On Stability of the MPCC Feasible Set

by:

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Abstract

The feasible set of mathematical programs with complementarity constraints (MPCC) is considered. We discuss local stability of the feasible set with respect to perturbations (up to first order) of the defining functions. Here, stability refers to homeomorphy invariance under small perturbations. For stability we propose a kind of Mangasarian-Fromovitz Condition (MFC) and its stronger version (SMFC). MFC is a natural Constraint Qualification for C-stationarity and SMFC is a generalization of the well-known Clarke’s maximal rank condition. It turns out that SMFC implies local stability. MFC and SMFC coincide in case that the number of complementarity constraints \( k \) equals to the dimension of the state space \( n \). Moreover, the equivalence of MFC and SMFC is also proven for the cases \( k = 2 \) as well as \( n \geq 2k \).

**Keywords**: Mathematical programs with complementarity constraints, Stability, (Strong) Mangasarian-Fromovitz Condition, Clarke’s Implicit Function Theorem.

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*Department of Mathematics – C, RWTH Aachen University, D-52056 Aachen, Germany, email: jongen@rwth-aachen.de

#School of Mathematics, The University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom, email: J.Ruckmann@bham.ac.uk, corresponding author

&Department of Mathematics – C, RWTH Aachen University, D-52056 Aachen, Germany, email: shikhman@mathc.rwth-aachen.de
1 Introduction

We consider the following mathematical programming problem with complementarity constraints (MPCC):

\[
\text{MPCC: } \min f(x) \text{ s.t. } x \in M[F_1, F_2]
\]

with

\[
M[F_1, F_2] := \{ x \in \mathbb{R}^n \mid F_1(x) \geq 0, F_2(x) \geq 0, F_1(x)^T F_2(x) = 0 \},
\]

where \( F_1 := (F_{1,i}, i = 1, \ldots, k)^T, F_2 := (F_{2,i}, i = 1, \ldots, k)^T \in C^1(\mathbb{R}^n, \mathbb{R}^k), \)
\( f \in C^1(\mathbb{R}^n, \mathbb{R}), k \leq n. \) Note that \( M[F_1, F_2] \) can be written as follows:

\[
M[F_1, F_2] = \{ x \in \mathbb{R}^n \mid \min \{ F_{1,i}(x), F_{2,i}(x) \} = 0, i = 1, \ldots, k \}.
\]

In this paper we deal with the local stability property of the feasible set \( M[F_1, F_2] \) with respect to \( C^1 \)-perturbations of the defining functions \( F_1 \) and \( F_2. \) Under \( C^1 \)-neighborhood of a function \( g \in C^1(\mathbb{R}^n, \mathbb{R}^l) \) we understand a subset of \( C^1(\mathbb{R}^n, \mathbb{R}^l), \) which contains for some \( \varepsilon > 0 \) the set

\[
\left\{ \tilde{g} \in C^1(\mathbb{R}^n, \mathbb{R}^l) \mid \sum_{i=1}^{l} \sup_{x \in \mathbb{R}^n} (|\tilde{g}_i(x) - g_i(x)| + \|\nabla \tilde{g}_i(x) - \nabla g_i(x)\|) \leq \varepsilon \right\}.
\]

Here, \( \| \cdot \| \) denotes the Euclidean norm on \( \mathbb{R}^n \) and \( \nabla g_i \) stands for the gradient of \( g_i \) as a column vector.

**Definition 1.1** The feasible set \( M[F_1, F_2] \) from (1) is called locally stable at \( \bar{x} \in M[F_1, F_2] \) if there exists a \( \mathbb{R}^n \)-neighborhood \( V \) of \( \bar{x} \) and a \( C^1 \)-neighborhood \( U \) of \( (F_1, F_2) \) in \( C^1(\mathbb{R}^n, \mathbb{R}^k) \times C^1(\mathbb{R}^n, \mathbb{R}^k) \) such that for every \( (\tilde{F}_1, \tilde{F}_2) \in U, \) the corresponding feasible set \( M[\tilde{F}_1, \tilde{F}_2] \cap V \) is homeomorphic with \( M[F_1, F_2] \cap V. \)

Our main goal is to characterize the local stability property of the feasible set \( M[F_1, F_2] \) in terms of the gradients of \( F_1 \) and \( F_2. \) In case of standard nonlinear programming (local) stability of the feasible set was studied in [5, 9]. In fact, for the feasible set

\[
M_{NLP}[h, g] := \{ x \in \mathbb{R}^n \mid h_i(x) = 0, i \in I, g_j(x) \geq 0, j \in J \}
\]
with \( h_i, g_j \in C^1(\mathbb{R}^n, \mathbb{R}), |I| < n, |J| < \infty \),

the local stability property at \( \bar{x} \in M_{NLP}[h, g] \) is characterized by the Mangasarian-Fromovitz Constraint Qualification (MFCQ), i.e.:

1. \( \nabla h_i(\bar{x}), i \in I \) are linearly independent,
2. there exists a \( \xi \in \mathbb{R}^n \) satisfying
   \[
   \nabla h_i(\bar{x})\xi = 0, \quad i \in I, \\
   \nabla g_j(\bar{x})\xi > 0, \quad j \in J_0(\bar{x}) := \{ j \in J \mid g_j(\bar{x}) = 0 \}.
   \]

Assume that the following Assumption A1 holds throughout the whole article.

**Assumption A1** For every \( \bar{x} \in M[F_1, F_2] \) and \( i \in \{1, \ldots, k\} \) the set of vectors \( \{\nabla F_{j,i}(\bar{x}) \mid F_{j,i}(\bar{x}) = 0, j = 1, 2\} \) is linearly independent.

Further, we define for \( \bar{x} \in M[F_1, F_2] \) and \( i = 1, \ldots, k \) the (non-empty) convex hull

\[
C_i(\bar{x}) := \text{conv}\{\nabla F_{j,i}(\bar{x}) \mid F_{j,i}(\bar{x}) = 0\}.
\]

**Definition 1.2 (MFC and SMFC)**

The Mangasarian-Fromovitz Condition (MFC) is said to hold at \( \bar{x} \in M[F_1, F_2] \) if any \( k \) vectors \( (w_1, \ldots, w_k) \in C_1(\bar{x}) \times \cdots \times C_k(\bar{x}) \) are linearly independent.

The Strong Mangasarian-Fromovitz Condition (SMFC) is said to hold at \( \bar{x} \in M[F_1, F_2] \) if there exists a \( k \)-dimensional linear subspace \( E \) of \( \mathbb{R}^n \) such that any \( k \) vectors \( (u_1, \ldots, u_k) \in P_E(C_1(\bar{x})) \times \cdots \times P_E(C_k(\bar{x})) \) are linearly independent, where \( P_E : \mathbb{R}^n \rightarrow E \) denotes the orthogonal projection.

The article is organized as follows. In Section 2 some relations between MFC, SMFC, Linear Independence Constraint Qualification (LICQ) (cf. [17]), Mordukhovich’s extremal principle and metric regularity are discussed. Several guiding examples in 2 and 3 dimensions are given.

In Section 3 we prove that SMFC implies local stability (cf. Theorem 3.2). Moreover, Theorem 3.3 shows that SMFC is equivalent to MFC for \( k = 2 \) as well as for \( n \geq 2k \).
Our notation is standard. The $n$-dimensional Euclidean space is denoted by $\mathbb{R}^n$, its nonpositive orthant by $\mathbb{R}^n_-$, a closed ball with radius $\varepsilon > 0$ and center $\bar{x} \in \mathbb{R}^n$ by $\bar{B}(\bar{x}, \varepsilon)$, the topological boundary of $B(0,1)$ by $\mathbb{S}^{n-1}$, the distance from $x \in \mathbb{R}^n$ to $K \subset \mathbb{R}^n$ by $\text{dist}(x,K) = \inf_{y \in K} \|x - y\|$ with the convention $\text{dist}(x,\emptyset) = \infty$. Given an arbitrary set $K \subset \mathbb{R}^n$, $\text{span}(K)$ denotes the set of all linear (convex, nonnegative) combinations of elements of $K$ respectively. The polar of $K$ is defined by $K^* := \{v \in \mathbb{R}^n \mid v^T w \leq 0 \text{ for all } w \in K\}$. $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^k$ denotes a multivalued map defined on $\mathbb{R}^n$ with $T(x) \subset \mathbb{R}^k$, $x \in \mathbb{R}^n$. The graph of $T$ is $\text{gph} \ T = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^k \mid y \in T(x)\}$ and the inverse of $T$ is $T^{-1} : \mathbb{R}^k \rightrightarrows \mathbb{R}^n$, given by $T^{-1}(y) = \{x \in \mathbb{R}^n \mid y \in T(x)\}$.

2 Conceptional relations and examples

To start with, we notice that MFC is a natural Constraint Qualification for the Clarke stationarity.

**Definition 2.1 (cf. [2, 17])** A point $\bar{x} \in M[F_1, F_2]$ is called Clarke stationary (C-stationary) if there exist real numbers $\lambda_{j,i}$, $j = 1, 2$, $i = 1, \ldots, k$ such that

$$
\nabla f(\bar{x}) + k \sum_{i=1}^k (\lambda_{1,i} \nabla F_{1,i}(\bar{x}) + \lambda_{2,i} \nabla F_{2,i}(\bar{x})) = 0,
$$

$$
F_{j,i}(\bar{x})\lambda_{j,i} = 0 \text{ for every } j = 1, 2, \ i = 1, \ldots, k,
$$

$$
\lambda_{1,i}\lambda_{2,i} \geq 0 \text{ for every } i \in \{1, \ldots, k\} \text{ with } F_{1,i}(\bar{x}) = F_{2,i}(\bar{x}) = 0.
$$

As an immediate consequence of Clarke’s stationarity condition for programs with locally Lipschitz functions in [1] and Lemma 1 in [17] we obtain

**Proposition 2.1 (MFC and C-stationarity)**

If $\bar{x}$ is a local minimizer of the MPCC and MFC holds at $\bar{x}$, then $\bar{x}$ is C-stationary.
For more details on C-stationarity and other stationarity concepts, such as W-, A-, M-, and S-stationarity, see [3], [11], [13], [17], [18].

It is clear that SMFC implies MFC. Moreover, these two conditions coincide for \( n = k \).

Further, we recall the well-known LICQ for MPCC (e.g. [17]), which is said to hold at \( \bar{x} \in M[F_1, F_2] \) if

\[
\{ \nabla F_{i,j}(\bar{x}) \mid F_{i,j}(\bar{x}) = 0, i = 1, \ldots, k, j = 1, 2 \} \text{ are linearly independent.}
\]

LICQ can be equivalently formulated in terms of transversal intersection of stratified sets (see [8]). As pointed out in [17], LICQ is a generic constraint qualification. However, LICQ is not necessary for local stability as one can see from the following Example 2.1. In this and all further examples only the local stability in 0 is of interest.

**Example 2.1 (2D, stable: one point \( \rightarrow \) one point)**

The set \( M^{2.1} := \{ (x, y) \in \mathbb{R}^2 \mid \min\{x, y\} = 0, \min\{x - y, 2x - y\} = 0 \} \) is a singleton and it is locally stable at 0 (see Figure 1). However, LICQ does not hold at 0. □

![Figure 1: Illustration of Example 2.1](image)

In this sense, LICQ appears to be too restrictive. It comes from the fact that LICQ does not impose the combinatorial structure of the complementarity constraints. Additionally, we notice that LICQ implies MFC.

Another condition, we intend to discuss, comes from the exact Mordukhovich’s extremal principle (cf. [4, 12]).
Let $\Omega \subset \mathbb{R}^n$ be any arbitrary closed set and $\bar{x} \in \Omega$. The nonempty cone

$$T(\bar{x}, \Omega) := \limsup_{\tau \searrow 0} \frac{\Omega - \bar{x}}{\tau}$$

is called the contingent (also Bouligand or tangent) cone to $\Omega$ at $x$.

The Fréchet normal cone is defined via polarization as follows:

$$\hat{N}(\bar{x}, \Omega) := (T(\bar{x}, \Omega))^\circ.$$ 

Finally, the limiting normal cone (also called Mordukhovich normal cone) is defined by

$$N(\bar{x}, \Omega) := \limsup_{x' \xrightarrow{\Omega, \bar{x}} \Omega} \hat{N}(x', \Omega)$$

$$= \left\{ \lim_{k \to \infty} w_k \mid \text{there exist } x_k \xrightarrow{\bar{x}}, x_k \in \Omega, w_k \in \hat{N}(x_k, \Omega) \right\}.$$

We recall the finite-dimensional version of the exact Mordukhovich’s extremal principle.

**Theorem 2.1 (Exact Extremal Principle in finite dimensions, cf. [12])**

Let $\Omega_i, i = 1, \ldots, k$ be nonempty closed subsets of $\mathbb{R}^n$ and $\bar{x} \in \bigcap_{i=1}^k \Omega_i$.

Suppose that the following condition is satisfied:

$$(\triangle) \quad \text{For all } x_i^* \in N(\bar{x}, \Omega_i), i = 1, \ldots, k :$$

$$\sum_{i=1}^k x_i^* = 0 \text{ implies } x_i^* = 0, i = 1, \ldots, k.$$ 

Then, there exists a neighborhood $V$ of $\bar{x}$ such that for all sufficiently small $a_i \in \mathbb{R}^n, i = 1, \ldots, k$ it holds:

$$\bigcap_{i=1}^k (\Omega_i - a_i) \cap V \neq \emptyset.$$
Actually, Theorem 2.1 provides a sufficient condition for the property that the intersection remains nonempty with respect to translations. In order to apply Theorem 2.1 in our setting we say that the Exact Extremal Principle Condition (EEPC) holds at $\bar{x} \in M[F_1, F_2]$ if and only if $(\triangle)$ is fulfilled for $\Omega_i := M_i$, where

$$M_i := \{ x \in \mathbb{R}^n \mid F_{1,i}(x) \geq 0, F_{2,i}(x) \geq 0, F_{1,i}(x)F_{2,i}(x) = 0 \}.$$  

**Proposition 2.2 (MFC implies EEPC)**

If MFC holds at $\bar{x} \in M[F_1, F_2]$ then EEPC also holds at $\bar{x}$.

**Proof.** Let $i \in \{1, \ldots, k\}$ be fixed. We provide a representation formula for $N(\bar{x}, M_i)$. We restrict ourselves to the interesting case that $F_{1,i}(\bar{x}) = F_{2,i}(\bar{x}) = 0$. Due to Assumption A1 we choose vectors $\xi_1, \ldots, \xi_{n-2} \in \mathbb{R}^n$ which form - together with the vectors $\nabla F_{1,i}(\bar{x}), \nabla F_{2,i}(\bar{x})$ - a basis for $\mathbb{R}^n$.

Next we put $y = \Phi(x)$ as follows:

$$y_1 := F_{1,i}(x), \ y_2 := F_{2,i}(x), \ y_3 := \xi_1^T(x - \bar{x}), \ldots, \ y_n := \xi_{n-2}^T(x - \bar{x}).$$

Note that $\Phi(\bar{x}) = 0$ and $D\Phi(\bar{x})$ is nonsingular. Therefore, $\Phi$ maps $M_i$ diffeomorphically to $K := \{ y \in \mathbb{R}^n \mid y_1 \geq 0, y_2 \geq 0, y_1y_2 = 0 \}$ locally at $\bar{x}$. Setting $L := \{ y \in \mathbb{R}^2 \mid y_1 \geq 0, y_2 \geq 0, y_1y_2 = 0 \}$ Proposition 6.41 from [16] yields:

$$N(0, K) = N(0, L \times \mathbb{R}^{n-2}) = N(0, L) \times N(0, \mathbb{R}^{n-2}).$$

From [4] and [13] we conclude that $N(0, L) = \mathbb{R}^2 \cup L$. Clearly, $N(0, \mathbb{R}^{n-2}) = \{0_{n-2}\}$. Altogether, we get

$$N(0, K) = \mathbb{R}^2 \cup L \times \{0_{n-2}\}.$$ 

Using Exercise 6.7 (change of coordinates) from [16] we get:

$$N(\bar{x}, M_i) = \{ \beta_1 \nabla F_{1,i}(\bar{x}) + \beta_2 \nabla F_{2,i}(\bar{x}) \mid \text{either } \beta_1 < 0, \beta_2 < 0 \text{ or } \beta_1\beta_2 = 0 \}. \quad (2)$$

Analogously, we obtain:

$$\hat{N}(\bar{x}, M_i) = \{ \beta_1 \nabla F_{1,i}(\bar{x}) + \beta_2 \nabla F_{2,i}(\bar{x}) \mid \beta_1 \leq 0, \beta_2 \leq 0 \}. \quad (3)$$

The representation (3) yields that MFC is equivalent to the following condition:
For all \( x_i^* \in \pm \hat{N}(\bar{x}, M_i), \ i = 1, \ldots, k: \)
\[
\sum_{i=1}^{k} x_i^* = 0 \implies x_i^* = 0, \ i = 1, \ldots, k.
\]

Since \( N(\bar{x}, M_i) \subset \pm \hat{N}(\bar{x}, M_i), \ i = 1, \ldots, k, \) (cf. (2) and (3)), the proposition follows immediately. \( \square \)

**Corollary 2.1 (MFC via Fréchet normal cones)**

MFC is equivalent to the following condition:

For all \( x_i^* \in \pm \hat{N}(\bar{x}, M_i), \ i = 1, \ldots, k: \)
\[
\sum_{i=1}^{k} x_i^* = 0 \implies x_i^* = 0, \ i = 1, \ldots, k,
\]

where \( M_i = \{ x \in \mathbb{R}^n \mid F_{1,i}(x) \geq 0, F_{2,i}(x) \geq 0, F_{1,i}(x)F_{2,i}(x) = 0 \} \). \( \square \)

As we show by Example 2.2, EEPC is not sufficient for \( M[F_1, F_2] \) to be locally stable at 0. In this and all further examples in 3D we understand under "two-star", "three-star" and "four-star" subsets of \( \mathbb{R}^3 \) as depicted in Figure 2 up to a homeomorphism.

**Figure 2:** "two-star" \quad "three-star" \quad "four-star"

**Example 2.2 (3D, nonstable: "four-star" \( \rightarrow 2 \) "two-stars")**

Consider the "four-star" subset
\[
M^{2.2} := \{ (x, y, z) \in \mathbb{R}^3 \mid \min\{x, y\} = 0, \min\{x+y-\sqrt{2}z, x+y+\sqrt{2}z\} = 0 \}
\]
(see Figure 3). After an appropriate perturbation the resulting set would have two path-connected components. Therefore, $M^{2.2}$ is not locally stable at 0.

$$\min\{x, y\} = 0$$

$$\min\{x + y - \sqrt{2}z, x + y + \sqrt{2}z\} = 0$$

**Figure 3:** Illustration of Example 2.2

To show that EEPC holds at 0 we set

$$M^{2.2}_1 := \{(x, y, z) \in \mathbb{R}^3 \mid \min\{x, y\} = 0\},$$

$$M^{2.2}_2 := \{(x, y, z) \in \mathbb{R}^3 \mid \min\{x + y - \sqrt{2}z, x + y + \sqrt{2}z\} = 0\}.$$ 

and obtain due to (2) from the proof of Proposition 2.2:

$$N(0, M^{2.2}_1) = \{(\beta_1, \beta_2)^T \in \mathbb{R}^3 \mid \text{either } \beta_1 < 0, \beta_2 < 0 \text{ or } \beta_1 \beta_2 = 0\}.$$  

$$N(0, M^{2.2}_2) = \{\beta_1 \begin{pmatrix} 1 \\ 1 \\ -\sqrt{2} \end{pmatrix} + \beta_2 \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} \mid \text{either } \beta_1 < 0, \beta_2 < 0 \text{ or } \beta_1 \beta_2 = 0\}.$$ 

From the above representations of $N(0, M^{2.2}_1)$ and $N(0, M^{2.2}_2)$ it is easy to see that EEPC (but not MFC) is satisfied at $0 \in M^{2.2}. \square$

The next stability concept we would like to discuss here is metric regularity. We recall that a multi-valued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^k$ is called *metrically regular* (with rank $L > 0$) at $(\bar{x}, \bar{y}) \in \text{gph } T$ if, for certain neighborhoods $U$ and $V$ of $\bar{x}$ and $\bar{y}$, respectively, it holds:

$$\text{dist}(x, T^{-1}(y)) \leq L \text{dist}(y, T(x)) \text{ for all } x \in U, y \in V.$$ 

Further, a multi-valued map $S : \mathbb{R}^k \rightrightarrows \mathbb{R}^n$ is called *pseudo-Lipschitz* (with rank $L > 0$) at $(\bar{y}, \bar{x}) \in \text{gph } S$ if there are neighborhoods $U$ and $V$ of $\bar{x}$ and $\bar{y}$, respectively, such that, given any points $(y, x) \in (V \times U) \cap \text{gph } S$, it holds:

$$\text{dist}(x, S(y')) \leq L\|y' - y\| \text{ for all } y' \in V.$$
It holds (cf. [6]) that $T$ is metrically regular at $(\bar{x}, \bar{y}) \in \text{gph} \ T$ if and only if $T^{-1}$ is pseudo-Lipschitz at $(\bar{y}, \bar{x})$.

It is well-known from [14] that the solution map $S(y, z) := \{x \in \mathbb{R}^n \mid h(x) = y, g(x) \leq z\}$, $(g, h) \in C^1(\mathbb{R}^n, \mathbb{R}^{k+m})$, is pseudo-Lipschitz at $(0, 0, \bar{x})$ if and only if MFCQ is satisfied at $\bar{x} \in S(0, 0)$. It means, therefore, that the local stability of $M_{NLP}[h, g] (= S(0, 0))$ at $\bar{x} \in M_{NLP}[h, g]$ is equivalent to the metric regularity of

$$S^{-1}(x) = \{(h(x), z) \mid g(x) \leq z\}$$

at $(\bar{x}, 0, 0)$.

To apply this idea in our setting we say that the Metric Regularity Condition (MRC) holds at $\bar{x} \in M_{F_1, F_2}$ if and only if

$$G : \begin{cases} \mathbb{R}^n & \to \mathbb{R}^k, \\ x & \mapsto (\min \{F_{1,i}(x), F_{2,i}(x)\})_{i=1,\ldots,k} \end{cases}$$

is metrically regular at $(\bar{x}, 0)$.

**Proposition 2.3** (MRC is equivalent to EEPC)

*MRC holds at $\bar{x} \in M_{F_1, F_2}$ if and only if EEPC holds at $\bar{x}$.*

**Proof.** MRC holds at $\bar{x} \in M_{F_1, F_2}$ if and only if the solution map $S(y) := \{x \in \mathbb{R}^n \mid G(x) = y\}$, $y \in \mathbb{R}^k$, is pseudo-Lipschitz at $(0, \bar{x})$. Setting

$$F : \begin{cases} \mathbb{R}^n & \to \mathbb{R}^{2k}, \\ x & \mapsto (F_{1,i}(x), F_{2,i}(x))_{i=1,\ldots,k} \end{cases}$$

and $D_i := \{(a_i, b_i) \in \mathbb{R}^2 \mid a_i \geq 0, b_i \geq 0, ab = 0\}$, $i = 1, \ldots, k$ we obtain:

$$S(y) = \{x \in \mathbb{R}^n \mid F(x) - y \in D_1 \times \cdots \times D_k\},$$

$$S^{-1}(x) = F(x) - D_1 \times \cdots \times D_k.$$

Therefore, MRC holds at $\bar{x} \in M_{F_1, F_2}$ if and only if $F(\cdot) - D_1 \times \cdots \times D_k$ is metrically regular at $(\bar{x}, 0)$. Since $F \in C^1(\mathbb{R}^n, \mathbb{R}^{2k})$ and $D_1 \times \cdots \times D_k$ is
closed, we can apply Example 9.44 from [16]. Due to this Example 9.44 the constraint qualification

$$u \in N(F(\bar{x}), D_1 \times \cdots \times D_k), \nabla^T F(\bar{x}) u \implies u = 0$$

is equivalent to the metric regularity of $F(\cdot) - D_1 \times \cdots \times D_k$ at $(\bar{x}, 0)$. Since $N(F(\bar{x}), D_1 \times \cdots \times D_k) = N(F_{1,1}(\bar{x}), F_{2,1}(\bar{x}), D_1) \times \cdots \times N(F_{1,k}(\bar{x}), F_{2,k}(\bar{x}), D_k)$ and $N(0, D_i) = \mathbb{R}^2 \cup D_i$, the formula (2) allows to conclude that the constraint qualification (4) is equivalent to EEPC. □

Proposition 2.3 and Example 2.2 show that MRC is not sufficient for $M[F_1, F_2]$ being locally stable.

At the end of this Section we briefly mention other 2- and 3-dimensional examples with 2 linear constraints respectively. These examples illustrate which phenomena might occur in general. They mainly highlight the possibilities arising with respect to the stability property of the feasible set $M[F_1, F_2]$ in low dimensions.

**Example 2.3 (2D, nonstable: one point $\rightarrow$ empty, two points)**

The set $M^{2.3} := \{(x, y) \in \mathbb{R}^2 \mid \min\{x, y\} = 0, \min\{-x, -y\} = 0\}$ is a singleton (see Figure 4a). Note that MFC is not satisfied at 0. After an appropriate perturbation $M^{2.3}$ either becomes empty or contains at least two points. □

**Example 2.4 (2D, nonstable: one point $\rightarrow$ two points)**

The set $M^{2.4} := \{(x, y) \in \mathbb{R}^2 \mid \min\{x, y\} = 0, \min\{-x + y, x + y\} = 0\}$ is a singleton (see Figure 4b). Note that MFC is not satisfied at 0. After an appropriate perturbation $M^{2.4}$ contains at least two points. □
Example 2.5 (3D, nonstable: "three-star" $\longrightarrow$ 1 or 2 "two-stars")

The set $M^{2.5} := \{ (x, y, z) \in \mathbb{R}^3 \mid \min \{ x, y \} = 0, \min \{ y - z, y + z \} = 0 \}$ is a "three-star" (see Figure 5a). Note that MFC is not satisfied at 0. After an appropriate perturbation $M^{2.5}$ either has two path-connected components or is a "two-star". □

Example 2.6 (3D, stable: "two-star" $\longrightarrow$ "two-star")

The set $M^{2.6} := \{ (x, y, z) \in \mathbb{R}^3 \mid \min \{ x, y \} = 0, \min \{ x - y + z, -x + y + z \} = 0 \}$ is a "two-star" (see Figure 5b). Note that MFC holds at 0. After any sufficiently small perturbation $M^{2.6}$ remains to be a "two-star". □
It is easy to see that in all these examples MFC holds at 0 if and only if the corresponding feasible set is locally stable. Moreover, these examples emphasize that the locally stable case corresponds to a feasible set being a Lipschitz manifold (see Corollary 3.1 in forthcoming section). This consideration, actually, motivated the introduction of SMFC, which is a generalization of the well-known maximal rank condition of Clarke (cf. [1]).

3 Sufficiency part

We intend to prove that SMFC implies local stability of the feasible set $M[F_1, F_2]$ (cf. Definition 1.1). The main idea is to show that under SMFC $M[F_1, F_2]$ appears to be a $k$-dimensional Lipschitz manifold.

Therefore, we recall briefly the notion of the Clarke’s generalized Jacobian and the corresponding Implicit Function Theorem (cf. [1]).

For a vector-valued function $G = (g_1, \ldots, g_k) : \mathbb{R}^n \to \mathbb{R}^k$ with $g_i$ being Lipschitz near $\bar{x} \in \mathbb{R}^n$, the set

$$\partial G(\bar{x}) := \text{conv}\{\lim DG(x_i) \mid x_i \to \bar{x}, x_i \notin \Omega_G\}$$

is called the Clarke’s generalized Jacobian, where $\Omega_G \subset \mathbb{R}^n$ denotes the set of points at which $G$ fails to be differentiable.

**Theorem 3.1 (Implicit Function Theorem, [1])** Let $G : \mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}^k$ be Lipschitz near $(\bar{y}, \bar{z}) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$ with $G(\bar{y}, \bar{z}) = 0$. Suppose that

$$\pi_z \partial G(\bar{y}, \bar{z}) := \{M \in \mathbb{R}^{k \times k} \mid \text{there exists } N \in \mathbb{R}^{k \times n} \text{ with } [N, M] \in \partial G(\bar{y}, \bar{z})\}$$

is of maximal rank, i.e. contains merely non-singular matrices. Then there exist a $\mathbb{R}^{n-k}$-neighborhood $Y$ of $\bar{y}$, a $\mathbb{R}^k$-neighborhood $Z$ of $\bar{z}$ and a Lipschitz function $\zeta : Y \to Z$ such that $\zeta(\bar{y}) = \bar{z}$ and for every $(y, z) \in Y \times Z$ it holds:

$$G(y, z) = 0 \text{ if and only if } z = \zeta(y).$$

However, Example 3.1 illustrates that Theorem 3.1 can not be applied directly in general just for the linear case of a stable $M[F_1, F_2]$.
Example 3.1 (3D, stable: IFT is not applicable)

Consider the set \( M_{3.1} := \{ (x, y, z) \in \mathbb{R}^3 \mid \min \{x, y\} = 0, \min \{-y + z, z\} = 0 \} \) (see Figure 6). This example shows that although \( M[F_1, F_2] \) is a Lipschitz manifold, it can not be parameterized by means of any splitting of \( \mathbb{R}^3 \) in the standard basis. Therefore, Theorem 3.1 can not be applied directly. □

\[
\begin{array}{c}
\min \{-y + z, z\} = 0 \\
\min \{x, y\} = 0
\end{array}
\]

Figure 6: Illustration of Example 3.1

Indeed, Example 3.1 suggests to firstly perform a linear coordinate transformation in order to make Theorem 3.1 applicable. Exactly this idea is incorporated in SMFC and allows to prove the following result.

Theorem 3.2 (Local stability under SMFC)

If SMFC holds at \( x \in M[F_1, F_2] \), then the feasible set \( M[F_1, F_2] \) is locally stable at \( \bar{x} \).

Proof. Let \( \bar{x} \in M[F_1, F_2] \). Since SMFC holds at \( \bar{x} \), there exists a \( k \)-dimensional linear subspace \( E \) of \( \mathbb{R}^n \) such that any \( k \) vectors \( (u_1, \ldots, u_k) \in P_E(C_1(\bar{x})) \times \cdots \times P_E(C_k(\bar{x})) \) are linearly independent. W.l.o.g., we may assume that \( E = \{0_{n-k}\} \times \mathbb{R}^k \).

Setting \( g_i := \min\{F_{1,i}, F_{2,i}\}, i = 1, \ldots, k \) we define

\[
G: \begin{cases} \\
\mathbb{R}^{n-k} \times \mathbb{R}^k & \rightarrow & \mathbb{R}^k, \\
(y, z) & \mapsto & (g_1(y, z), \ldots, g_k(y, z)).
\end{cases}
\]

Let \( \bar{y}, \bar{z} \in \mathbb{R}^{n-k} \times \mathbb{R}^k \). We obtain from \( \partial g_i(\bar{x}) = C_i(\bar{x}), i = 1, \ldots, k \), and the choice of \( E \) that

\[
\pi_z \partial G(\bar{y}, \bar{z}) = P_E(C_1(\bar{x})) \times \cdots \times P_E(C_k(\bar{x})).
\]
Hence, due to SMFC $\pi_z \partial G(\tilde{y}, \tilde{z})$ is of maximal rank and Theorem 3.1 can be applied. Then there exist a compact $\mathbb{R}^{n-k}$-neighborhood $Y$ of $\tilde{y}$, a $\mathbb{R}^k$-neighborhood $Z$ of $\tilde{z}$ and a Lipschitz function $\zeta : Y \to Z$ such that $\zeta(\tilde{y}) = \tilde{z}$ and for every $(y, z) \in Y \times Z$ it holds:

$$G(y, z) = 0 \text{ if and only if } z = \zeta(y).$$

For $\varepsilon > 0$ we set

$$K_\varepsilon := \left( \{ (y, \zeta(y)) \mid y \in Y \} + \bar{B}_{\mathbb{R}^n}(0, \varepsilon) \right) \cap (Y \times Z)$$

an $\varepsilon$-tube around $M[F_1, F_2] \cap (Y \times Z)$. Due to the compactness of $Y$, continuity reasonings and stability of SMFC within the space of $C^1$-functions (taking $Y$ smaller if needed) there exists $\varepsilon > 0$ such that:

1. $K_\varepsilon \subset Y \times Z$ and $K_\varepsilon$ is compact,
2. there exists a $C^1$-neighborhood $U$ of $(F_1, F_2)$ in $C^1(\mathbb{R}^n, \mathbb{R}^k) \times C^1(\mathbb{R}^n, \mathbb{R}^k)$ such that for every $(\tilde{F}_1, \tilde{F}_2) \in U$ it holds:

$$M[\tilde{F}_1, \tilde{F}_2] \cap (Y \times Z) \subset K_\varepsilon.$$

We assume $U$ to be a ball of radius $r > 0$ in $C^1(\mathbb{R}^n, \mathbb{R}^k) \times C^1(\mathbb{R}^n, \mathbb{R}^k)$.

3. SMFC is fulfilled at every $x \in M[\tilde{F}_1, \tilde{F}_2] \cap (Y \times Z)$ for every $(\tilde{F}_1, \tilde{F}_2) \in U$ with the same $k$-dimensional linear subspace $E$.

Let now $(\tilde{F}_1, \tilde{F}_2) \in U$ be arbitrary, but fixed. Setting $\tilde{g}_i := \min\{\tilde{F}_{1,i}, \tilde{F}_{2,i}\}$, $i = 1, \ldots, k$ we define

$$\tilde{G} : \mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}^k, \quad (y, z) \mapsto (\tilde{g}_1(y, z), \ldots, \tilde{g}_k(y, z)).$$

Our aim is to show that for every fixed $\tilde{y} \in Y$ the equation $\tilde{G}(\tilde{y}, z) = 0$ is uniquely solvable with $(\tilde{y}, z) \in K_\varepsilon$. For that, we set for $(t, y, z) \in [0, 1] \times \mathbb{R}^{n-k} \times \mathbb{R}^k$

$$H_{1,i}(t, y, z) := (1 - t)F_{1,i}(y, z) + t\tilde{F}_{1,i}(y, z), \quad H_{2,i}(t, y, z) := (1 - t)F_{2,i}(y, z) + t\tilde{F}_{2,i}(y, z), \quad i = 1, \ldots, k.$$
This conclusion allows us to apply Theorem 3.1 for the equation
\[ \tilde{\text{solvable with }} (y, z) \]
These considerations allow us to claim, that
\[ (y, z) \]
Next, we fix \( \tilde{y} \in Y \) and consider the equation \( H(t, \tilde{y}, z) = 0 \) near its solution \((0, \tilde{y}, \zeta(\tilde{y}))\). Since \((\tilde{y}, \zeta(\tilde{y})) \in M[F_1, F_2] \cap (Y \times Z)\) we obtain from \((\bullet, \bullet, \bullet)\) that SMFC holds at \((\tilde{y}, \zeta(\tilde{y}))\). It means that
\[ \pi_z \partial H(0, \tilde{y}, \zeta(\tilde{y})) = \pi_z \partial G(\tilde{y}, \zeta(\tilde{y})) \]
is of maximal rank and Theorem 3.1 can be applied for \( H(t, \tilde{y}, z) = 0 \) near its solution \((0, \tilde{y}, \zeta(\tilde{y}))\). Thus, we obtain for every \( t \in [0, \delta) \), \( 0 < \delta \leq 1 \) a solution \( z(t) \) such that \( H(t, \tilde{y}, z(t)) = 0 \). Since \((H_1(t, \cdot, \cdot), H_2(t, \cdot, \cdot)) \in U\), \((\bullet, \bullet)\) yields that \((\tilde{y}, z(t)) \in K_\varepsilon\) for every \( t \in [0, \delta) \). Hereby, \( \delta \) is taken smaller if needed.

These considerations allow us to claim, that
\[ \tilde{t} := \sup \{ \tilde{t} \in [0, 1) \mid \text{for every } t \in [0, \tilde{t}) \text{ there exists at least one } (\tilde{y}, z(t)) \in K_\varepsilon \text{ such that } H(t, \tilde{y}, z(t)) = 0 \} \]
is well-defined.

Assume that \( \tilde{t} \neq 1 \). Then, there is a sequence of solutions \( z(t_m), t_m \in [0, \tilde{t}), t_m \to \tilde{t} \) such that \((\tilde{y}, z(t_m)) \in K_\varepsilon\) and \( H(t_m, \tilde{y}, z(t_m)) = 0 \). We use the compactness of \( K_\varepsilon \) from \((\bullet)\) to obtain the existence of \( \tilde{z} \) with \((\tilde{y}, \tilde{z}) \in K_\varepsilon\) and \( z_m \to \tilde{z} \). Hence, due to the continuity we get in the limit \( H(t, \tilde{y}, \tilde{z}) = 0 \). This conclusion allows us to apply Theorem 3.1 for the equation \( H(t, \tilde{y}, z) = 0 \) near \((\tilde{t}, \tilde{y}, \tilde{z})\) to extend the solution for \( t > \tilde{t} \). This yields a contradiction with the definition of \( \tilde{t} \).

So, we claim that \( \tilde{t} = 1 \) and as above we obtain: \( \tilde{G}(\tilde{y}, z) \equiv H(1, \tilde{y}, z) = 0 \) is solvable with \((\tilde{y}, z) \in K_\varepsilon\).

The unique solvability of \( \tilde{G}(\tilde{y}, z) = 0 \) for \((\tilde{y}, z) \in K_\varepsilon\) can be proven by contradiction using analogous arguments. One has only to follow different solutions by applying Theorem 3.1 successively until the unique solution \((0, \tilde{y}, \zeta(\tilde{y}))\) of \( G(\tilde{y}, z) \equiv H(t, \tilde{y}, z) = 0 \) will be reached.
Altogether, it is proven: For every \( \tilde{y} \in Y \) the equation \( \tilde{G}(\tilde{y}, z) = 0 \) is uniquely solvable with \( (\tilde{y}, z(\tilde{y})) \in K_\varepsilon \). From (●●) one can immediately see that \( \tilde{G}(\tilde{y}, z) = 0 \) is uniquely solvable, actually, in \( Z \). Therefore, \( M[\tilde{F}_1, \tilde{F}_2] \cap (Y \times Z) = \{(y, z(y)) \mid y \in Y\} \). Hereby, \( z : Y \to Z \) is Lipschitz due to (●●●) and Theorem 3.1, which is applicable locally around every \( \tilde{x} \in M[\tilde{F}_1, \tilde{F}_2] \cap (Y \times Z) \).

It remains to add that \( M[F_1, F_2] \cap (Y \times Z) \) and \( M[\tilde{F}_1, \tilde{F}_2] \cap (Y \times Z) \), both being Lipschitz graphs on \( Y \), are homeomorphic with \( \mathbb{R}^{n-k} \) and, thus, with each other. □

From the proof of Theorem 3.2 we deduce the following Corollary 3.1.

**Corollary 3.1** If SMFC holds at every \( \bar{x} \in M[F_1, F_2] \), then the feasible set \( M[F_1, F_2] \) is a \((n-k)\)-dimensional Lipschitz manifold.

**Proof.** With notation as in Theorem 3.2 we define a Lipschitz homeomorphism

\[
\varphi : \begin{cases} 
Y & \to (Y \times Z) \cap M[F_1, F_2], \\
y & \mapsto (y, \zeta(y)).
\end{cases}
\]

Thus, \( M[F_1, F_2] \) is locally Lipschitz homeomorphic with \( \mathbb{R}^{n-k} \) at \( \bar{x} \). Since \( \bar{x} \in M[F_1, F_2] \) was arbitrary, \( M[F_1, F_2] \) is a \((n-k)\)-dimensional Lipschitz manifold. □

Now we intend to show that SMFC is equivalent to MFC at least for the case \( k = 2 \) as well as for \( n \geq 2k \). It follows mainly from the following (linear-algebraic) Lemma 3.1.

**Lemma 3.1** Let \( C_i := \text{conv}\{v_{j,i} \in \mathbb{R}^n \mid j = 1, 2\}, i = 1, \ldots, k \) and for every \( i \in \{1, \ldots, k\} \) let \( v_{1,i}, v_{2,i} \) be linearly independent. For \( k = 2, n \geq 2 \) as well as for \( n \geq 2k \) the following conditions (A) and (B) are equivalent:

(A) any \( k \) vectors \( (w_1, \ldots, w_k) \in C_1 \times \cdots \times C_k \) are linearly independent.

(B) there exists a \( k \)-dimensional linear subspace \( E \) of \( \mathbb{R}^n \) such that any \( k \) vectors \( (u_1, \ldots, u_k) \in P_E(C_1) \times \cdots \times P_E(C_k) \) are linearly independent, where \( P_E : \mathbb{R}^n \to E \) denotes the orthogonal projection.
Proof. The nontrivial part is to prove that (A) implies (B) for \( k = 2, n > k \).
At first let \( k \) be given in general, not necessarily equal 2.

For proving (A) implies (B) it is sufficient to show: if (A) holds then

\[
(\ast) \quad \text{there exists a} \ (n-1)\text{-dimensional linear subspace } T \text{ of } \mathbb{R}^n \text{ such that any } k \text{ vectors } (u_1, \ldots, u_k) \in P_T(C_1) \times \cdots \times P_T(C_k) \text{ are linearly independent.}
\]

Indeed, in this case we only need to apply (\ast) \( n-k \) times successively to get a \( k \)-dimensional linear subspace \( E \) with the desired property from (B).

Suppose now that (A) is fulfilled, but (\ast) does not hold. It means:

\[
(\ast\ast) \quad \text{for every} \ (n-1)\text{-dimensional linear subspace } W \text{ of } \mathbb{R}^n \text{ there exist } u_i^W \in P_W(C_i), \alpha_i^W \in \mathbb{R} \text{ (not all equal 0), } i = 1, \ldots, k \text{ such that}
\]

\[
\sum_{i=1}^{k} \alpha_i^W u_i^W = 0.
\]

Every \((n-1)\)-dimensional linear subspace \( W \) of \( \mathbb{R}^n \) can be described as \( \{x \in \mathbb{R}^n \mid \eta^T \cdot x = 0\}, \eta \in \mathbb{S}^{n-1} \). The orthogonal projection of \( C_i \) on \( W \) is, thus, \( P_W(C_i) = \{u - (u \cdot \eta)\eta \mid u \in C_i\} \).

Hence, for all \( \eta \in \mathbb{S}^{n-1} \) there exist \( u_i^\eta \in C_i, \alpha_i^\eta \in \mathbb{R} \) (not all equal 0), \( i = 1, \ldots, k \) such that

\[
\sum_{i=1}^{k} \alpha_i^\eta (u_i^\eta - (u_i^\eta \cdot \eta)\eta) = 0. \quad (5)
\]

Setting \( u^\eta := \sum_{i=1}^{k} \alpha_i^\eta u_i^\eta \) we obtain from (5):

\[
u^\eta = (u^\eta \cdot \eta)\eta \quad \text{for all } \eta \in \mathbb{S}^{n-1}. \quad (6)
\]

Case (a): \( u^\eta \cdot \eta = 0 \) for at least one \( \eta \in \mathbb{S}^{n-1} \)

Then, (6) provides that \( u^\eta = 0 \) and, thus, \( u_i^\eta \in C_i, i = 1, \ldots, k \) are linearly dependent. This claim yields a contradiction to (A).
Case (b): \( \eta^T \eta \neq 0 \) for all \( \eta \in \mathbb{S}^{n-1} \)

By setting \( D := \left\{ \sum_{i=1}^{k} \beta_i u_i \mid (\beta_i)_{i=1}^{k} \in \mathbb{R}^k \setminus \{0\}, \ u_i \in C_i \right\} \) we obtain \( u^T \in D \).

From (6) we have \( \eta = \frac{1}{u^T \eta} u^T \in D \) and, therefore:

\[
\mathbb{R}^n \setminus \{0\} \subset D. \tag{7}
\]

Now we consider three cases with respect to the dimension \( n \).

Case (b.1): \( n > 2k \)

Since \( C_i \) is a convex combination of at most two vectors, (7) yields a contradiction because of the dimensionality reasoning.

Case (b.2): \( n = 2k \)

Suppose, first, that \( (v_{j,i})_{j=1,2, i=1,...,k} \) are linearly independent.

Define a linear coordinate transformation \( L : \mathbb{R}^n \rightarrow \mathbb{R}^n \) as follows:

\[
L(v_{1,i}) = e_{2i-1} + e_{2i}, \ L(v_{2,i}) = e_{2i-1}, \ i = 1,...,k,
\]

whereby \( e_m \) denotes the \( m \)-th standard basis vector for \( 1 \leq m \leq n \).

Taking these coordinates, we obtain: \( C_i = \{e_{2i-1} + \lambda_i e_{2i} \mid \lambda_i \in [0,1]\} \). Thus, setting \( T := \{l_n\}^\perp \), where \( l_n \) is the last column of \( L \), we produce a contradiction to the assumption (**) .

Let us, contrarily, assume now that \( v_{j,i}, j = 1,2, i = 1,...,k \) are linearly dependent. In this case all \( C_i, i = 1,...,k \) lie in the \( (n-1) \)-dimensional linear subspace \( T := \{x \in \mathbb{R}^n \mid \nu^T \cdot x = 0\} \), where \( \nu \perp \text{span}\{v_{j,i}, j = 1,2, i = 1,...,k\} \). This fact allows us to perform the dimensional reduction directly and to proceed to the case \( n < 2k \).

Case (b.3): \( n < 2k \)

For \( a \in \{-1,1\}^k \) we set \( K_a := \text{cone}\{a_i v_{1,i}, a_i v_{2,i} \mid i = 1,...,k\} \). Obviously,

\[
D \cup \{0\} = \bigcup_{a \in \{-1,1\}^k} K_a. \tag{8}
\]
Due to the theorem about alternatives (e.g. [15]) we claim as follows:

$$\text{int} \ (K_a^o) = \emptyset \ \text{if and only if} \ (A) \ \text{does not hold.}$$

Here, int($K_a^o$) denotes the interior of the polar cone of $K_a$.

Thus, we may assume that

$$\text{int}(K_a^o) \neq \emptyset \ \text{for all} \ a \in \{-1,1\}^k. \quad (9)$$

To proceed to a contradiction we now exploit that $k = 2$. Note that $n = 3$ in this case. We remind that due to (9) $K_a$ properly lies in a half space for all $a \in \{-1,1\}^k$. Setting $\{-1,1\}^2 =: \{a^1, -a^1, a^2, -a^2\}$ we can strictly separate $K_{a^1}$ and $K_{-a^2}$ by a plane $\gamma_1 \ni 0, \ l = 1, 2$. Since $0 \in \gamma_1 \cap \gamma_2$, there exists $\xi \in \gamma_1 \cap \gamma_2, \xi \neq 0$ such that $\xi \notin \bigcup_{a \in \{-1,1\}^2} K_a$ by construction. This yields a contradiction to $\mathbb{R}^n = \bigcup_{a \in \{-1,1\}^2} K_a$ coming from (8) and (7). $\square$

**Theorem 3.3** Let $k = 2$ or $n \geq 2k$. Then, SMFC is equivalent to MFC.

**Proof.** It is straightforward to see that the conclusion can be obtained by application of Lemma 3.1. We have to adjust the proof of Lemma 3.1 only for the case, if exactly one constraint in $\min\{F_{1,i}(\bar{x}), F_{2,i}(\bar{x})\} = 0$ is active (i.e. $F_{1,i}(\bar{x}) = 0, F_{2,i}(\bar{x}) > 0$ or vice versa). For that we define $C_i$ from Lemma 3.1 to be $C_i(\bar{x})$. It would not change the proof of Lemma 3.1 as much. One can imagine that only the so-called biactive set of constraints is crucial (cf. [11], [18]). $\square$

The question, whether SMFC is equivalent to MFC for $k < n < 2k, \ k > 2$, is still open and is subject to current research.

**References**


