \{(U, V) : U, V \in \mathcal{B} \text{ and } U \cap V = \varnothing\}.$$

For any pair $(U, V) \in C$ we let $S_{(U,V)}$ be the set of all utility functions that separates the beliefs of $U$ and $V$. That is, the set of all functions $y : \Omega \times A \to \mathbb{R}$ for which $\mathrm{br}_y(U) \cap \mathrm{br}_y(V) = \varnothing$. It follows from Lemma 2 that the set $S_{(U,V)}$ is open and dense. Hence the set,

$$S = \bigcap_{(U,V) \in C} S_{(U,V)},$$

is open dense and contains the function $w$. We shall show that every function in $S$ satisfies $\epsilon$-approximate revelation. Let $y \in S$ and let $\mu \in \Delta(\Omega)$. There exists a set $V$ such that $\mu \in V$. Let $T = \{U \in \mathcal{B} : U \cap V = \varnothing\}$. By definition of $S$, we must have that, $\mathrm{br}_y(V) \cap \mathrm{br}_y(U) = \varnothing$ for every $U \in T$. Hence $\mathrm{br}_y(\mu) \cap \mathrm{br}_y(\nu) = \varnothing$ for every $\nu \in L := \cup_{U \in T} U$. As $\mathcal{B}$ is a cover of $\Delta(\Omega)$ of balls with radius $\frac{\epsilon}{4}$ the complement of $L$ must be contained in $\mathrm{B}_\epsilon(\mu)$, the closed ball of radius $\epsilon$ around $\mu$. Therefore if $a \in \mathrm{br}_y(\mu) \cap \mathrm{br}_y(\nu)$ then $\nu \in \mathrm{B}_\epsilon$. This demonstrates $\epsilon$-approximate revelation.

Since $S$ is an open set, and the set of smooth functions is dense in the space of all continuous functions we can find a function $v$ with a distance of at most $\delta$ of $w$ that lies in $S$. $v$ is a smooth function that satisfies $\epsilon$-approximate revelation. $\square$

## A.4 Proof of Theorem 4

We shall assume that each player has a utility function in the set $\mathcal{U}$ determined by Theorem 1.

**First Step:** We shall show that each player reveals his partition element at infinity to all of his neighbors in any equilibrium $\sigma$.

As in Rosenberg et. al. [35], we let $A_i^*$ be the random set of limit actions player $i$ is playing. That is the compact set of all accumulation points to the random sequence of actions $(a_i^t)_{t \in \mathbb{N}}$ that player $i$ takes along the equilibrium path. We shall show that the partition element of each player $i$ is measurable with respect to $A_i^*$.

Let $i \in V$. For $P_i \in \mathcal{P}_i$ define

$$C(P_i) = conv\{\lambda(\mathrm{d}\omega | P_i \cap \hat{P}) : \hat{P} \in \bigvee_{j \in N_i \setminus i} \mathcal{P}_j\}.^{30} \tag{12}$$

Given that $P_i \in \mathcal{P}_i$ is the chosen partition element of player $i$, the set $C(P_i) \subset \Delta(\Omega)$ contains the set of all beliefs player $i$ might have along any equilibrium path of any equilibrium $\sigma$. That is, for any equilibrium strategy $\sigma$ one can verify that

[30] For a set $A$ let $conv(A)$ be the convex hull of $A$.

conditional on $P_i$ being chosen it is always the case that

$$\forall t \ \Pr_\sigma(\mathrm{d}\omega|I_i^t) \in C(P_i).$$

This follows from the fact that $(\Pr_\sigma(\mathrm{d}\omega|I_i^t))_{t\in\mathbb{N}}$ is a belief martingale. In addition, for every $P_i \in \mathcal{P}_i$ the set $C(P_i)$ is a closed set as a convex combination of finitely many elements. Note that for every two distinct partition elements $P_i, P_i' \in \mathcal{P}_i$, $C(P_i) \cap C(P_i') = \varnothing$. To see this note that for any element $\mu_i \in C(P_i)$ it holds that $\mu_i(P_i) = 1$ and $\mu_i(P_i') = 0$.

We let $A(P_i) \subset A$ be the compact set of best responses of player $i$ to a belief that lies in $C(P_i)$.

$$A(P_i) = \{a \in A : a \in \mathrm{br}_{u_i}(\mu_i),\ \mu_i \in C(P_i)\}.$$

We know from Theorem 1 that $A(P_i) \cap A(P_i') = \varnothing$ for every player $i$ and any two different partition elements $P_i, P_i' \in \mathcal{P}_i$.

It follows from Proposition 2.1. of Rosenberg et. al. [35] that in every equilibrium $\sigma$ any limit action is a best response to the limit belief of player $i$. That is, let $q_i^\infty$ be player $i$'s limit belief. Proposition 2.1. states that

$$\Pr_\sigma(A_i^* \subset \mathrm{br}_{u_i}(q_i^\infty)) = 1. \tag{13}$$

Hence in particular it follows that

$$\Pr_\sigma(A_i^* \subset A(P_i)) = 1.$$ [31]

Therefore, the partition of player $i$ is measurable with respect to $A_i^*$. As a result at infinity the sequence of actions of player $i$ reveals his partition element to all of his neighbors. Therefore for each player $i$, and an equilibrium $\sigma$ the limit information $I_i^\infty$ contains his partition element and the partition element of all his neighbors.

**The inductive step:**

We let $N_i^k$ be the set of players within distance of $k$ from player $i$ including $i$. Assume that at equilibrium each player's limit information $I_i^\infty$ contains the join

[31] We slightly abuse notation and treat $A(P_i)$ as a random set that corresponds to the chosen partition element $P_i \in \mathcal{P}_i$

element generated by the partitions of all players in $N_i^{k-1}$. Similarly to step 1 we shall show that each element in $\bigvee_{j\in N_i^{k-1}} \mathcal{P}_i$ is measurable with respect to agent $i$'s limit action at equilibrium. We then deduce that the limit information of each player $i$ contains $\bigvee_{j\in N_i^k} \mathcal{P}_i$. For each join element $P_i = (P_j)_{j\in N_i^{k-1}} \in \bigvee_{j\in N_i^{k-1}} \mathcal{P}_j$ define

$$C(P_i) = conv\{\lambda(\mathrm{d}\omega|P_i \cap \hat{P}) : \hat{P} \in \bigvee_{j\in N_i^k \setminus N_i^{k-1}} \mathcal{P}_j\}.$$

As in step 1 for every player $i$ and two distinct elements $P_i, P_i' \in \bigvee_{j\in N_i^{k-1}} \mathcal{P}_j$ it must hold that $C(P_i) \cap C(P_i') = \varnothing$. We let $A(P_i)$ be the compact set of best responses of player $i$ to a belief in $C(P_i)$. Since $u_i \in \mathcal{V}$ by Proposition 1 it holds that $A(P_i) \cap A(P_i') = \varnothing$ for any two distinct elements $P_i, P_i' \in \bigvee_{j\in N_i^{k-1}} \mathcal{P}_j$.

It follows from the induction hypothesis that for every player $i$, and an equilibrium strategy $\sigma$ the limit belief of player $i$ lies in $C(P_i)$ that is,

$$\Pr_\sigma(q_i^\infty \in C(P_i)) = 1.$$

Again it follows from Proposition 2.1 that $\Pr_\sigma(A_i^* \subset \mathrm{br}_{u_i}(q_i^\infty)) = 1$. In particular we have that

$$\Pr_\sigma(A_i^* \subset A(P_i)) = 1.$$

Hence the limit action of player $i$ reveals $P_i$. Moreover, for each agent $j \in N_i \setminus i$ of $i$'s neighbors the partition $\bigvee_{l\in N_j^{k-1}} \mathcal{P}_l$ is measurable with respect to $A_j^*$. Hence, at infinity agent $i$ knows $\bigvee_{l\in N_j^{k-1}} \mathcal{P}_l$. Thus in equilibrium the limit information that is revealed to player $i$ is precisely $\bigvee_{j\in N_i^k} \mathcal{P}_j$. This completes the inductive step and the proof of the theorem. $\square$

## A.5 Proof of Theorem 6

### A.5.1 Proof of Proposition 2

We prove the proposition by induction. The proposition trivially holds for the first agent. Assume it holds up to the $n$th agent. We let $\bar{s}_n = (s_1, \ldots, s_n)$ and as before $h_{n+1} = (\sigma_1(s_1), \ldots, \sigma_n(s_n, h_n))$. Note that by the assumption of Proposition 2 we have

$$\Pr_\sigma(\mathrm{d}\omega|h_n, \sigma_n(h_n, s_n)) = f_n^\sigma(h_n, s_n).$$

Equation (1) and the induction hypothesis for agent $n$ yields,

$$f_n^\sigma(h_n, s_n) = \mathrm{Pr}_\sigma(\mathrm{d}\omega|\overline{s}_n).$$

Together we get the following equality:

$$\mathrm{Pr}_\sigma(\mathrm{d}\omega|h_{n+1}) = \mathrm{Pr}_\sigma(\mathrm{d}\omega|\overline{s}_n). \tag{14}$$

**Lemma 4.** *Under the above conditions* $\mathrm{Pr}_\sigma(s_{n+1}|\overline{s}_n) = \mathrm{Pr}_\sigma(s_{n+1}|h_{n+1})$.

*Proof.* Let $\phi$ be the distribution on the signal spaces $S_1 \times \ldots \times S_n$ generated by $\lambda$ and the functions $F_1, \ldots, F_n$. In order to prove the lemma it is sufficient to show that for every subset $T \subset S_1 \times \ldots \times S_n$ and $R \subset S_{n+1}$:

$$\int_T \phi(\mathrm{d}\overline{s}_n) \int_R \mathrm{Pr}_\sigma(\mathrm{d}s_{n+1}|\overline{s}_n) = \int_T \phi(\mathrm{d}\overline{s}_n)) \int_R \mathrm{Pr}_\sigma(\mathrm{d}s_{n+1}|h_{n+1}).$$

To see this, for any sets $T, R$ as above we have,

$$\begin{aligned}
& \int_T \phi(\mathrm{d}\overline{s}_n) \int_R \mathrm{Pr}_\sigma(\mathrm{d}s_{n+1}|\overline{s}_n) \\
= & \int_T \phi(\mathrm{d}\overline{s}_n) \int_\Omega \mathrm{Pr}_\sigma(\mathrm{d}\omega|\overline{s}_n) \int_R \mathrm{Pr}_\sigma(\mathrm{d}s_{n+1}|\omega, \overline{s}_n) && (15) \\
= & \int_T \phi(\mathrm{d}\overline{s}_n) \int_\Omega \mathrm{Pr}_\sigma(\mathrm{d}\omega|h_{n+1}) \int_R \mathrm{Pr}_\sigma(\mathrm{d}s_{n+1}|\omega, h_{n+1}) && (16) \\
= & \int_T \phi(\mathrm{d}\overline{s}_n) \int_R \mathrm{Pr}_\sigma(\mathrm{d}s_{n+1}|h_{n+1}) && (17)
\end{aligned}$$

Equation (15) follows from the law of iterated expectation (or simply Bayes rule). Since conditional on the state of the world the signal $s_{n+1}$ is independent of the other signals $\overline{s}_n$, and the history $h_{n+1}$, we can deduce that,

$$\mathrm{Pr}_\sigma(\mathrm{d}s_{n+1}|\omega, \overline{s}_n) = \mathrm{Pr}_\sigma(\mathrm{d}s_{n+1}|\omega) = \mathrm{Pr}_\sigma(\mathrm{d}s_{n+1}|h_{n+1}, \omega). \tag{18}$$

This and Equation (14) demonstrates Equation (16). Equation (17) follows again from the law of iterated expectation. □

***Proof of Proposition 2.*** We turn to the proof of the inductive step. Let $T, R$ be as above, and $B \subset \Omega$. Let $\psi$ be the prior measure over the set of signals $S_1 \times, \ldots, \times S_{n+1}$ we shall show that:

$$\int_{T \times R} \psi(\mathrm{d}\overline{s}_{n+1}) \int_B \mathrm{Pr}_\sigma(\mathrm{d}\omega|\overline{s}_{n+1}) = \int_{T \times R} \psi(\mathrm{d}\overline{s}_{n+1}) \int_B \mathrm{Pr}_\sigma(\mathrm{d}\omega|h_{n+1}, s_{n+1}),$$

which is equivalent to equation (1). We have the following sequence of equalities:

$$
\begin{aligned}
&\int_{T\times R} \psi(\mathrm{d}\overline{s}_{n+1}) \int_B \mathrm{Pr}_\sigma(\mathrm{d}\omega|\overline{s}_{n+1}) \\
=\;& \int_T \phi(\mathrm{d}\overline{s}_n) \int_B \mathrm{Pr}_\sigma(\mathrm{d}\omega|\overline{s}_n) \int_R \mathrm{Pr}_\sigma(\mathrm{d}s_{n+1}|\omega, \overline{s}_n) && (19)\\
=\;& \int_T \phi(\mathrm{d}\overline{s}_n) \int_B \mathrm{Pr}_\sigma(\mathrm{d}\omega|h_{n+1}) \int_R \mathrm{Pr}_\sigma(\mathrm{d}s_{n+1}|\omega, h_{n+1}) && (20)\\
=\;& \int_T \phi(\mathrm{d}\overline{s}_n) \int_R \mathrm{Pr}_\sigma(\mathrm{d}s_{n+1}|h_{n+1}) \int_B \mathrm{Pr}_\sigma(\mathrm{d}\omega|h_{n+1}, s_{n+1}) && (21)\\
=\;& \int_T \phi(\mathrm{d}\overline{s}_n) \int_R \mathrm{Pr}_\sigma(\mathrm{d}s_{n+1}|\overline{s}_n) \int_B \mathrm{Pr}_\sigma(\mathrm{d}\omega|h_{n+1}, s_{n+1}) && (22)\\
=\;& \int_{T\times R} \psi(\mathrm{d}\overline{s}_{n+1}) \int_B \mathrm{Pr}_\sigma(\mathrm{d}w|h_{n+1}, s_{n+1}) && (23)
\end{aligned}
$$

Equation (19) follows from Bayes rule. Equation (20) follows from equation (18) and Equation 14 which is the induction hypothesis. Equation (21) follows from Bayes rule. Equation (22) follows from the lemma. Equation (23) follows again be an application of Bayes rule.

□

### A.5.2 Proof of Theorem 6

We start by focusing on the first decision maker agent 1. Let, $f_1(s_1) = Pr(\mathrm{d}\omega|s_1) \in \Delta(\Omega)$ represent the agent's posterior belief over the set of states after he receives signal $s \in S$. Let $\psi$ be the prior distribution over the set of signals for agent 1 that is generated by $\lambda$ and $F_1$. Assume that $u \in \mathcal{U}$, the generic set of utilities that we obtain from Theorem 1. A strategy of agent 1, $\sigma_1 : S_1 \to A$ is an equilibrium strategy if and only if

$$\sigma_1(s_1) \in \mathrm{br}_u(f_1(s_1)) \text{ for } \psi \text{ almost every } s_1 \in S.$$

By Theorem 1 there exists unique prediction function $Q : A \to \Delta(\Omega)$ for $u$. By the previous condition and the definition of $Q$,

$$Q(\sigma_1(s_1)) = f_1(s_1) \text{ for } \psi \text{ almost every } s_1 \in S.$$

Therefore we can deduce that,

$$\mathrm{Pr}_{\sigma_1}(\mathrm{d}\omega|\sigma_1(s_1)) = Q(\sigma_1(s)) = f_1(s) \text{ for almost every } s \in S.$$

In particular, $\Pr_{\sigma_1}(\mathrm{d}\omega|\sigma_1(s)) = \Pr_{\sigma_1}(\mathrm{d}\omega|s_1)$.

By Proposition 2, we have $\Pr_{\sigma_1}(\mathrm{d}\omega|\sigma_1(s_1), s_2) = \Pr_{\sigma}(\mathrm{d}\omega|s_1, s_2)$ for almost every $s_2 \in S$.

We can now set the signal for agent 2 to be $\overline{s}_2 = (\sigma_1(s_1), s_2)$. One can apply similar consideration to those of agent 1 and have that for every equilibrium strategies of agents 1 and 2, $\sigma = (\sigma_1, \sigma_2)$

1. $\Pr_{\sigma}(\mathrm{d}\omega|\sigma_1(s_1)) = \Pr_{\sigma}(\mathrm{d}\omega|s_1) = Q(\sigma_1(s))$.

2. $\Pr_{\sigma}(\mathrm{d}\omega|\sigma_1(s), \sigma_2(h_2, s_2)) = \Pr_{\sigma}(\mathrm{d}\omega|s_1, s_2) = Q(\sigma_2(h_2, s_2))$.

We proceed by induction. For every agent $n$ and equilibrium strategies $(\sigma_1, \ldots, \sigma_n)$ we have by the characteristics of $u$ that,

$$\Pr_{\sigma}(\mathrm{d}\omega|\sigma_1(s_1), \ldots, \sigma_n(h_n, s_n)) = \Pr_{\sigma}(\mathrm{d}\omega|\sigma_1(s_1), \ldots, s_n).$$

Hence by Proposition 2:

$$\Pr_{\sigma}(\mathrm{d}\omega|\sigma_1(s_1), \ldots, \sigma_n(h_n, s_n)) = \Pr_{\sigma}(\mathrm{d}\omega|s_1, \ldots, s_n) = Q(\sigma_n(h_n, s_n)).$$

This concludes the proof of Theorem 6. □

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