Noncooperative oligopoly in markets with a continuum of traders

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We show the existence of a pure strategy Cournot–Nash equilibrium for a model of noncooperative exchange where large traders, represented as atoms, and small traders, represented by an atomless part, are allowed to buy and sell all commodities.

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1. Introduction

In a seminal paper, Shitovitz (1973) studied oligopoly in a general equilibrium framework by modeling trade as a cooperative game between large traders, represented as atoms, and small traders, represented by an atomless part. The model analyzed by Shitovitz (1973) admits the counterintuitive possibility that the core allocations are competitive despite the presence of atoms.

Taking inspiration from Shitovitz (1973), Okuno et al. (1980) proposed to study the Cournot–Nash equilibria of a model of simultaneous, noncooperative exchange between large traders and small traders as the appropriate setting for the analysis of oligopoly in general equilibrium. The model of noncooperative exchange they used belongs to a line of research initiated by Shapley and Shubik (1977) (see also Dubey and Shubik, 1978; Mas-Colell, 1982; Sahi and Yao, 1989; Amir et al., 1990; Peck et al., 1992; Dubey and Shapley, 1994, among others).

The approach proposed and adopted by Okuno et al. (1980) contrasts with a line of research, initiated by Gabszewicz and Vial (1972) (see also Roberts and Sonnenschein, 1977; Roberts, 1980; Mas-Colell, 1982; Dierker and Grodal, 1986; Codognato and Gabszewicz, 1993; d’Aspremont et al., 1997; Gabszewicz and Michel, 1997; Shitovitz, 1997, among others), in that, in the models belonging to this line, large traders interact sequentially with small traders. Indeed, small traders are assumed to be price takers and submit Walras demand correspondences whereas large traders influence prices by manipulating the Walras price correspondence. Okuno et al. (1980) observed that these models “have been deficient in that they have simply assumed a priori that certain agents behave as price takers while others act noncompetitively, with no formal explanation being given as to why a particular agent should behave one way or the other. Shitovitz’s approach

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represents an important contribution in pointing to an explicit formulation leading to such differences in behavior. [...] we seek to explore the use of this type of model in studying issues of oligopoly in a general equilibrium framework. A specific focus of our work is in illuminating how either perfectly or imperfectly competitive behavior may emerge endogenously [...] depending on the characteristics of the agent and his place in the economy" (see p. 22). These underlying conceptual problems have an important consequence for the existence of equilibria in the models belonging to the line of research initiated by Gabszewicz and Vial (1972); as the Walras price correspondence may fail to be continuous, equilibria may not exist even in mixed strategies.

In this paper, we follow the noncooperative approach to oligopoly in general equilibrium proposed by Okuno et al. (1980). In their paper, no trader is allowed to be both buyer and seller of any commodity. Furthermore, their paper contains no general existence result. Here, we are able to generalize their approach showing that, under standard assumptions on traders' endowments and preferences, and allowing for a general model of trade, there exists a pure strategy Cournot–Nash equilibrium.

We do this by using the model of noncooperative exchange originally proposed by Lloyd S. Shapley and further analyzed by Sahi and Yao (1989) in exchange economies with a finite number of traders, and Codognato and Ghosal (2000) in exchange economies with an atomless continuum of traders. In this model, traders send out bids, i.e., quantity signals, which indicate how much of each commodity they are willing to put up for trade. Every bid of any commodity is tagged by the name of some other commodity for which it has to be exchanged. The rule of price formation requires that a single price system, which equates the value of the total amount of bids for any commodity to the value of the total amount available of that commodity, is used to clear the markets.

We use this model to study exchange with a mixed measure space of traders consisting of atoms, which represent the large traders, and an atomless part, which represents the small traders. The model has the feature that, since no small trader is able to manipulate prices, the best reply of each small trader attains a commodity bundle which maximizes his utility subject to his budget constraint at the prevailing market clearing prices.

In the proof of the existence theorem, we synthesize the techniques used by Sahi and Yao (1989) in finite exchange economies and the techniques used to show the existence of noncooperative equilibria in nonatomic games (see, for instance, Schneider, 1973 and Khan, 1985). Our proof, however, requires us to solve new technical issues. First, in order to show that there exists a best reply to a traders' strategy profile we have to generalize a result used by Aumann (1966) to prove his existence theorem. Second, in order to assure that a Cournot–Nash equilibrium is the limit of a sequence of perturbed Cournot–Nash equilibria, we have to use a version of the Fatou's lemma in several dimensions proved by Artstein (1979).

The paper is organized as follows. In Section 2, we introduce the mathematical model. In Section 3, we state and prove the existence theorem. In Section 4, we discuss the assumptions of the model by means of two examples.

2. The mathematical model

We shall consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space \((T, T, \mu)\), where \(T\) is the set of traders, \(T\) is the \(\sigma\)-algebra of all \(\mu\)-measurable subsets of \(T\), and \(\mu\) is a real valued, nonnegative, countably additive measure defined on \(T\). We assume that \((T, T, \mu)\) is finite, i.e., \(\mu(T) < \infty\). This implies that the measure space \((T, T, \mu)\) contains at most countably many atoms. Let \(T_1\) denote the set of atoms and \(T_0 = T \setminus T_1\) the atomless part of \(T\). A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "all" traders, or "each" trader, or "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. The word "integrable" is to be understood in the sense of Lebesgue.

In the exchange economy, there are \(I\) different commodities. A commodity bundle is a point in \(R^I_+\). An assignment of commodity bundles to traders is an integrable function \(x: T \rightarrow R^I_+\). There is a fixed initial assignment \(w\), satisfying the following assumption.

Assumption 1. \(w(t) > 0\), for all \(t \in T\).

The preferences of each trader \(t \in T\) are described by a utility function \(u_t: R^I_+ \rightarrow R\), satisfying the following assumptions.

Assumption 2. \(u_t: R^I_+ \rightarrow R\) is continuous, strongly monotone in the interior of \(R^I_+\), quasi-concave, for all \(t \in T_0\), and concave, for all \(t \in T_1\).

Let \(B\) denote the Borel \(\sigma\)-algebra of \(R^I_+\). Moreover, let \(T \otimes B\) denote the \(\sigma\)-algebra generated by the sets \(E \times F\) such that \(E \in T\) and \(F \in B\).

Assumption 3. \(u: T \times R^I_+ \rightarrow R\) given by \(u(t, x) = u_t(x)\), for each \(t \in T\) and for each \(x \in R^I_+\), is \(T \otimes B\) measurable.

We also need the following assumption (see Sahi and Yao, 1989).

\(w(t) > 0\), for all \(t \in T\).
Assumption 4. There are at least two traders in $T_1$ for whom $w(t) \succ 0$; $u_t$ is continuously differentiable in $R^l_+$; $\{x \in R^l_+; ut(x) = ut(w(t))\} \subset R^l_+$.

A price vector is a vector $p \in R^l_+$. According to Aumann (1966), we define, for each $p \in R^l_+$, a correspondence $\Delta_p : T \to \mathcal{P}(R^l)$ such that, for each $t \in T$, $\Delta_p(t) = \{x \in R^l_+; px \leq pw(t)\}$, a correspondence $\Gamma_p : T \to \mathcal{P}(R^l)$ such that, for each $t \in T$, $\Gamma_p(t) = \{x \in R^l_+; \text{ for all } y \in \Delta_p(t), ut(x) \geq ut(y)\}$, and finally a correspondence $X_p : T \to \mathcal{P}(R^l)$ such that, for each $t \in T$, $X_p(t) = \Delta_p(t) \cap \Gamma_p(t)$. The following proposition generalizes a result used by Aumann (1966) to prove his existence theorem.

**Proposition.** Under Assumptions 1, 2, and 3, for each $p \in R^l_+$, there exists an integrable function $x_p : T \to R^l_+$ such that, for each $t \in T$, $x_p(t) \in X_p(t)$.

**Proof.** Let $p \in R^l_+$. Then, Assumption 2 implies that, for each $t \in T$, $x_p(t) \neq \emptyset$. Let $Q_+$ denote the subset of $R_+$ consisting of all the nonnegative rational points. Then, the graph of the correspondence $\Delta_p$, \[(t, x) = x \in \Delta_p(t)\] is a subset of $T \otimes B$ as \[(t, x) \in T \otimes B \text{ if and only if } \sum_{i=1}^l p^l x^l \leq \sum_{i=1}^l p^l w^l(t) = T \times R^l_+ \setminus \bigcup_{q \in Q_+} \{t; q > p^l w^l(t)\} \times \{x; \sum_{i=1}^l p^l x^l > q\}\]. Let $Q_+$ denote the subset of $R^l_+$ consisting of all the nonnegative rational points. Then, the graph of the correspondence $\Gamma_p$, \[(t, x) = x \in \Gamma_p(t)\] is a subset of $T \otimes B$ as \[(t, x) \in T \otimes B \text{ if and only if } \sum_{i=1}^l p^l x^l \leq \sum_{i=1}^l p^l w^l(t) \times R^l_+ \bigcup \{t; x^l \geq ut(l)\}\]. Hence, the graph of the correspondence $X_p$, \[(t, x) = x \in X_p(t)\] is a subset of $T \otimes B$ as \[(t, x) \in T \otimes B \text{ if and only if } x \in \Delta_p(t) \cap \Gamma_p(t)\]. But then, by the Measurable Choice Theorem in Aumann (1969), there exists a measurable function $x_p$ such that, for each $t \in T$, $x_p(t) \in X_p(t)$, which is also integrable as $x_p(t) \in \sum_{i=1}^l p^l x^l(t) \leq \frac{1}{p^l}$, $j = 1, \ldots, l$, for all $t \in T$. \(\Box\)

Let $b \in R^l_+$ be a vector such that $b = (b_{11}, b_{12}, \ldots, b_{ll})$. A strategy correspondence is a correspondence $B : T \to \mathcal{P}(R^l_+)$ such that, for each $t \in T$, $B(t) = \{b \in R^l_+ ; \sum_{j=1}^l b_{ij} = w^l(t), i = 1, \ldots, l\}$. A strategy selection is an integrable function $b : T \to R^l_+$ such that, for all $t \in T$, $b(t) \in B(t)$. For each $t \in T$, $b_j(t)$, $i, j = 1, \ldots, l$, represents the amount of commodity $i$ that trader $t$ offers in exchange for commodity $j$. Given a strategy selection $b$, we define the aggregate matrix $B = (b_j(t) \mu_t)$. Moreover, we denote by $b \setminus b(t)$ a strategy selection obtained by replacing $b(t)$ in $b$ with $b(t) \in B(t)$.

Then, we introduce two further definitions (see Sahi and Yao, 1989).

**Definition 1.** A nonnegative square matrix $A$ is said to be irreducible if, for every pair $i, j$, with $i \neq j$, there is a positive integer $k = k(i, j)$ such that $a_{ij}^k > 0$, where $a_{ij}$ denotes the $ij$th entry of the $k$th power $A^k$ of $A$.

**Definition 2.** Given a strategy selection $b$, a price vector $p$ is market clearing if

$$p \in R^l_+, \quad \sum_{i=1}^l p^l b_{ij} = p^l \left( \sum_{i=1}^l b_{ij} \right), \quad j = 1, \ldots, l. \tag{1}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector $p$ satisfying (1) if and only if $B$ is irreducible. Denote by $p(b)$ the function which associates, with each strategy selection $b$ such that $B$ is irreducible, the unique, up to a scalar multiple, clearing price vector $p$.

Given a strategy selection $b$ such that $p$ is market clearing and unique, up to a scalar multiple, consider the assignment determined as follows:

$$x^l(t, b(t), p(b)) = p^l(t) - \sum_{j=1}^l b_{ji}(t) \sum_{i=1}^l b_{ij}(t) \frac{p^l(b)}{p^l(b)},$$

for all $t \in T$, $j = 1, \ldots, l$. It is easy to verify that this assignment is an allocation. Given a strategy selection $b$, the traders’ final holdings are defined as

$$x^l(t) = x^l(t, b(t), p(b)) \quad \text{if } p \text{ is market clearing and unique,}$$

$$x^l(t) = w^l(t) \quad \text{otherwise,}$$

for all $t \in T$, $j = 1, \ldots, l$.

This reformulation of the Shapley’s model allows us to define the following concept of Cournot–Nash equilibrium for exchange economies with an atomless part (see Codognato and Ghosal, 2000).
Definition 3. A strategy selection \( \hat{b} \) such that \( \hat{b} \) is irreducible is a Cournot–Nash equilibrium if
\[
u_t(x(t, \hat{b}(t), p(\hat{b}))) \geq \nu_t(x(t, b(t), p(\hat{b} \setminus b(t))))
\]
for all \( t \in T \) and for all \( b(t) \in B(t) \).

3. The existence theorem

We are now able to state and prove our existence theorem.

Theorem. Under Assumptions 1, 2, 3, and 4, there exists a Cournot–Nash equilibrium \( \hat{b} \).


In the first step of the proof, we show the existence of a slightly perturbed Cournot–Nash equilibrium. More precisely, in this part we generalize Sahi and Yao’s approach by exploiting, as in Schmeidler (1973), the Kakutani–Fan–Glicksberg Theorem (see Theorem 17.55 in Aliprantis and Burkinshaw, 2006, p. 583). To this end, we adapt to our framework some of the arguments provided by Khan (1985) to generalize Schmeidler’s existence theorem.

This step of the proof consists in stating and proving three lemmas. To this end, we need to introduce the following preliminary concepts.

Given \( \epsilon > 0 \), we define the aggregate bid matrix \( \hat{B}^\epsilon \) to be \( \hat{B}^\epsilon = (\int_{E} b_{ij}(t) d\mu + \epsilon) \). Clearly, the matrix \( \hat{B}^\epsilon \) is irreducible. The interpretation is that an outside agency places fixed bids of \( \epsilon \) for each pair of commodities \( (i, j) \). Given \( \epsilon > 0 \), we denote by \( p^\epsilon(b) \) the function which associates, with each strategy selection \( b \), the unique, up to a scalar multiple, price vector which satisfies
\[
\sum_{i=1}^{l} p^\epsilon(b_{ij}) = p^\epsilon\left(\sum_{i=1}^{l} (b_{ji} + \epsilon)\right), \quad j = 1, \ldots, l.
\]

Definition 4. Given \( \epsilon > 0 \), a strategy selection \( \hat{b}^\epsilon \) is an \( \epsilon \)-Cournot–Nash equilibrium if
\[
u_t(x(t, \hat{b}^\epsilon(t), p(\hat{b}^\epsilon))) \geq \nu_t(x(t, b(t), p(\hat{b} \setminus b(t))))
\]
for all \( t \in T \) and for all \( b(t) \in B(t) \).

We neglect, as usual, the distinction between integrable functions and equivalence classes of such functions and denote by \( L_1(\mu, R^d) \) the set of integrable functions taking values in \( R^d \) and by \( L_1(\mu, B(\cdot)) \) the set of strategy selections. Note that the locally convex Hausdorff space we shall be working in is \( L_1(\mu, R^d) \), endowed with its weak topology.

The first lemma provides us with the properties of \( L_1(\mu, B(\cdot)) \) required for the application of the Kakutani–Fan–Glicksberg Theorem.

Lemma 1. The set \( L_1(\mu, B(\cdot)) \) is nonempty, convex and weakly compact.

Proof. For each \( i = 1, \ldots, l \), let \( \lambda_{ij} \geq 0 \). \( \sum_{i=1}^{l} \lambda_{ij} = 1 \). Since \( \omega \) is an assignment, the function \( b : T \to R^d_+ \) such that, for each \( t \in T \), \( b_{ij}(t) = \lambda_{ij} \omega(t) \), \( i, j = 1, \ldots, l \) belongs to \( L_1(\mu, B(\cdot)) \). The fact that \( L_1(\mu, R^d) \) is a vector space and the fact that, for each \( t \in T \), \( B(t) \) is convex imply that \( L_1(\mu, B(\cdot)) \) is convex. Finally, the fact \( L_1(\mu, B(\cdot)) \) is weakly compact can be proven following Khan (1985). First, notice that \( \sup_{E \subset L_1(\mu, B(\cdot)))} \lambda \int_{E} |b_{ij}(t)| d\mu < \infty \), \( i, j = 1, \ldots, l \). For each \( \epsilon > 0 \), \( \int_{E} |b_{ij}(t)| d\mu \leq \epsilon \), \( i, j = 1, \ldots, l \). This, by the Dunford–Pettis Theorem (see Diestel, 1984, p. 93), in turn implies that \( L_1(\mu, B(\cdot)) \) is weakly compact closure. Now, let \( \{b^n\} \) be a convergent sequence of \( L_1(\mu, B(\cdot)) \). Since \( L_1(\mu, R^d) \) is complete, \( \{b^n\} \) converges in the mean to an integrable function \( b \). But then, there exists a subsequence \( \{b^{n_k}\} \) of \( \{b^n\} \) such that \( b^{n_k}(t) \) converges to \( b(t) \), for all \( t \in T \) (see Theorem 25.5 in Aliprantis and Burkinshaw, 1998, p. 203). The compactness of \( B(t) \), for each \( t \in T \), implies that \( b \in L_1(\mu, B(\cdot)) \). Hence \( L_1(\mu, B(\cdot)) \) is norm closed and, since it is also convex, it is weakly closed (see Corollary 4 in Diestel, 1984, p. 12). □

Now, given \( \epsilon > 0 \), let \( \alpha^\epsilon : L_1(\mu, B(\cdot)) \to L_1(\mu, B(\cdot)) \) be a correspondence such that \( \alpha^\epsilon(b) = \{b \in L_1(\mu, B(\cdot)) : b(t) \in \alpha^\epsilon(b)\} \), for all \( t \in T \) where, for each \( t \in T \), the correspondence \( \alpha^\epsilon : L_1(\mu, B(\cdot)) \to B(t) \) is such that \( \alpha^\epsilon(b) = \arg\max_{f \in [x(t, b(t), p^\epsilon(b \setminus b(t)))]}: f(b(t) \in B(t)) \).

The second lemma provides us with the properties of the correspondence \( \alpha^\epsilon \), required for the application of the Kakutani–Fan–Glicksberg Theorem. A crucial role in the proof is played by the generalization of Aumann’s results, provided
by the Proposition in the previous section, and by Lemma 5 in Codognato and Ghosal (2000), used to prove an equivalence à la Aumann between the set of the Cournot–Nash equilibrium allocations and the set of the Walras equilibrium allocations for exchange economies with an atomless continuum of traders.

**Lemma 2.** Given $\epsilon > 0$, the correspondence $\alpha^\epsilon : L_1(\mu, B(\cdot)) \to L_1(\mu, B(\cdot))$ is such that the set $\alpha^\epsilon(b)$ is nonempty and convex, for all $b \in L_1(\mu, B(\cdot))$, and it has a weakly closed graph.

**Proof.** Let $\epsilon > 0$ be given. Consider a trader $t \in T_1$. By Lemmas 4 and 5 in Sahi and Yao (1989), we know that $\alpha^\epsilon$ is an upper hemicontinuous correspondence such that, for all $b \in L_1(\mu, B(\cdot))$, $\alpha^\epsilon_1(b)$ is nonempty, compact and convex. Now, consider a trader $t \in T_0$. Given $b \in L_1(\mu, B(\cdot))$, it is immediate to verify that $u_b(\langle t, b(t), p^b(\cdot, b(t)) \rangle) = u_b(\langle t, b(t), p^b(b(t)) \rangle)$, for all $b \in B(t)$. Therefore, for all $b \in L_1(\mu, B(\cdot))$, $\alpha^\epsilon_1(b)$ is nonempty and compact, by the continuity of the function $u_b(\langle t, b(t), p^b(b(t)) \rangle)$ over the compact set $B(t)$, and convex, by Assumption 2. The upper hemicontinuity of $\alpha^\epsilon_1$ is a straightforward consequence of the Berge Maximum Theorem (see Theorem 17.31 in Aliprantis and Border, 2006, p. 579). Now, given a strategy selection $b \in L_1(\mu, B(\cdot))$, by the Proposition, there exists an integrable function $x_{p^b}(t)$ such that, for each $t \in T$, $x_{p^b}(t) \in x_{p^b}(t)$. By Lemma 5 in Codognato and Ghosal (2000), for each $t \in T$, there exist $t^j(t) \geq 0$, $\sum_j t^j(t) = 1$, such that

$$x_{p^b}(t) = \lambda^j(t) \sum_{j=1}^\infty p^j_{\epsilon}(b)w^j(t) \quad j = 1, \ldots, l.$$  

Define a function $\lambda : T \to R_1$, such that $\lambda(t) = \lambda(t)$, for each $t \in T$. Since $x_{p^b}(t)$ and $w$ are measurable functions and $\sum_{j=1}^\infty p^j(b)w^j(t) \gg 0$, for all $t \in T$, $\lambda$ is a measurable function. Now, define a function $b^\epsilon : T \to R_1$, such that $b^\epsilon_{ij}(t) = w^j(t)\lambda(t)$, $j = 1, \ldots, l$, for all $t \in T_0$ and $b^\epsilon(t) \in \alpha^\epsilon_1(b)$, for all $t \in T_1$. By Theorem 2 in Codognato and Ghosal (2000), $\alpha^\epsilon_1(t) \in \alpha^\epsilon_2(b)$, for each $t \in T_0$. Moreover, the restriction of $b^\epsilon$ to $T_0$ is measurable as the restriction of $\lambda$ to $T_0$ is measurable and the restriction of $b^\epsilon$ to $T_1$ is measurable as $T_1$ consists solely of at most countably many atoms. Then, the function $b^\epsilon$ is integrable as $\sum_{ij} b^\epsilon_{ij}(t) \leq w^j(t)$, for all $t \in T$. But then, $\alpha^\epsilon(b)$ is nonempty. The convexity of $\alpha^\epsilon(b)$ is a straightforward consequence of the convexity of $\alpha^\epsilon_1(b)$, for all $t \in T$. Finally, the fact that $\alpha^\epsilon$ has a weakly closed graph can be proven following Khan (1985). Let $(b^\epsilon_1, b^\epsilon_2)$ be a net converging to $(b, b^\epsilon)$ where $b^\epsilon_2 \in \alpha^\epsilon_2(b^\epsilon)$. The weak closure of the set $(b^\epsilon, b^\epsilon_2)$ is weakly compact as it is a subset of the set $L_1(\mu, B(\cdot)) \times L_1(\mu, B(\cdot))$ and then, by the Eberlein–Smulian Theorem (see Theorem 6.34 in Aliprantis and Border, 2006, p. 241), it is weakly sequentially compact. This, in turn, implies that there exists a sequence $(b_1, b_2, \ldots)$, extracted from the net $(b^\epsilon, b^\epsilon_2)$, which converges weakly to $(b, b^\epsilon)$ (see Problem 17L in Kelley and Namioka, 1963, p. 165). Now, for each $t \in T$, denote by $L_1(b^\epsilon(t))$ the set of the limit points of the sequence $(b^\epsilon(t))$ and by $coL_1(b^\epsilon(t))$ the set of the convex combinations of these limit points. Given a $t \in T$, let $b(t)$ be a limit point of the sequence $(b^\epsilon(t))$. Then, there is a subsequence $(b^\epsilon_{1k}(t))$ of the sequence $(b^\epsilon(t))$ which converges to $b(t)$. Now, consider the subsequence $(b^\epsilon_{1k}(t))$ of the sequence $(b^\epsilon(t))$. It has a subsequence $(b^\epsilon_{1k}(t))$ which converges weakly to $b_1$ as $L_1(\mu, B(\cdot))$ is weakly sequentially compact by the Eberlein–Smulian Theorem. Therefore there exists a sequence $(b^\epsilon_{1k}(t), b^\epsilon_{1k}(t))$ which converges weakly to $(b_1, b_1)$ and which is such that $L_1(b^\epsilon_{1k}(t)) \in \alpha^\epsilon_1(b^\epsilon_{1k}(t))$. Hence, $b(t) \in \alpha^\epsilon_1(b)$, by the Closed Graph Theorem (see Theorem 17.11 in Aliprantis and Border, 2006, p. 561), as $L_1(\mu, B(\cdot))$ is weakly compact and $\alpha_1$ is upper hemicontinuous and compact-valued. But then, $L_1(b^\epsilon_1(t)) \subseteq b_1^\epsilon(t)$, for all $t \in T$. This, together with the fact that $\alpha^\epsilon_1(b)$ is convex, implies that $coL_1(b^\epsilon_1(t)) \subseteq b_1^\epsilon(t)$, for all $t \in T$. Since the sequence $(b^\epsilon(t))$ converges weakly to $b^\epsilon$ and is uniformly integrable, by Proposition C in Artstein (1979), we have that $b^\epsilon(t) \in coL_1(b^\epsilon_1(t))$, for all $t \in T$. But then, $b^\epsilon(t) \in \alpha^\epsilon_1(b)$, for all $t \in T$. □

Now, we are ready prove the existence of an $\epsilon$-Cournot–Nash equilibrium.

**Lemma 3.** For each $\epsilon > 0$, there exists an $\epsilon$-Cournot–Nash equilibrium $b^\epsilon$.

**Proof.** $L_1(\mu, B(\cdot))$ is nonempty and weakly compact, by Lemma 1. Moreover, for each $\epsilon > 0$, the correspondence $\alpha^\epsilon$ has nonempty convex values and a weakly closed graph, by Lemma 2. Therefore, for each $\epsilon > 0$, by the Kakutani–Fan–Glicksberg Theorem, there exists a fixed point $b^\epsilon$ of the correspondence $\alpha^\epsilon$ and hence an $\epsilon$-Cournot–Nash equilibrium. □

The second step of the proof consists in defining the notion of a $\delta$-positive strategy correspondence and showing the existence of a $\delta$-positive $\epsilon$-Cournot–Nash equilibrium. This step is needed to build a sequence of $\epsilon$-Cournot–Nash equilibria whose limit is a Cournot–Nash equilibrium and is based on the assumption that there exist at least two atoms who hold a strictly positive amount of all commodities.

Let $T_1 \subseteq T_1$ be a set consisting of two traders in $T_1$ for whom Assumption 4 holds. Moreover, let $\delta = \min_{t \in T_1} \{ \frac{1}{T} \min \{ w_1(t), \ldots, w_l(t) \} \}$. We say that the correspondence $b^\delta : T \to R^l$ is a $\delta$-positive strategy correspondence if $b^\delta(t) = b(t) \cap \{ b \in R^l : \sum_{j=1}^l \sum_{i=1}^l p_{ij}(b) + b_j \geq \delta \}$, for each $j \subseteq \{1, \ldots, l\}$, for each $t \in T_1$, $b^\delta(t) = b(t)$, for the remaining traders $t \in T$. Moreover, we say that an $\epsilon$-Cournot–Nash equilibrium $b^\epsilon$ is $\delta$-positive if, for all $t \in T$, $b^\epsilon(t) \in b^\delta(t)$. Given $\epsilon > 0$, let $\alpha^{\delta, \epsilon} : L_1(\mu, B(\cdot)) \to L_1(\mu, B(\cdot))$ be a correspondence such that $\alpha^{\delta, \epsilon}(b) = b \in L_1(\mu, B(\cdot))$, $b(t) \in \alpha^{\delta, \epsilon}(b)$, for all $t \in T$, where, for each $t \in T$, $\alpha^{\delta, \epsilon}(b) = \alpha^{\epsilon}(b) \cap b^\delta(t)$. The following lemma is a strengthening of Lemma 3.
Lemma 4. For each $\epsilon > 0$, there exists a $\delta$-positive $\epsilon$-Cournot–Nash equilibrium $\hat{b}^\epsilon$.

Proof. Let $\epsilon > 0$ be given. By Lemma 6 in Sahi and Yao (1989), we know that, for each $b \in L_1(\mu, B(\cdot))$, $\alpha_{\epsilon \delta}^t(b)$ is nonempty, for each $t \in T$. But then, by the same argument of Lemma 2, $\alpha_{\epsilon \delta}^t(b)$ is nonempty. The convexity of $\alpha_{\epsilon \delta}^t(b)$ is a straightforward consequence of the convexity of $\alpha_{\epsilon \delta}^t(b)$ and $B^t(\cdot)$, for all $t \in T$. For each $t \in T$, $\alpha_{\epsilon \delta}^t$ is upper hemicontinuous and compact valued as it is the intersection of the correspondence $\alpha_{\epsilon \delta}^t$, which is upper hemicontinuous and compact valued by Lemma 2, and the continuous and compact valued correspondence which assigns to each strategy selection $b \in L_1(\mu, B(\cdot))$ the strategy set $B(t)$ (see Theorem 17.25 in Aliprantis and Border, 2006, p. 567). Therefore, $\alpha_{\epsilon \delta}^t$ has a weakly closed graph, by the same argument used in the proof of Lemma 2. But then, as in the proof of Lemma 3, by the Kakutani–Fan–Glicksberg Theorem, there exists a fixed point $\hat{b}^\epsilon$ of the correspondence $\alpha_{\epsilon \delta}^t$. and hence a $\delta$-positive $\epsilon$-Cournot–Nash equilibrium.

Finally, in the last step of the proof, we show that there exists the limit of a sequence of $\delta$-positive $\epsilon$-Cournot–Nash equilibria and that this limit is a Cournot–Nash equilibrium. In this part of the proof, the assumption that there exist at least two atoms with a continuously differentiable utility function plays a fundamental role as it guarantees, by Lemma 9 in Sahi and Yao (1989), that the limit of the sequence of prices exists and is a strictly positive price vector. Adapting Say and Yao’s approach to our framework raises new technical issues, due to the presence of the atomless part. We handle them by using a generalization of the Fatou’s lemma in several dimensions provided by Artstein (1979) and we add a third case, concerning the atomless part, to the two cases, concerning atoms, considered by Sahi and Yao (1989).

Let $\epsilon_n = \frac{\alpha}{n}, n = 1, 2, \ldots$. By Lemma 4, for each $n = 1, 2, \ldots$, there is a $\delta$-positive $\epsilon_n$-Cournot–Nash equilibrium $\hat{b}^\epsilon_n$. The fact that the sequence $\{f_T \hat{b}^\epsilon_n(t) \mu(t)\}$ belongs to the compact set $\{b_j \in \mathbb{R}^T : b_j \leq f_T w(t) \mu(t), i, j = 1, \ldots, l\} = \alpha_{\epsilon \delta}^t$, for each $j \in \{1, \ldots, l\}$ and the sequence $\{\hat{b}^\epsilon_n\}$, where $\hat{p}^\epsilon_n = p(\hat{b}^\epsilon_n)$, for each $n = 1, 2, \ldots$, belongs, by Lemma 9 in Sahi and Yao (1989), to a compact set $P$, implies that there is a subsequence $\{f_T \hat{b}^\epsilon_n(t) \mu(t)\}$ of the sequence $\{f_T \hat{b}^\epsilon_n(t) \mu(t)\}$ which converges to an element of the set $\{b_j \in \mathbb{R}^T : b_j \leq f_T w(t) \mu(t), i, j = 1, \ldots, l\}$ and a sequence $\{\hat{p}^\epsilon_n\}$ of the sequence $\{\hat{p}^\epsilon_n\}$ which converges to an element of the set $P$. Since the sequence $\{\hat{b}^\epsilon_n\}$ satisfies the assumptions of Theorem A in Artstein (1979), there is a function $\hat{b}$ such that $\hat{b}(t)$ is a limit point of the sequence $\{\hat{b}^\epsilon_n(t)\}$, for all $t \in T$, and such that the sequence $\{f_T \hat{b}^\epsilon_n(t) \mu(t)\}$ converges to $f_T \hat{b}(t) \mu(t)$. Moreover, for all $t \in T$, $\hat{b}(t) \in B^t(\cdot)$ because $\hat{b}(t)$ is the limit of a subsequence of $\{\hat{b}^\epsilon_n\}$. Since the sequence $\{\hat{p}^\epsilon_n\}$ converges to a price vector $\hat{p} \in P$, by the continuity of (2), $\hat{p}$ and $f_T \hat{b}(t) \mu(t)$ must satisfy (1). Moreover, since, by Lemma 9 in Sahi and Yao (1989), $\hat{p} \geq 0$, Lemma 1 in Sahi and Yao (1989) implies that $\hat{b}$ is completely reducible. But then, since $\hat{b} \in L_1(\mu, B(\cdot))$, by Remark 3 in Sahi and Yao (1989), $\hat{B}$ must be irreducible. In order to conclude that $\hat{b}$ is a $\delta$-positive Cournot–Nash equilibrium, we have to show that $u_t(x(t, \hat{b}(t), \hat{p})) = u_t(x(t, \hat{b}(t), \hat{p}(b(\hat{b}(t))))$, for all $t \in T$ and for all $b(t) \in B(t)$. Let $\hat{B} \setminus b(t)$ denote the aggregate matrix corresponding to the strategy selection $\hat{b}(t)$ and let $\hat{B} = B(t)$ denote the aggregate matrix corresponding to the strategy selection $\hat{b}^\epsilon_n \setminus b(t)$, for each $n = 1, 2, \ldots$. As in Sahi and Yao (1989), we proceed by considering the following possible cases.

Case 1. $t \in T_1$ and $b(t) \in B(t)$ is such that $\hat{B} \setminus b(t)$ is completely reducible. Clearly, $\hat{b}^\epsilon_n \setminus b(t)$ is irreducible, for each $n = 1, 2, \ldots$, and so is $\hat{b}(t)$, by Remark 3 in Sahi and Yao (1989). Consider a subsequence $\{\hat{b}^\epsilon_n(t) \mu(t)\}$ of the sequence $\{\hat{b}^\epsilon_n(t) \mu(t)\}$ which converges to $\hat{b}(t)$ and a subsequence $\{f_T \hat{b}^\epsilon_n(t) \mu(t)\}$ of the sequence $\{f_T \hat{b}^\epsilon_n(t) \mu(t)\}$ which converges to a point $\hat{b} \in \{b_j \in \mathbb{R}^T : b_j \leq f_T w(t) \mu(t), i, j = 1, \ldots, l\}$. Therefore, the sequence $\{f_T \hat{b}^\epsilon_n(t) \mu(t) + \hat{b}^\epsilon_n \hat{p}(t) \mu(t)\}$ converges to $\hat{b} + \hat{b}(t) \mu(t)$. Then, $\hat{b} = f_T \hat{b}(t) \mu(t)$ as the sequence $\{f_T \hat{b}^\epsilon_n(t) \mu(t)\}$ converges to $f_T \hat{b}(t) \mu(t)$. But then, the sequence $\{f_T \hat{b}^\epsilon_n \setminus b(t) \mu(t)\}$ converges to $f_T \hat{b}(t) \mu(t)$. This and the fact that, by Lemma 2 in Sahi and Yao (1989), prices are cofactors, imply that the sequence $\{p^\epsilon_n(\hat{b}^\epsilon_n \setminus b(t))\}$ converges to $p(b(\hat{b}(t)))$. Consequently, the sequence $\{x(t, b(t), \hat{p}(b(\hat{b}(t))))\}$ converges to $x(t, b(t), p(b(\hat{b}(t))))$. The fact that the sequence $\{x(t, \hat{b}^\epsilon_n(t), \hat{p}(\hat{b}^\epsilon_n))\}$ converges to $x(t, \hat{b}(t), \hat{p})$ and that $u_t(x(t, \hat{b}^\epsilon_n(t), \hat{p}(\hat{b}^\epsilon_n))) \geq u_t(x(t, \hat{b}(t), \hat{p}(\hat{b}^\epsilon_n(t))))$, for each $n = 1, 2, \ldots$, allow us to conclude, by Assumption 2, that $u_t(x(t, \hat{b}(t), \hat{p})) \geq u_t(x(t, b(t), p(b \setminus b(t))))$.

Case 2. $t \in T_1$ and $b(t) \in B(t)$ is not completely reducible. Consider a subsequence $\{\hat{b}^\epsilon_n(t) \mu(t)\}$ of the sequence $\{\hat{b}^\epsilon_n(t) \mu(t)\}$ which converges to $\hat{b}(t)$. The fact that the sequence $\{x(t, \hat{b}^\epsilon_n(t), \hat{p}(\hat{b}^\epsilon_n))\}$ converges to $x(t, \hat{b}(t), \hat{p})$ and the fact that $u_t(x(t, \hat{b}^\epsilon_n(t), \hat{p}(\hat{b}^\epsilon_n))) \geq u_t(w(t))$, for each $n = 1, 2, \ldots$, imply, by Assumption 2, that $u_t(x(t, \hat{b}(t), \hat{p})) \geq u_t(x(t, b(t), p(b \setminus b(t))))$.

Case 3. $t \in T_0$ and $b(t) \in B(t)$. Clearly, the matrix $\hat{B} \setminus b(t)$ is irreducible, $p^\epsilon_n(\hat{b}^\epsilon_n \setminus b(t)) = p^\epsilon_n(\hat{b}^\epsilon_n)$, for each $n = 1, 2, \ldots$, and $p(\hat{b} \setminus b(t)) = p(b)$. Consider a subsequence $\{\hat{b}^\epsilon_n(t) \mu(t)\}$ of the sequence $\{\hat{b}^\epsilon_n(t) \mu(t)\}$ which converges to $\hat{b}(t)$. The fact
that the sequence \( \{x(\tau, b(\tau), \hat{p}^t)\} \) converges to \( x(\tau, b(\tau), \hat{p}) \), the fact that the sequence \( \{x(\tau, \tilde{b}^{t_n}(\tau), \tilde{p}^t)\} \) converges to \( x(\tau, \tilde{b}(\tau), \tilde{p}) \) and the fact that \( u_t(x(\tau, \tilde{b}^{t_n}(\tau), \tilde{p}^t)) \geq u_t(x(\tau, b(\tau), \hat{p}^t)) \), for each \( n = 1, 2, \ldots, \) imply that \( u_t(x(\tau, b(\tau), \hat{p})) \geq u_t(x(\tau, b(\tau), \tilde{p})) \). □

### 4. Discussion of the model

We have shown the existence of a Cournot–Nash equilibrium under the assumption that there are at least two atoms with strictly positive endowments of all commodities. We provide now an example which shows that the violation of this assumption may lead to a nonexistence result.

**Example 1.** Consider the following exchange economy: \( I = 2, T = T_1 \cup T_0, \) where \( T_1 = \{2, 3\} \) and \( T_0 = [0, 1] \), \( w(2) = (1, 0), \) \( w(3) = (0, 1), \) \( u_2(\cdot) \) and \( u_3(\cdot) \) satisfy Assumptions 2 and 4, \( w(t) = (0, 1), \) for all \( t \in T_0, u_t(\cdot) \) satisfies Assumptions 2 and 3, for all \( t \in T_0. \) This exchange economy does not admit any Cournot–Nash equilibrium.

**Proof.** Suppose that there exists a Cournot–Nash equilibrium \( \hat{b}. \) This implies that \( \bar{b}_{12}(2) > 0 \) and \( \bar{b}_{21}(3) + \int_{T_0} \bar{b}_{21}(t) d\mu > 0 \) as the matrix \( \tilde{B} \) is irreducible. According to (1), the ratio of prices is

\[
\frac{\hat{p}_1}{\hat{p}_2} = \frac{\bar{b}_{21}(3) + \int_{T_0} \bar{b}_{21}(t) d\mu}{\bar{b}_{12}(2)}.
\]

The final holdings of atom 2 are therefore determined as follows:

\[
x^1(2, \hat{b}(2), p(\hat{b})) = 1 - \bar{b}_{12}(2),
\]

\[
x^2(2, \hat{b}(2), p(\hat{b})) = \bar{b}_{21}(3) + \int_{T_0} \bar{b}_{21}(t) d\mu.
\]

Then, \( \bar{b}_{12}(2) > 0 \) cannot be a best reply of atom 2 as \( u_2(\cdot) \) is strongly monotone, by Assumption 2. This generates a contradiction. □

Aumann (1966) proved the existence of a Walras equilibrium for exchange economies with an atomless continuum of traders under Assumptions 1, 2, and 3, and a further assumption requiring that each commodity is held, in the aggregate, by the atomless part. The next example shows that a Cournot–Nash equilibrium may not exist for exchange economies with atoms and an atomless part which satisfies Aumann’s assumptions.

**Example 2.** Consider the following exchange economy: \( I = 2, T = T_1 \cup T_0, \) where \( T_1 = \{2, 3\} \) and \( T_0 = [0, 1] \), \( w(2) = (1, 0), \) \( w(3) = (0, 1), \) \( u_2(\cdot) \) and \( u_3(\cdot) \) satisfy Assumptions 2 and 4, \( w(t) = (0, 1), \) for all \( t \in [0, 1], w(t) = (0, 1), \) for all \( t \in [\frac{1}{2}, 1], u_t(\cdot) = kx^1 + x^2, \) for all \( t \in [0, \frac{1}{2}], u_t(\cdot) = x^1 + kx^2, \) for all \( t \in [\frac{1}{2}, 1], k > 1. \) This exchange economy does not admit any Cournot–Nash equilibrium.

**Proof.** Suppose that there exists a Cournot–Nash equilibrium \( \hat{b}. \) This implies that \( \bar{b}_{12} = \bar{b}_{12}(2) + \int_{T_0} \bar{b}_{12}(t) d\mu > 0 \) and \( \bar{b}_{21} = \bar{b}_{21}(3) + \int_{T_0} \bar{b}_{21}(t) d\mu > 0 \) as the matrix \( \tilde{B} \) is irreducible. According to (1), the ratio of prices is

\[
\frac{\hat{p}_1}{\hat{p}_2} = \frac{\bar{b}_{21}}{\bar{b}_{12}}.
\]

The final holdings of traders belonging to the atomless part are therefore determined as follows:

\[
x^1(t, \hat{b}(t), p(\hat{b})) = 1 - \bar{b}_{12}(t),
\]

\[
x^2(t, \hat{b}(t), p(\hat{b})) = \frac{\bar{b}_{21} - \bar{b}_{21}(t)}{\bar{b}_{12}},
\]

for all \( t \in [0, \frac{1}{2}], \) and

\[
x^1(t, \hat{b}(t), p(\hat{b})) = \frac{\bar{b}_{21}(t)}{\bar{b}_{21}},
\]

\[
x^2(t, \hat{b}(t), p(\hat{b})) = 1 - \bar{b}_{21}(t),
\]

for all \( t \in [\frac{1}{2}, 1]. \)
for all \( t \in \left[ \frac{1}{2}, 1 \right] \). This implies that, for all \( t \in [0, \frac{1}{2}] \), \( \tilde{b}_{12}(t) = 0 \) if \( k > \frac{\hat{b}_{12}}{b_{12}} \), and, for all \( t \in [\frac{1}{2}, 1] \), \( \tilde{b}_{21}(t) = 0 \) if \( k > \frac{\hat{b}_{21}}{b_{21}} \).

Suppose that \( \int_{0}^{1} \tilde{b}_{12}(t) d\mu > 0 \) and \( \int_{0}^{1} \tilde{b}_{21}(t) d\mu > 0 \). Then, \( k \leq \frac{b_{12}}{b_{21}} \) and \( k \leq \frac{b_{21}}{b_{12}} \). But then, \( k^{2} \tilde{b}_{21} \leq \tilde{b}_{21} \) with \( k^{2} > 1 \), a contradiction. This implies that \( \int_{0}^{1} \tilde{b}_{21}(t) d\mu = 0 \) or \( \int_{0}^{1} \tilde{b}_{21}(t) d\mu = 0 \). Suppose, without loss of generality, that \( \int_{0}^{1} \tilde{b}_{12}(t) d\mu = 0 \). This implies that \( \tilde{b}_{12}(2) > 0 \) as the matrix \( \tilde{B} \) is irreducible. The final holdings of atom 2 are therefore determined as follows:

\[
x^{1}(2, \tilde{b}(2), p(\tilde{b})) = 1 - \tilde{b}_{12}(2),
\]

\[
x^{2}(2, \tilde{b}(2), p(\tilde{b})) = \tilde{b}_{21}(3) + \frac{1}{2} \int_{0}^{1} \tilde{b}_{21}(t) d\mu.
\]

Then, \( \tilde{b}_{12}(2) > 0 \) cannot be a best reply of atom 2 as \( u_{2}(\cdot) \) is strongly monotone, by Assumption 2. This generates a contradiction. \( \square \)

References


